Econometrica, Vol. 82, No. 1 (January, 2014), 1–39

EXPECTED UNCERTAIN UTILITY THEORY

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We introduce and analyze *expected uncertain utility* (*EUU*) *theory*. A prior and an *interval utility* characterize an EUU decision maker. The decision maker transforms each uncertain prospect into an *interval-valued* prospect that assigns an interval of prizes to each state. She then ranks prospects according to their expected interval utilities. We define uncertainty aversion for EUU, use the EUU model to address the Ellsberg Paradox and other ambiguity evidence, and relate EUU theory to existing models.

KEYWORDS: Ambiguity, Ellsberg, subjective probability.

1. INTRODUCTION

WE CONSIDER AN AGENT who must choose between Savage acts that associate a monetary prize to every state of nature. The agent has a prior μ on a σ-algebra $\mathcal{E}$ of *ideal events*. Ideal events capture aspects of the uncertainty that the agent can quantify without difficulty. For $\mathcal{E}$-measurable acts (*ideal acts*), the agent is an expected utility maximizer. Therefore, the utility of an $\mathcal{E}$-measurable act f is

$$W(f) = \int v(f)\, d\mu \tag{1}$$

for some von Neumann–Morgenstern (vNM) utility index v.

When confronted with a non-ideal act, f, the agent forms an ideal lower bound $[f]_1$ and an ideal upper bound $[f]_2$. These bounds represent the range of possible outcomes implied by uncertainty that cannot be quantified. The utility of act f is

$$W(f) = \int u\big([f]_1, [f]_2\big)\, d\mu \tag{2}$$

where $u(x, y)$ is the utility of an unquantifiable uncertain prospect with prizes between x and y. We refer to the utility function W as an *expected uncertain utility* (EUU) and to the utility index u as an *interval utility*.

When f is ideal, the lower and upper bounds coincide and (2) reduces to the expected utility formula (1) with utility index v such that $v(x) = u(x, x)$. The purpose of the extension to non-ideal acts is to accommodate well-documented deviations from expected utility theory. EUU theory interprets these deviations as instances in which the decision maker cannot quantify all aspects of the relevant uncertainty.

[1]The paper was presented as the Fisher–Schultz Lecture at the 2009 Econometric Society European Meetings in Barcelona. This research was supported by Grants from the National Science Foundation. We are grateful to Adriano Basso, Jay Lu, Asen Kochov, Peter Wakker, and three anonymous referees for their comments and suggestions.

 DOI: 10.3982/ECTA9188

EUU theory is closely tied to Savage's model (Savage (1954)) and, as a result, mirrors that model's separation between *uncertainty perception* and *uncertainty attitude*: the prior μ measures uncertainty perception, while the interval utility u measures uncertainty attitude. Despite its closeness to subjective expected utility theory, EUU is flexible enough to accommodate Ellsberg-style and Allais-style evidence. The former identifies behavior inconsistent with any single subjective prior over the event space, while the latter deals with systematic violations of the independence axiom assuming preferences are consistent with a subjective probability assessment.

In this paper, we provide a Savage-style representation theorem for EUU theory and use it to address Ellsberg-style experiments. Section 2 introduces the model, the axioms, and the representation theorem. Section 3 defines comparative measures of uncertainty aversion and the uncertainty of events, and relates these measures to the parameters of the model. Just as Savage's theorem, our representation theorem requires a rich (continuum) state space. To address experimental evidence and to relate our model to the literature, it is convenient to restrict to acts that are measurable with respect to a fixed finite partition of the state space. We introduce the discrete version of EUU theory in Section 4, and provide a detailed discussion of the related literature in Section 5. Section 6 uses discrete EUU and the comparative measures to address Ellsberg-style evidence, and Section 7 shows how EUU accommodates variations of Ellsberg experiments due to Machina (2009). In a companion paper (Gul and Pesendorfer (2013)), we show how EUU can be used to address Allais-style evidence and evidence showing that, ceteris paribus, decision makers prefer uncertain prospects that depend on familiar rather than unfamiliar events.

2. MODEL AND AXIOMS

The interval $X = [l, m]$, $l < m$, is the set of monetary prizes and Ω is the state space. An act is a function $f : \Omega \to X$ and $\mathcal{F}$ is the set of all acts. For any property P, let {P} denote the set of all $\omega \in \Omega$ at which P holds. For example, $\{f > g\} = \{\omega \mid f(\omega) > g(\omega)\}$. For $\{P\} = \Omega$, we simply write P; that is, $f \in [x, y]$ means $\{\omega \mid f(\omega) \in [x, y]\} = \Omega$. We identify $x \in X$ with the constant act $f = x$. Consider the following six axioms for binary relations on $\mathcal{F}$:

AXIOM 1: *The binary relation $\succeq$ is complete and transitive*.

Axiom 2 is a natural consequence of the fact that acts yield monetary prizes.[2]

AXIOM 2: *If $f > g$, then $f \succ g$*.

[2]Though natural, the assumption is not implied by the Savage axioms and cannot be satisfied in the Savage model with a countable state space; see Wakker (1993).

For any $f, g \in \mathcal{F}$ and $A \subset \Omega$, let fAg denote the act that agrees with f on A and with g on the A^c, the complement of A; that is, fAg is the unique act h such that $A \subset \{h = f\}$ and $A^c \subset \{h = g\}$. Ideal events are events E such that Savage's sure thing principle holds for E and E^c.

DEFINITION: An event E is *ideal* if $[fEh \succeq gEh$ and $hEf \succeq hEg]$ implies $[fEh' \succeq gEh'$ and $h'Ef \succeq h'Eg]$ for all acts f, g, h, and h'.

An event A is *null* if $fAh \sim gAh$ for all $f, g, h \in \mathcal{F}$. Let $\mathcal{E}$ be the set of all ideal events and E, E', E_i, etc. denote elements of $\mathcal{E}$. Let $\mathcal{E}_+ \subset \mathcal{E}$ denote the set of ideal events that are not null. Our main hypothesis is that the agent uses elements of $\mathcal{E}_+$ to quantify the uncertainty of all events. More precisely, if A contains exactly the same elements of $\mathcal{E}_+$ as B and A^c contains exactly the same elements of $\mathcal{E}_+$ as B^c, then the agent is indifferent between identical bets on A and B. Axiom 3, below, formalizes this hypothesis; the axiom is weaker since it applies the hypothesis only to a subset of events, but our representation implies that it holds for all events.

An event is *diffuse* if it and its complement contains no element of $\mathcal{E}_+$.

DEFINITION: An event D is *diffuse* if $E \cap D \neq \emptyset \neq E \cap D^c$ for every $E \in \mathcal{E}_+$.

Let $\mathcal{D}$ be the set of all diffuse events; let D, D', D_i, etc. denote elements of $\mathcal{D}$ and note that they represent events whose likelihood cannot be bounded by elements of $\mathcal{E}_+$. Axiom 3 requires that the decision maker is indifferent between betting on $E \cap D_1$ and $E \cap D_2$ if $E \in \mathcal{E}$ and $D_1, D_2 \in \mathcal{D}$. Notice that $E \cap D_1$ and $E \cap D_2$ contain no element of $\mathcal{E}_+$, while $(E \cap D_1)^c$ and $(E \cap D_2)^c$ both contain exactly those elements of $\mathcal{E}_+$ that are contained in E^c. Thus, Axiom 3, below, is an implication of our main hypothesis.

AXIOM 3: *$yE \cap Dx \sim yE \cap D'x$ for all x, y, E, D, and D'.*

One consequence of Axiom 3 is that it permits the partitioning of Ω into a finite collection of sets $D_1, \dots, D_n$ such that $y(D_j \cup D_k)x \sim yD_ix$ for all i, j, and k. Note that Savage's theory allows for a similar possibility for infinite collections of sets. Diffuse sets are limiting events that play a similar role in EUU theory as arbitrarily unlikely events do in Savage's theory. They allow us to calibrate the uncertainty of events.

Axiom 4, below, is Savage's comparative probability axiom (P4) applied to ideal events.

AXIOM 4: *If $y > x$ and $w > z$, then $yEx \succeq yE'x$ implies $wEz \succeq wE'z$.*

Axiom 5 is Savage's divisibility axiom for ideal events. It serves the same role here as in Savage. Its statement below is a little simpler than Savage's original statement because our setting has a best and a worst prize. Let $\mathcal{F}^o$ denote the set of simple acts, that is, acts such that $f(\Omega)$ is finite. The simple act, $f \in \mathcal{F}^o$, is ideal if $f^{-1}(x) \in \mathcal{E}$ for all x. Let $\mathcal{F}^e$ denote the set of ideal simple acts.

AXIOM 5: *If $f, g \in \mathcal{F}^e$ and $f \succ g$, then there exists a partition $E_1, \dots, E_n$ of Ω such that $lE_i f \succ mE_i g$ for all i.*

Axiom 6, below, is a strengthening of Savage's dominance condition adapted to our setting. We use it to extend the representation from simple acts to all acts, to establish continuity of u, and to guarantee countable additivity of the prior μ. For ideal acts $f \in \mathcal{F}^e$, Axiom 6(i) implies Arrow's (1970) monotone continuity axiom, the standard axiom for ensuring the countable additivity of the probability measure in subjective expected utility theory.

AXIOM 6: *Let $g \succeq f_n \succeq h$ for all n. Then,* (i) *$f_n \in \mathcal{F}^e$ converges pointwise to f implies $g \succeq f \succeq h$.* (ii) *$f_n \in \mathcal{F}$ converges uniformly to f implies $g \succeq f \succeq h$.*

Axiom 6(ii) is what would be required to get a continuous von Neumann–Morgenstern utility index when proving Savage's theorem in a setting with real-valued prizes. Here, it serves a similar role; it ensures the continuity of the interval utility.

Theorem 1, below, is our main result. It establishes the equivalence of the six axioms to the existence of an EUU representation. The EUU representation has two parameters, a prior μ and an interval utility u that assigns a utility to a prize interval. A countably additive probability measure μ on some σ-algebra $\mathcal{E}_\mu$ is a *prior* if it is complete and non-atomic.[3] Let

$$I = \{[x, y] \mid l \le x \le y \le m\}$$

be the set of all prize intervals. Each prize interval $[x, y]$ can be identified by its end points $(x, y) \in \mathbb{R}^2$. Therefore, given any function $u : I \to \mathbb{R}$, we write $u(x, y)$ rather than the more cumbersome $u([x, y])$. Such a function is an *interval utility* if it is continuous and $u(x, y) > u(x', y')$ whenever $x > x'$ and $y > y'$.

Let $\mathcal{F}_\mu$ be the set of all $\mathcal{E}_\mu$-measurable acts and let $\mathbf{F}_\mu$ be the set of $\mathcal{E}_\mu$-measurable functions $\mathbf{f} : \Omega \to I$. We refer to elements of $\mathbf{F}_\mu$ as interval acts. For $\mathbf{f} \in \mathbf{F}$, let $\mathbf{f}_i$ denote the ith coordinate of $\mathbf{f}$ so that $\mathbf{f}(\omega) = [\mathbf{f}_1(\omega), \mathbf{f}_2(\omega)]$, $\mathbf{f}_1(\omega) \le \mathbf{f}_2(\omega)$ and $\mathbf{f}_i \in \mathcal{F}_\mu$.

DEFINITION: The interval act $\mathbf{f} \in \mathbf{F}_\mu$ is the envelope of $f \in \mathcal{F}$ if (i) $f \in \mathbf{f}$ and (ii) $f \in \mathbf{g}$ and $\mathbf{g} \in \mathbf{F}_\mu$ imply $\mu\{\mathbf{f} \subset \mathbf{g}\} = 1$.

[3]A prior is complete if $A \subset E$ and $\mu(E) = 0$ implies $A \in \mathcal{E}_\mu$. It is non-atomic if $\mu(A) > 0$ implies $0 < \mu(B) < \mu(A)$ for some $B \subset A$.

By definition, f's envelope, if it exists, is unique up to sets of measure zero.[4] Lemma 1, below, shows that every act has an envelope and that the mapping from acts to envelopes is onto.[5]

LEMMA 1: *Fix a prior μ. Then, every act $f \in \mathcal{F}$ has an envelope and, conversely, for any $\mathbf{f} \in \mathbf{F}_\mu$, there is $f \in \mathcal{F}$ such that $\mathbf{f}$ is f's envelope.*

Henceforth, we let $[f] = ([f]_1, [f]_2)$ denote the envelope of f. A preference $\succeq$ is an *expected uncertain utility* (EUU) if there exist a prior μ and an interval utility u such that the function W defined as

$$W(f) = \int u[f]\, d\mu \tag{3}$$

represents $\succeq$. We write $W = (\mu, u)$ if W, u, μ satisfy equation (3) and let $\succeq^u_\mu$ denote the EUU preference associated with (μ, u).

THEOREM 1: *The binary relation $\succeq$ satisfies Axioms 1–6 if and only if there are a prior μ and an interval utility u such that $\succeq = \succeq^u_\mu$.*

Routine arguments ensure that the prior is unique and the interval utility is unique up to a positive affine transformation for any $\succeq^u_\mu$. The set of ideal events $\mathcal{E}$ for $\succeq^u_\mu$ is the σ-algebra $\mathcal{E}_\mu$.[6] Hence, $\mathcal{F}_\mu$ is the set of ideal acts $\mathcal{F}^e$ and, since $[f]_1 = [f]_2 = f$ for $f \in \mathcal{F}_\mu$, the restriction of $\succeq$ to ideal events is a subjective expected utility preference.

In subjective expected utility theory, the prior measures the decision maker's uncertainty perception. With it, any act can be mapped to a lottery over prizes such that the utility of the act is equal to the expected utility of the lottery. EUU allows an analogous two-step evaluation of acts. With the prior, each act can be mapped into a lottery over prize intervals such that the utility of the act is equal to the expected utility of the interval lottery.

For any set Y, a *probability on* Y is a function $q : Y \to [0, 1]$ such that $\{y : q(y) > 0\}$ is a finite set and $\sum_Y q(y) = 1$. An *interval lottery* is a probability on I. Let Λ be the set of interval lotteries. For the prior μ and the simple act $f \in \mathcal{F}^o$, let $\lambda^f_\mu := \mu \circ [f]^{-1}$; that is, $\lambda^f_\mu(x, y) = \mu\{f = [x, y]\}$ for all $[x, y] \in I$.

[4]In probability theory, the standard term for $[f]_1$ is *maximal measurable minorant* of f and $[f]_2$ is the *minimal measurable majorant* (van der Waart and Wellner (1996)). We use the more concise term *envelope* for the pair $[f]_1, [f]_2$ for brevity.

[5]To establish that for every envelope there is an act with that envelope, we use the Banach–Kuratowski theorem (Birkhoff (1967)) which uses the continuum hypothesis. The continuum hypothesis is needed for the lemma to hold for any prior. However, in particular examples of priors, such as the example below, the lemma can be verified directly.

[6]See Lemma B11 for a proof of this assertion.

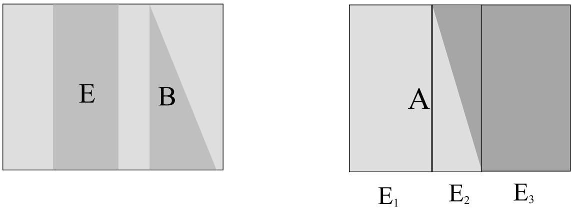

FIGURE 1.—Example.

Then,

$$\int u[f]\,d\mu = \sum_I u(x, y)\lambda_\mu^f(x, y).$$

The following example illustrates the mapping from acts to envelopes and interval lotteries.

EXAMPLE: Let $\Omega = [0, 1] \times [0, 1]$ be the unit square and let $\mathcal{E}_0$ be the smallest σ-algebra that contains all events of the form $[a, b] \times [0, 1]$ for $0 \leq a \leq b \leq 1$. That is, $\mathcal{E}_0$ contains all full-height rectangles as illustrated by the set E in the left panel of Figure 1. The set B depicted in the same figure is not an element of $\mathcal{E}_0$.

Let μ_0 be the unique measure on $\mathcal{E}_0$ that satisfies

$$\mu_0\big([a, b] \times [0, 1]\big) = b - a$$

and let the agent's prior, μ, be the completion of $(\mathcal{E}_0, \mu_0)$.[7] For example, the act $\hat{f} = xEy$ is ideal and has utility $W(\hat{f}) = \mu(E)u(x, x) + (1 - \mu(E))u(y, y)$.

Next, consider the (not ideal) act $f = xAy$, where $x < y$ and A is the set depicted in the right panel of Figure 1. The act f yields prize x on the light shaded region and y on the dark shaded region. The envelope of f is $[f]_1 = xE_1 \cup E_2y$, $[f]_2 = xE_1y$, where the sets E_1, E_2, and E_3 are as depicted in the right panel of Figure 1. Therefore, the utility of f is $W(f) = u(x, x)\mu(E_1) + u(x, y)\mu(E_2) + u(y, y)\mu(E_3)$. The interval lottery λ_μ^f assigns probability $\mu(E_1)$ to (x, x), $\mu(E_2)$ to (x, y), and $\mu(E_3)$ to (y, y) and, thus, the utility of f is the expected utility of the interval lottery λ_μ^f:

$$W(f) = u(x, x)\lambda_\mu^f(x, x) + u(x, y)\lambda_\mu^f(x, y) + u(y, y)\lambda_\mu^f(y, y).$$

Our main hypothesis implies that the agent is indifferent between all comparable bets on diffuse events. Consider the following four events R, B, G, Y in Figure 2.

[7]Therefore, if B is a (Lebesgue-)measure zero subset of $[0, 1]$, then any subset of $B \times [0, 1]$ is in $\mathcal{E}_\mu$.

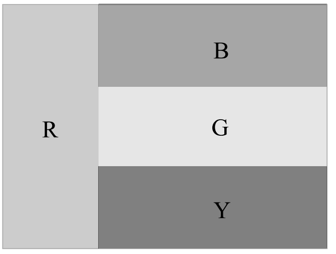

FIGURE 2.—Ideal and diffuse sets.

The event R is ideal ($R \in \mathcal{E}_\mu$) because it corresponds to a full-height rectangle, whereas the events B, G, and Y are diffuse subsets of the complement of R. Axiom 3 then implies that the agent is indifferent between betting on B, G, or Y and, moreover, is also indifferent between betting on $B \cup Y$ and Y. Such failures of strict monotonicity also occur in other models of ambiguity. For example, consider an α-maxmin expected utility maximizer[8] who confronts an Ellsberg-style urn which is known to have one red ball and three balls that are blue, green, or yellow. Assume the agent's set of priors contains all probabilities that are consistent with a $1/4$-probability of red; in particular, the set contains priors that assign probability zero to any pair of non-red colors. In that case, a bet on B yields the same utility as a comparable bet on $B \cup Y$.

The α-maxmin model offers a simple fix to avoid this failure of strict monotonicity: consider only those sets of probabilities that assign at least $\beta \in (0, 1/4]$ to each event B, G, Y. Expected uncertain utility theory offers an analogous fix: consider only priors such that each event B, G, Y contains an ideal event that has probability β.[9] With this restriction, a bet on $B \cup Y$ is strictly preferred to an equivalent bet on Y. More generally, for any fixed finite partition of the state space, we can choose a prior such that the betting preference is strictly monotone. The fixed finite partition represents a setting with a discrete state space that researchers typically use when relating the theory to evidence.[10] Of course, for any fixed prior (on the infinite state space Ω), there are nested diffuse sets. Therefore, for any fixed prior, there will exist some finite partition that yields a failure of strict monotonicity.

[8]See Section 4 for a definition of α-maxmin expected utility.

[9]To construct this new prior, we add ideal events that are not full-height rectangles.

[10]Theorem 4, below, provides a representation for the discrete model. Section 5 shows that discrete EUU has enough flexibility to be consistent with all common variants of the Ellsberg paradox. Nonetheless, EUU betting preferences are no more permissive than the betting preferences of other ambiguity models: for the case of two prizes, Theorem 5 implies that discrete EUU is a special case of α-maxmin utility.

3. ATTITUDE AND PERCEPTION

In expected utility theory, the prior describes an agent's risk perception; that is, it determines how each act gets mapped to a lottery. The prior plays the same role in EUU; it determines how each act is mapped to an *interval* lottery. The mapping from acts to interval lotteries is onto just like the mapping from acts to lotteries in Savage's model. That is, given any prior, any interval lottery can be generated by some simple act.

LEMMA 2: *For any $\lambda \in \Lambda$ and prior μ, there is $f \in \mathcal{F}^o$ such that $\lambda_\mu^f = \lambda$.*

Lemma 2 implies that, irrespective of the prior, each EUU decision maker confronts the same range of prospects. For any pair of priors μ and $\bar{\mu}$ and any act $f \in \mathcal{F}^o$, we can find $\bar{f} \in \mathcal{F}^o$ such that $\lambda_\mu^f = \lambda_{\bar{\mu}}^{\bar{f}}$. Thus, the agent with prior $\bar{\mu}$ perceives the same risk and uncertainty from act $\bar{f}$ as the agent with prior μ does from act f. This enables EUU to compare the attitudes of agents with different priors. The EUU preferences $\succeq_\mu^u$ and $\succeq_{\bar{\mu}}^{\bar{u}}$ have *the same attitude* if $\lambda_{\bar{\mu}}^{\bar{f}} = \lambda_\mu^f$, $\lambda_{\bar{\mu}}^{\bar{g}} = \lambda_\mu^g$ implies

$$f \succeq_\mu^u g \quad \text{if and only if} \quad \bar{f} \succeq_{\bar{\mu}}^{\bar{u}} \bar{g}.$$

Lemma 3, below, shows how the EUU model achieves separation between uncertainty perception and attitude. Consider two EUU agents with identical priors. How these agents rank acts depends only on their uncertainty attitudes (i.e., interval utilities). When the two agents have different priors $\mu, \bar{\mu}$, we can still isolate the uncertainty attitude by controlling for the uncertainty they perceive ($\lambda_\mu = \lambda_{\bar{\mu}}$). Lemma 3 establishes that two agents have the same uncertainty attitude if and only if one's interval utility is a positive affine transformation of the other's interval utility.

LEMMA 3: *The preference $\succeq_\mu^u$ has the same attitude as $\succeq_{\bar{\mu}}^{\bar{u}}$ if and only if $\bar{u} = \beta_1 u + \beta_2$ for some $\beta_1, \beta_2 \in \mathbb{R}$ with $\beta_1 > 0$.*

It follows from Lemmas 2 and 3 that each interval utility u induces a preference relation $\succeq^u$ on Λ. That is, $\lambda \succeq^u \lambda'$ if and only if, for all μ, f, g, $\lambda_\mu^f = \lambda$ and $\lambda_\mu^f = \lambda'$ imply $f \succeq_\mu^u g$. Henceforth, we will call this preference *the attitude* $\succeq^u$.

DEFINITION: The attitude $\succeq^u$ is *more cautious than* $\succeq^{\bar{u}}$ if $x \succeq^{\bar{u}} \lambda$ implies $x \succeq^u \lambda$.

For the interval utility u, let $v_u(x) = u(x, x)$. Hence, $v_u : X \to \mathbb{R}$ is a vNM utility index on I. For $x, y \in X$ such that $x < y$, let σ_u^{xy} be the unique $\sigma \in [0, 1]$ that satisfies

$$u(x, y) = v_u\big(\sigma x + (1 - \sigma)y\big).$$

The quantity $\sigma_u^{xy} x + (1 - \sigma_u^{xy}) y$ is the certainty equivalent of the uncertain interval $[x, y]$; σ_u^{xy} is well defined because v_u is strictly increasing, continuous, and satisfies $0 \leq \sigma_u^{xy} \leq 1$.

THEOREM 2: *The attitude $\succeq^u$ is more cautious than $\succeq^{\bar{u}}$ if and only if $v_u \circ v_{\bar{u}}^{-1}$ is concave and $\sigma_u^{xy} \geq \sigma_{\bar{u}}^{xy}$ for all $x < y$.*

Suppose $\succeq^u$ and $\succeq^{\bar{u}}$ are equally cautious; that is, each is (weakly) more cautious than the other. Then, Theorem 2 ensures that both $v_u \circ v_{\bar{u}}^{-1}$ and its inverse are concave and therefore $v_u \circ v_{\bar{u}}^{-1}$ is affine. Also, $\sigma_u^{xy} = \sigma_{\bar{u}}^{xy}$ for all $x < y$ and therefore $\succeq^u = \succeq^{\bar{u}}$. Hence, similarly to expected utility theory, two individuals that have the same level of cautiousness (risk aversion in expected utility theory) have the same ranking of interval lotteries (regular lotteries in expected utility theory).

The function v_u describes the interval utility for degenerate intervals $[x, x]$. As Theorem 2 shows, the more cautious preference has a more concave v_u. This part of the comparative measure corresponds to the standard comparative measure of risk aversion for expected utility maximizers. For non-degenerate intervals, the more cautious interval utility has a lower certainty equivalent than the less cautious interval utility. This is the novel part that generalizes risk aversion to include *uncertainty aversion*. Next, we separate uncertainty aversion from risk aversion. To do so, we take advantage of an insight from Epstein (1999) and use ideal acts (and diffuse acts) as benchmarks.[11]

Recall that $\mathcal{F}^e$ are the ideal acts. Let $\Lambda^e = \{\lambda \in \Lambda \mid \lambda_\mu^f \text{ for } f \in \mathcal{F}^e\}$. Hence, Λ^e is the set of ideal interval lotteries, that is, the set of all interval lotteries that can be generated by ideal acts. By using ideal interval lotteries as benchmarks, we can identify the decision maker's uncertainty attitude.

DEFINITION: The attitude $\succeq^u$ is *more uncertainty averse than* $\succeq^{\bar{u}}$ if $\lambda \succeq^{\bar{u}} \lambda'$ implies $\lambda \succeq^u \lambda'$ for all $\lambda \in \Lambda^e$.

Our definition differs from Epstein's (1999) since it accommodates differences in priors by comparing acts that yield the same interval lottery. The following corollary to Theorem 2 characterizes comparative uncertainty aversion. In particular, it establishes that one attitude is more uncertainty averse than another if and only if it is more cautious than the other and the two have the same ranking of ideal acts.

[11]Epstein (1999) used as a benchmark a subset of acts for which the agent is probabilistically sophisticated but not necessarily an expected utility maximizer. By contrast, our definition uses ideal acts as a benchmark. Ghirardato and Marinacci (2002) used constant acts as benchmarks but restricted to bi-separable preferences and required that the preferences being compared are cardinally symmetric. Our preferences are not bi-separable; as a result, their definition of cardinal symmetry cannot be applied to our model.

COROLLARY 1: *The attitude $\succeq^u$ is more uncertainty averse than $\succeq^{\bar{u}}$ if and only if $\succeq^u$ is more cautious than $\succeq^{\bar{u}}$ and v_u is a positive affine transformation of $v_{\bar{u}}$.*

Next, we use our comparative measure of uncertainty aversion to derive a comparative measure for the *uncertainty of events*. First, fix a prior μ and consider two events $A, B \subset \Omega$. If, for every interval utility u, the EUU preference $\succeq^u_\mu$ prefers betting on B to betting on A, then B is a better bet; that is, B dominates A.

DEFINITION: Event B *dominates* A if $x < y$ implies $yBx \succeq^u_\mu yAx$ for every interval utility u; A and B are *comparable* if neither dominates the other.

Between any two ideal events, the one with the higher probability will dominate the other and hence two different EUU decision makers always rank bets on ideal events the same way. With non-ideal events, it is possible for some EUU decision makers to prefer betting on A while others with the same perception of uncertainty prefer betting on B. This difference in behavior reflects the difference in the decision makers' uncertainty attitude and the differing levels of uncertainty associated with these events. If more uncertainty averse u's prefer B to A while less uncertainty averse ones have the opposite ranking, then we say A is more uncertain than B.

DEFINITION: Event A is *more uncertain* than B if A and B are comparable and if $x < y$, $\succeq^{u_1}$ more uncertainty averse than $\succeq^{u_2}$, and $yBx \succeq^{u_2}_\mu yAx$ imply $yBx \succeq^{u_1}_\mu yAx$.

For any $A \subset \Omega$, the *inner probability* of the event A is defined as

$$\mu_*(A) = \sup_{\substack{E \in \mathcal{E}_\mu \\ E \subset A}} \mu(E).$$

For ideal events $\mu_*(E) + \mu_*(E^c) = 1$, while for general events $\mu_*(A) + \mu_*(A^c) \leq 1$.

THEOREM 3: *Event B dominates A if and only if $\mu_*(A) \leq \mu_*(B)$ and $\mu_*(A^c) \geq \mu_*(B^c)$; A is more uncertain than B if and only if $\mu_*(A) < \mu_*(B)$ and $\mu_*(A^c) < \mu_*(B^c)$.*

The difference $1 - (\mu_*(A) + \mu_*(A^c))$ represents the probability mass the agent cannot distribute to A or to A^c. Theorem 3 shows that when A is more uncertain than B, this difference is greater for A than for B.

4. EUU IN A DISCRETE SETTING

To prove Theorem 1, above, we require an infinite state space. However, in applications and when comparing the EUU model to existing alternatives, it is convenient to use a discrete state space. Let $S = \{1, \dots, n\}$ be the finite state space, let $\mathcal{P}$ be the set of non-empty subsets of S, and let a, a', b, b', etc., denote elements of $\mathcal{P}$. We interpret S as a partition of the original state space Ω. The onto function $\rho : \Omega \to S$ describes this partition so that state s in the discrete model corresponds to the event $\rho^{-1}(s) \subset \Omega$ in the original model. With slight abuse of terminology, we refer to ρ as a partition of S and define $\rho^{-1}(a) := \bigcup_{s\in a} \rho^{-1}(s)$. Let $\phi : S \to [l, m]$ be a discrete act and let Φ be the set of discrete acts. The act $\phi \in \Phi$ in the discrete model corresponds to the act $f = \phi \circ \rho$ in the original model.

As in the original model, the utility function for the discrete state space has two parameters, the interval utility u and a probability that reflects the agent's prior in the discrete model. To see how the discrete prior is derived from the original prior, consider the case with two states $S = \{1, 2\}$: the event $A = \rho^{-1}(1) \subset \Omega$ corresponds to state 1 and A^c corresponds to state 2. The act $\phi(1) = x$, $\phi(2) = y$ in the discrete model corresponds to xAy in the original model. As illustrated in the right panel of Figure 1, the expected uncertain utility depends on the probability of three events; the event E_1 is the maximal ideal subset of A; the event E_3 is the maximal ideal subset of A^c; and the ideal set E_2 represents the residual. The values $\mu(E_1)$, $\mu(E_2)$, and $\mu(E_3)$ define a probability π on the non-empty *subsets* of $S = \{1, 2\}$, where

$$\pi\{1\} = \mu(E_1),$$
$$\pi\{2\} = \mu(E_3),$$
$$\pi\{1, 2\} = \mu(E_2).$$

An agent with prior μ cannot apportion the probability $\mu(E_2)$ to event A or event A^c. The probability $\pi\{1, 2\}$ in the discrete model corresponds to $\mu(E_2)$ of the original model, that is, the part of the probability of the event $\{1, 2\}$ that cannot be apportioned to state 1 or to state 2. For $x < y$, the utility of the discrete act ϕ is

$$U(\phi) = \pi\{1\}u(x, x) + \pi\{1, 2\}u(x, y) + \pi\{2\}u(y, y).$$

For the general case with $n \geq 2$, let $\mathcal{P}$ be the set of all non-empty subsets of S and let Π be the set of all probabilities on $\mathcal{P}$. A preference $\succeq$ (on Φ) is a discrete EUU if there are an interval utility u and a probability $\pi \in \Pi$ such that

$$U(\phi) = \sum_{a\in\mathcal{P}} u\Big(\min_{s\in a} \phi(s), \max_{s\in a} \phi(s)\Big)\pi(a) \tag{4}$$

represents $\succeq$. Henceforth, write $U = (u, \pi)$ if U, π, u satisfy equation (4), and we let $\succeq_\pi^u$ denote the discrete EUU that this U represents.

THEOREM 4: *Fix $W = (u, \mu)$, S and, for any $\pi \in \Pi$, let $U_\pi = (u, \pi)$. Then, for every partition ρ, there is a unique π such that $W(\phi \circ \rho) = U_\pi(\phi)$ for all $\phi \in \Phi$. Conversely, for every π, there is a partition ρ such that $W(\phi \circ \rho) = U_\pi(\phi)$ for all $\phi \in \Phi$.*

Theorem 4 shows that the prior μ in the original model does not constrain the prior π in the discrete model. For any $\pi \in \Pi$, there is a partition that generates this prior. One special case is the discrete prior corresponding to a partition of the original state space into ideal subsets. In that case, $\pi(a) = 0$ for all non-singleton a and

$$U(\phi) = \sum_{s \in S} v_u\big(\phi(s)\big)\pi\big(\{s\}\big).$$

Thus, $U = (u, \pi)$ is an expected utility function. If $\pi(a) > 0$ for some non-singleton a, the quantity $\pi(a)$ reflects the decision maker's inability to reduce the uncertainty of the event a to uncertainty about its components. Given any probability π on $\mathcal{P}$, define the capacity[12] π_* such that $\pi_*(\emptyset) = 0$ and, for all $a \subset S$,

$$\pi_*(a) = \sum_{b \in \mathcal{P}, b \subset a} \pi(b).$$

The prior π corresponding to partition ρ satisfies

$$\pi_*(a) = \mu_*\big(\rho^{-1}(a)\big)$$

for all $a \in \mathcal{P}$. The co-capacity π^* of π_* is defined as $\pi^*(a) = 1 - \pi_*(a^c)$ for all $a \subset S$. Note that

$$\pi^*(a) = \sum_{b \in \mathcal{P}, b \cap a \neq \emptyset} \pi(b) = 1 - \mu_*\big(\rho^{-1}\big(a^c\big)\big).$$

The functions π, π_*, π^* are the central concepts of Dempster–Shafer theory (Dempster (1967), Shafer (1976)). The probability π is called a *basic belief assignment*; the capacity π_* is a *belief function* and interpreted as a lower bound on the probability of event a; π^* is the *plausibility function* and interpreted as an upper bound on the probability of the event a. Dempster (1967) introduced these concepts to generalize Bayesian inference by adding a degree of confidence to probabilistic statements. In Dempster's interpretation, the basic

[12] A function κ is a *capacity* if (i) $\kappa(\emptyset) = 0$, $\kappa(S) = 1$ and (ii) $a \subset b$ implies $\kappa(a) \leq \kappa(b)$.

belief assignment is a probability over a related (auxiliary) state space which he identified with $\mathcal{P}$. Then, $\pi_*(a) = \sum_{b \subset a} \pi(a)$ is the probability of the set of auxiliary states that imply a (and hence, a lower bound on the "true" probability of a); $\pi^*(a) = \sum_{b \cap a \neq \emptyset} \pi(a)$ is the probability of the set of auxiliary states that are consistent with a (and hence, an upper bound on the "true" probability of a). Dempster (1967) and Shafer (1976) were concerned with the relationship between basic belief assignments and belief (and possibility) functions[13] and how to update belief functions in light of new information; by contrast, our results relate belief functions to preferences over acts. Thus, our model provides a subjective foundation for the Dempster–Shafer theory of evidence.

Let Δ^S be the set of all probabilities on S and let $\Delta_\pi \subset \Delta^S$ be the core of the capacity π_*.[14] That is,

$$\Delta_\pi = \left\{ p \in \Delta^S \,\middle|\, \pi_*(a) \leq \sum_{s \in a} p(s) \text{ for all } a \subset S \right\}.$$

Theorem 3 describes how the discrete prior, π, measures the uncertainty of events. Translated to the discrete setting, Theorem 3 implies the following corollary:

COROLLARY 2: *Event a is more uncertain than event b if and only if $\pi_*(b) > \pi_*(a)$ and $\pi^*(a) > \pi^*(b)$.*

Thus, in the language of Dempster–Shafer theory, event a is more uncertain than event b if it has a greater gap between belief and plausibility. For example, assume there are three states, $S = \{1, 2, 3\}$, and $\pi\{1\} = 1/3$, $\pi\{2\} = \pi\{3\} = \alpha \in [0, 1/3)$, and $\pi\{2, 3\} = 2/3$. State 1 is ideal since $\pi_*\{1\} = 1/3 = \pi^*\{1\}$ and, since $\pi_*(\{2\}) = \alpha$, $\pi^*\{2\} = 2/3 - \alpha$, state 2 is more uncertain than state 1.

Our next result relates EUU to Choquet expected utility (Schmeidler (1989)) and to α-maxmin expected utility. A binary relation $\succeq$ on Φ is a Choquet expected utility (CEU) preference if there exist a capacity κ and a continuous, strictly increasing function $v : X \to \mathbb{R}$ such that the function $V : \Phi \to \mathbb{R}$ defined by $V(\phi) = \int v(\phi)\, d\kappa$ represents $\succeq$, where the integral above denotes the Choquet integral. We write $\succeq_{\kappa v}$ for a CEU preference with parameters κ and v and $V = (v, \kappa)$ if $V(\phi) = \int v(\phi)\, d\kappa$ for all ϕ.

For $\alpha \in [0, 1]$, the binary relation $\succeq$ on Φ is an α-maxmin expected utility (α-MEU) preference if there exist a compact set of probabilities $\Delta \subset \Delta^S$ and a continuous, strictly increasing $v : X \to \mathbb{R}$ such that the function V defined by

$$V(\phi) = \alpha \min_{p \in \Delta} \sum_{s \in S} v\big(\phi(s)\big) p(s) + (1 - \alpha) \max_{p \in \Delta} \sum_{s \in S} v\big(\phi(s)\big) p(s)$$

[13]Dempster (1967) showed that each one of the three functions π, π^*, π_* implies a unique value for the other two.

[14]See Schmeidler (1989) for a definition of the core of a capacity. Schmeidler showed that every convex capacity has a non-empty core. Since π_* is convex, it follows that Δ_π is non-empty.

represents $\succeq$. We let $\succeq_{\alpha\Delta v}$ denote the α-MEU with parameters α, Δ, v and write $V = (v, \Delta, \alpha)$ if the equation above holds for all ϕ.

Theorem 5 gives a condition on the interval utility so that the EUU preference is in the intersection of CEU and α-MEU preferences.

THEOREM 5: *Let $v : X \to \mathbb{R}$ be a continuous, strictly increasing function, $\alpha \in (0, 1)$ and $u(x, y) = \alpha v(x) + (1 - \alpha)v(y)$ for all $(x, y) \in I$. Then,*

$$\succeq_{\kappa v} = \succeq_{\alpha\Delta v} = \succeq_{\pi}^{u}$$

for $\kappa = \alpha\pi_* + (1 - \alpha)\pi^*$ and $\Delta = \Delta_\pi$.

Theorem 5 shows that when u is separable with the same utility index applied to the upper and lower bounds of the interval, EUU coincides with CEU and α-MEU. When $\alpha = 1$, the interval utility depends only on the lower bound and hence the discrete EUU preference coincides with the MEU preference (Gilboa and Schmeidler (1989)) with utility index v_u and the set of probabilities Δ_π. Theorems 4 and 5 can be combined to establish conditions under which a CEU or an α-MEU is a discrete EUU: If $\succeq_{\kappa v}$ is a CEU with a capacity that can be expressed as a convex combination of a belief function and its plausibility function, then $\succeq_{\kappa v}$ is a discrete EUU. Similarly, if $\succeq_{\alpha\Delta v}$ is an α-MEU with a set of probabilities that form the core of a belief function, then $\succeq_{\alpha\Delta v}$ is a discrete EUU.

We can apply our measure of uncertainty aversion to the preferences characterized in Theorem 5. For $u(x, y) = \alpha v(x) + (1 - \alpha)v(y)$, the parameter α measures uncertainty aversion. Since

$$v\big(\sigma_u^{xy} x + \big(1 - \sigma_u^{xy}\big) y\big) = \alpha v(x) + (1 - \alpha)v(y),$$

it follows that σ^{xy} increases as α increases. Therefore, (α, v) is more uncertainty averse than $(\bar{\alpha}, v)$ if $\alpha \geq \bar{\alpha}$.

5. UNCERTAINTY AND THE ELLSBERG PARADOX

In this section, we relate EUU theory to observed behavior in various versions of the Ellsberg experiment, Ellsberg (1961). Our goal is not only to show that EUU theory is flexible enough to accommodate the Ellsberg paradox but also to take advantage of the separation between uncertainty perception and uncertainty attitude to relate a decision maker's propensity for Ellsberg-paradox behavior to his uncertainty aversion parameter.

The Ellsberg experiment has two possible prizes $y = 1$ and $x = 0$. Given any event $b \subset S$, a *bet* is an act that delivers 1 if b occurs and 0 otherwise. Hence, we can identify each act with an event b. The experimenter elicits the decision makers' preferences over some collection of bets: $\mathbf{B} \subset 2^S$. Let $U = (u, \pi)$ be a discrete EUU utility. In the Ellsberg experiment, the interval utility affects

TABLE I
SINGLE-URN EXPERIMENT

	Config. 1	Config. 2	Config. 3	Config. 4
Ball 1	r	r	r	r
Ball 2	w	w	g	g
Ball 3	w	g	w	g

behavior only through the values $u(0,0)$, $u(1,1)$, and $u(0,1)$. We normalize $u(1,1)=1$, $u(0,0)=0$, and $u(0,1)=z$. Recall that z measures the agent's uncertainty aversion: lower values of z correspond to greater uncertainty aversion. Since the preference depends only on π and z, we write $\succeq_\pi^z$ rather than $\succeq_\pi^u$.

The subjects are told that one or more urns have each been filled with a fixed number of balls of various colors. An outcome is a configuration (one color for each ball in each urn) *and* a draw from each urn. Let $S=\{s_{it}\}$ where $t=1,\dots,k$ and $i=1,\dots,m$; the state $s_{it}\in S$ represents an outcome in which the balls were drawn from urns filled according to the tth configuration. Let $n=mk$ be the number of states.

For example, in the *single-urn experiment*, one ball is drawn from an urn that contains three balls. It is known that exactly one ball is red and the remaining two balls are either white or green. Let $S=\{s_{it}\}$ for $i=1,2,3$, $t=1,2,3,4$. Suppose the three balls are numbered $1,2,3$ and ball 1 is always red. Each column, t, depicts one possible color configuration and each row corresponds to a particular ball, 1, 2, or 3 being drawn. Table I describes the map from states to color draws.

In the *two-urn experiment*, urn I contains one red ball (ball 1) and one white ball (ball 2); urn II contains two balls that are red or white. One ball is drawn from each urn. Table II depicts the two-urn experiment. A column of Table II represents a color choice for balls 1 and 2 in urn II. For example, in column 1, both balls are white; in column 2, ball 1 is red and ball 2 is white, etc. A row represents a pair of draws (balls 1 or 2), one from each urn.

TABLE II
TWO-URN EXPERIMENT

	Config. 1	Config. 2	Config. 3	Config. 4
Ball 1 from I, ball 1 from II	rw	rr	rw	rr
Ball 1 from I, ball 2 from II	rw	rw	rr	rr
Ball 2 from I, ball 1 from II	ww	wr	ww	wr
Ball 2 from I, ball 2 from II	ww	ww	wr	wr

Let **B** be all combinations of color draws in S. For example, in the single-urn experiment, **B** is the algebra generated by the partition $\{\mathrm{r}, \mathrm{w}, \mathrm{g}\}$, while in the two-urn experiment, **B** is the algebra generated by the partition $\{\mathrm{rr}, \mathrm{rw}, \mathrm{wr}, \mathrm{ww}\}$. We say that two events are (*experimentally*) *comparable* if they contain the same number of states. For example, the single-color events r, w, and g are all comparable in the single-urn experiment because each contains four states.

The defining feature of an Ellsberg experiment is that, for some events $a \in \mathbf{B}$, the chance of winning at a in each configuration of the urn is fixed. For example, ex post (i.e., upon inspecting the contents of the urn), $a = \mathrm{g} \cup \mathrm{w}$ has a 2/3 chance of winning in every configuration in the single-urn experiment. We call such events experimentally unambiguous. In contrast, a bet on $b = \mathrm{r} \cup \mathrm{g}$ has a 2/3 chance of winning in two configurations, a 1/3 chance in one configuration, and is a sure winner in the final configuration. Hence, b is experimentally ambiguous in the single-urn experiment. Let $|a|$ denote the cardinality of the set a and let **B** be the algebra of subsets of S corresponding to all combinations of color draws. For any event a, let $a_t = \{s \in a \mid s = s_{it} \text{ for some } i\}$ be the outcomes in a associated with the tth possible configuration.

An event $a \in \mathbf{B}$ is *experimentally unambiguous* if $\min_t |a_t| = \max_t |a_t|$; otherwise, it is experimentally ambiguous. Let **A** be the collection of all experimentally unambiguous events in **B**. It is easy to see that complements of experimentally unambiguous events are experimentally unambiguous and disjoint unions of experimentally unambiguous events are experimentally unambiguous. Hence, we have the following lemma:

LEMMA 4: *The class* **A** *of experimentally unambiguous events is a* λ*-system*; *that is*, *it contains* S *and is closed under complements and disjoint unions*.

Intersections of experimentally unambiguous events need not be experimentally unambiguous. *Zhang's four-color urn* describes such a situation: one ball is drawn from an urn with two balls; balls are red, white, green, or orange. There is exactly one ball in each of the following two categories: (1) red or white and (2) red or green. It follows that there is also one ball in each of the following two categories: (3) orange or green and (4) orange or white. Table III describes Zhang's experiment.

TABLE III

ZHANG'S FOUR-COLOR URN

	Config. 1	Config. 2	Config. 3	Config. 4
Ball 1	r	o	w	g
Ball 2	o	r	g	w

All single-color events have two states and are experimentally ambiguous. Of the six two-color events, r ∪ o and w ∪ g are experimentally ambiguous. The remaining four are experimentally unambiguous. Thus, r ∪ w and r ∪ g are experimentally unambiguous but r is not.

Given **B**, we say that the experimentally unambiguous events **A** are a *finite source* for the preference $\succeq$ on Φ if

$$|a| \geq |b|, \quad a, b \in \mathbf{A} \quad \text{implies} \quad a \succeq b.$$

Thus, **A** is a finite source means that the agent is probabilistically sophisticated (in the sense of Machina and Schmeidler (1992)) on the class of experimentally unambiguous events and the probability measure $\eta : \mathbf{A} \to [0, 1]$ such that

$$\eta(a) = \frac{|a|}{n}$$

represents his betting preference. Our definition of a *source* mirrors Epstein and Zhang's (2001) definition of unambiguous events. Like Epstein and Zhang, we require that the agent be probabilistically sophisticated over an appropriate λ-system of events.

The collection **B** is an *Ellsberg experiment* if there exist $a \in \mathbf{A}$ and $b \in \mathbf{B} \setminus \mathbf{A}$ such that $|a| = |b|$. Given any Ellsberg experiment **B** and preference $\succeq$ on Φ, $(\mathbf{B}, \succeq)$ is an *Ellsberg Paradox* if **A** is a source for $\succeq$ and if

$$|a| = |b|, \quad a \in \mathbf{A}, b \notin \mathbf{A} \quad \text{implies} \quad a \succ b.$$

Thus, $(\mathbf{B}, \succeq)$ is an Ellsberg paradox means that probabilistic sophistication fails when the agent compares experimentally ambiguous and unambiguous events.

Theorem 6, below, shows that, for any Ellsberg experiment, there is an uncertainty perception π that renders each experimentally ambiguous event more π-uncertain than every comparable experimentally unambiguous event. Moreover, the experiment yields a paradox for any decision maker with that perception and greater uncertainty aversion than a benchmark.

THEOREM 6: *For any Ellsberg experiment*, **B**, *there are* π *and* $z^* > 0$ *such that*
(i) **A** *is a discrete source for* $\succeq_\pi^z$ *for all* z;
(ii) b *is more uncertain than* a *whenever* $a \in \mathbf{A}$, $b \in \mathbf{B} \setminus \mathbf{A}$, *and* $|a| = |b|$;
(iii) $(\mathbf{B}, \succeq_\pi^z)$ *is an Ellsberg paradox for all* $z < z^*$.

For the discrete setting analyzed in this section, Theorem 6 shows that EUU theory can address Ellsberg-style evidence, including versions of the Ellsberg experiment that require that the unambiguous events do not form a σ-algebra. Specifically, Theorem 6 implies that there is a discrete prior for the Zhang urn such that experimentally unambiguous events are a source with each of its elements less uncertain than comparable experimentally unambiguous events.[15]

Theorem 1 shows that the ideal events form a σ-algebra and, therefore, the experimentally unambiguous events in Zhang's experiment cannot be ideal. However, as Theorem 6 shows, probabilistic sophistication is not confined to the ideal events; the prior can be chosen so that the agent is probabilistically sophisticated over the experimentally unambiguous events even if those events are not closed under intersection. Put differently, Theorem 6 shows that the existence of a σ-algebra of ideal events presents no obstacle to addressing Ellsberg-style evidence.

6. MACHINA REVERSALS

The Ellsberg experiments analyzed in the previous section have only two prizes. In that case, a single parameter characterizes the interval utility and, as a result, each EUU is also a Choquet expected utility and an α-maxmin expected utility.[16] Recently, Machina (2009) examined variations of Ellsberg experiments with more than two prizes and showed that Choquet expected utility theory (and related models) cannot accommodate behavior that appears plausible and even natural. In the context of EUU theory, Machina's conjectured behavior is synonymous with the nonseparability of the interval utility u. To demonstrate this, we describe Machina's experiment below and show that EUU can accommodate the conjectured behavior if and only if the interval utility is nonseparable.

Assume a ball is drawn from an urn known to have 20 balls; 10 balls are marked 1 or 2 and 10 are marked 3 or 4. Let $S = \{1, 2, 3, 4\}$ be the state space and, hence, each discrete act $\phi \in \Phi$ corresponds to a vector $(\phi(1), \phi(2), \phi(3), \phi(4)) \in X^4$. Machina (2009) observed that if $\succeq$ is any Choquet expected utility such that

$$(x_1, x_2, x_3, x_4) \sim (x_2, x_1, x_3, x_4) \sim (x_2, x_1, x_4, x_3) \sim (x_4, x_3, x_2, x_1) \tag{5}$$

for all $x_1, x_2, x_3, x_4 \in X$, then we must have $(x_1, x_2, x_3, x_4) \sim (x_1, x_4, x_3, x_2)$ whenever $x_1 \geq x_3 \geq x_2 \geq x_4$. In particular, $(20, 10, 10, 0) \sim (20, 0, 10, 10)$. He noted that this indifference may be an undesirable restriction for a flexible model. Call it an *M-reversal* if a preference, $\succeq$ on Φ, is not indifferent between (x_1, x_2, x_3, x_4) and (x_1, x_4, x_3, x_2) for some $x_1 \geq x_3 \geq x_2 \geq x_4$, $x_i \in X$ despite satisfying (5).

[15]One example of such a prior is the following: there is $\alpha \geq 0$, $\beta > 0$ such that $\pi(a) = \alpha$ for all single-color events; $\pi(a) = \alpha + \beta$ for all experimentally unambiguous two-color events; $\pi(a) = 0$ for all other events.

[16]Let 0 and 1 be the two prizes. Normalize $u(0, 0) = 0$ and $u(1, 1) = 1$; choose the utility index v such that $v(0) = 0$, $v(1) = 1$; set $\alpha = u(0, 1)$ and apply Theorem 5.

Let Π^m be a collection of probabilities on the set of non-empty subsets of S that satisfy the following conditions: $\pi\{1,2\} = \pi\{3,4\} > 0$ and $\pi(a) = 0$ if a is any other two-state event; $\pi(a) = \pi(b)$ if a and b are both single-state events or three-state events. These conditions imply that any EUU with discrete prior in Π^m satisfies (5) and that the events $\{1,2\}$ and $\{3,4\}$ are less π-uncertain than other two-state events. The interval utility u is *separable* if there are $v_1, v_2 : X \to \mathbb{R}$ such that $u(x,y) = v_1(x) + v_2(y)$ for all $(x,y) \in I$.

THEOREM 7: *If* $\pi \in \Pi^m$, *then* $\succeq_\pi^u$ *has no M-reversals if and only if* u *is separable*.

Theorem 7 shows that Machina reversals occur if the interval utility is not separable. Experimental evidence reported in L'Haridon and Placido (2010) shows that 70% of subjects exhibit M-reversals, and of those subjects, roughly 2/3 prefer "packaging" the two extreme outcomes together. That is,

$$(20, 0, 10, 10) \succ (20, 10, 10, 0).$$

This pattern of preference is implied by an interval utility that satisfies

$$u(x_4, x_1) + u(x_3, x_2) > u(x_1, x_2) + u(x_3, x_4)$$

for $x_1 \geq x_3 \geq x_2 \geq x_4$. EUU is not the only theory that can accommodate Machina reversals. Baillon, L'Haridon, and Placido (2011) observed that Siniscalchi's (2009) vector-valued expected utility model permits them, while α-maxmin expected utility and Klibanoff, Marinacci, and Mukerji's (2005) smooth model of uncertainty rule them out.

7. RELATED LITERATURE

We can organize the literature on uncertainty and uncertainty aversion by grouping models according to the extent to which uncertainty/ambiguity is built into the choice objects. At one extreme, there are papers such as Gilboa (1987), Casadesus-Masanell, Klibanoff, and Ozdenoren (2000), Epstein and Zhang (2001), the current paper, and a number of others that study preferences over Savage acts over an unstructured state space. At the other extreme, there are papers that introduce novel choice objects designed to incorporate uncertainty that cannot be reduced to risk. The latter models are silent on how "real-life" prospects are reduced to these choice objects; that is, they do not describe how Savage acts can be mapped to the investigated choice objects. For example, Olszewski (2007) and Ahn (2008) considered sets of lotteries, and interpreted sets with a single lottery as situations in which the decision maker can reduce all uncertainty to risk, while sets with multiple lotteries depict Knightian uncertainty.

In this latter category is Jaffray (1989), who studied preferences over belief functions over prizes.[17] A belief function that assigns probability 0 to both singleton sets $\{x\}$, $\{y\}$ but assigns probability 1 to the set $\{x, y\}$ depicts a situation in which the decision maker knows that he will end up with either x or y but views any remaining uncertainty as irreducible to risk. To see the relationship between Jaffray's model and EUU theory, consider a discrete EUU $\succeq^u_\pi$. We can associate a capacity κ over the set of all non-empty subsets of X with each act ϕ in a natural way by letting

$$\kappa(Y) = \sum_{\{a|\phi(a)\subset Y\}} \pi(a).$$

It is easy to verify that the discrete EUU $\succeq^u_\pi$ is indifferent between two acts that yield the same capacity κ as defined above. Moreover, it can be shown that this induced preference will satisfy Jaffray's assumptions. Hence, EUU theory and Jaffray's model stand roughly in the same relationship as Savage's theory and von Neumann–Morgenstern theory: one takes as given lotteries (probability distribution in vNM theory, capacities in the Jaffray model) as the choice objects, the other starts with acts, shows that each act can be identified with a lottery in a natural way, and ensures that each preference (EU or EUU) over acts induces a preference over lotteries (EU or Jaffray).

In between these two classes of models, there are those that partially build-in a distinction between uncertainty and risk. Segal (1990), Klibanoff, Marinacci, and Mukerji (2005), Nau (2006), and Ergin and Gul (2009) achieved the desired effect by structuring the state space. In the first two papers, uncertainty resolves in two stages; the first stage represents ambiguity, the second stage risk. The remaining two papers assume that the state space has a product structure and identify one dimension with ambiguity and the other with risk.

The extensive literature on ambiguity models in the Anscombe–Aumann (1963) framework also falls into this intermediate category. This literature includes Schmeidler (1989), who introduced Choquet expected utility, Gilboa and Schmeidler (1989), who introduced maxmin expected utility, the generalizations of maxmin expected utility, such as α-maxmin expected utility preferences (see Ghirardato and Marinacci (2001)), variational preferences of Maccheroni, Marinacci, and Rustichini (2006), the general uncertainty averse preferences of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011), as well as Zhang's (2002) model of Choquet expected utility with inner probabilities, Lehrer's (2007) model of partially specified probabilities, and Siniscalchi's (2009) vector expected utility theory.

Theorem 5, above, relates EUU to Choquet expected utility and to α-maxmin expected utility. EUU theory is also related to Zhang (2002) and Lehrer (2007). Zhang (2002) studied preferences in the Savage setting, while

[17]As defined in the previous section, a belief function is a type of capacity.

Lehrer (2007) considered the Anscombe–Aumann framework with a finite state space. Ignoring the difference in the sets of prizes, in the state spaces, and in the underlying axioms, we can state the relationship between these two models and EUU theory as follows: each Lehrer representation can be identified with a Zhang representation (and vice versa) by identifying every partially specified probability with its inner probability extension to the set of all subsets of the state space. Then, it can be verified that every Lehrer/Zhang representation is equivalent to an EUU representation for some interval utility u such that $u(x, y) = v_u(x)$ for all x, y. Hence, Lehrer/Zhang preferences correspond to the subclass of EUU preferences for which the interval utility depends only on the lower end of the interval. Note that this is also the subclass of EUU preferences that are maxmin expected utility.

8. CONCLUSION

In this paper, we introduce Expected Uncertain Utility theory, an extension of subjective expected utility theory, to address well-known anomalies in choice behavior. By staying close to Savage's model, EUU theory replicates one of the main achievements of Savage's theory: the separation of uncertainty perception and uncertainty attitude. As in subjective expected utility theory, the agent's uncertainty perception is described by a prior. In EUU theory, the prior serves two roles: it specifies which aspects of the uncertainty can be quantified and it measures the uncertainty of those events that can be quantified. The theory uses a simple dichotomy that distinguishes perfectly quantifiable and totally unquantifiable uncertainty. Nonetheless, mixing those two elements yields sufficient flexibility to address Ellsberg-style evidence and, as we show in a companion paper, Allais-style evidence.

APPENDIX A: PRELIMINARY RESULTS

The prior μ is convex ranged if, for every $0 < r < 1$ and every $E \in \mathcal{E}_\mu$, there is $E' \subset E$, $E' \in \mathcal{E}_\mu$ such that $\mu(E') = r\mu(E)$. A standard result (Billingsley (1995, p. 35)) establishes that every countably additive non-atomic measure is convex ranged. Hence, every prior is convex ranged.

For the prior μ, let

$$\mu_*(A) = \sup_{\substack{E \in \mathcal{E}_\mu \\ E \subset A}} \mu(E).$$

Since μ is countably additive, it is straightforward to show that the supremum is attained. Call $E \in \mathcal{E}_\mu$ the *core* of A if $E \subset A$ and $\mu(E) = \mu_*(A)$, and note that it is unique up to a set of measure 0. A set D is *μ-diffuse* if $\mu_*(D) = \mu_*(D^c) = 0$. Let $\mathcal{D}_\mu$ be the set of all μ-diffuse sets.

Let $E \in \mathcal{E}_\mu$, $N = \{1, \dots, n\}$, and $\{A_i\}_{i \in N}$ be a finite partition of E. Let $\mathcal{N}$ be the set of all non-empty subsets of N and, for $J \in \mathcal{N}$, let $\mathcal{N}(J) = \{L \in \mathcal{N} \mid$

$L \subset J\}$. Let $A^J = \bigcup_{i \in J} A_i$, let C^J be the core of A^J, and let $C^N = E$. The *ideal split* $\{E_*^J\}_{J \in \mathcal{N}} \subset \mathcal{E}_\mu$ of $\{A_i\}_{i \in N}$ is inductively defined as follows: $E_*^{\{i\}} := C^{\{i\}}$ for all $i \in N$; for J such that $|J| > 1$,

$$E_*^J := C^J \setminus \left(\bigcup_{\substack{L \in N(J) \\ L \neq J}} E_*^L \right).$$

Note that $\{E_*^J\}$ is a partition of E that satisfies $\bigcup_{L \in \mathcal{N}(J)} E_*^L \subset A^J$ for all $J \in \mathcal{N}$ and $\mu_*(A^J) = \mu(C^J) = \sum_{L \in \mathcal{N}(J)} \mu(E_*^L)$.

For any act $f \in \mathcal{F}^o$ with range $\{x_1, \ldots, x_n\}$, let $\{E_*^J(f)\}$ be the ideal split of $\{f^{-1}(x_i)\}$.

LEMMA A1: *Let $f \in \mathcal{F}^o$ with range $\{x_1, \ldots, x_n\}$. Then, $\mathbf{f} \in \mathbf{F}_\mu$ such that $\mathbf{f}(\omega) = (\min_{i \in J} x_i, \max_{i \in J} x_i)$ for $\omega \in E_*^J(f)$ is an envelope of f.*

PROOF: Let $A_i = f^{-1}(x_i)$, $A^J = \bigcup_{i \in J} A_i$, and let C^J be the core of A^J. To show that $\mathbf{f}_1$ is a lower envelope, first note that $f(\omega) = x_i$ and $\omega \in E_*^J(f)$ implies $i \in J$. Hence, $\mathbf{f}_1 \leq f$. Let $g \in \mathcal{F}_\mu$ such that $\mu\{g > \mathbf{f}_1\} > 0$. Then, there exists $J \in \mathcal{N}$ such that $\mu(\{g > \mathbf{f}_1\} \cap E_*^J(f)) > 0$. Let $x_j = \min_{i \in J} x_i$, let $E = \{g > \mathbf{f}_1\} \cap E_*^J(f)$, and note that $E \in \mathcal{E}_\mu$ since $g \in \mathcal{F}_\mu$. We claim that $\min_{\omega \in E} f(\omega) = x_j$, thus proving that $g \not\leq f$. Suppose, to the contrary, that $\min_{\omega \in E} f(\omega) = x_i > x_j$. It follows that $E \subset C^I$ for $I = J \setminus \{j\}$. From the construction of the ideal split, it follows that $C^I = \bigcup_{L \in \mathcal{N}(I)} E_*^L(f)$ and therefore $E \not\subset E^J(f)$, yielding the desired contradiction. An analogous argument proves that $[f]_2$ is an upper envelope of f. *Q.E.D.*

By definition, the envelope of f is unique up to measure zero. In the following, *the* envelope of f, denoted $[f]$, refers to this equivalence class of interval acts.

LEMMA A2: *Assume the continuum hypothesis holds and μ is a prior. Then, $\mathcal{D}_\mu$ is non-empty and every $D \in \mathcal{D}_\mu$ can be partitioned into $D_1, D_2 \in \mathcal{D}_\mu$.*

PROOF: Birkhoff (1967, p. 266, Theorem 13) proved the following: under the continuum hypothesis, no nontrivial (i.e., not identically equal to 0), countably additive measure such that every singleton has measure 0 can be defined on the algebra of all subsets of the continuum. We will use Birkoff's result to establish that Ω must have a nonmeasurable subset. That is, there exists $A \subset \Omega$ such that $A \notin \mathcal{E}_\mu$.

Since μ is convex valued, we can construct a random variable, ψ, that has a uniform distribution on the interval $[0, 1]$ on this probability space. (For example, see the construction in Billingsley (1995, proof of Theorem 20.4, p. 265).) Define $\hat{\mu}(R) = \mu(\psi^{-1}(R))$ for every $R \subset [0, 1]$. If $\mathcal{E}_\mu$ contains every subset of

Ω, $\hat{\mu}$ defines a measure on the set of all subsets of the unit interval. Moreover, since ψ has a uniform distribution, $\hat{\mu}(\{x\}) = 0$ for all $x \in [0, 1]$, contradicting Birkoff's result.

Let $E_*^{\{1\}}(A), E_*^{\{2\}}(A), E_*^{\{1,2\}}(A)$ be an ideal split of $\{A_1, A_2\}$ for $A_1 = A$ and $A_2 = A^c$ and let $\alpha = \sup_{A\subset\Omega} \mu(E_*^{\{1,2\}}(A))$. By the argument above, there exists $A \subset \Omega$ such that $A \notin \mathcal{E}_\mu$. Hence, $\alpha > 0$.

To establish that α is attained, consider a sequence of sets $A(n)$ such that

$$\mu\big(E_*^{\{1,2\}}\big(A(n)\big)\big) > \alpha - \frac{1}{n}$$

for all $n = 1, 2, \dots$. Let $E(n) = \bigcup_{j\leq n}(E_*^{\{1,2\}}(A(j)))$ and set $E(0) = \emptyset$. Define $B(n) = [E(n) \cap A(n)] \setminus E(n-1)$ and let $A = \bigcup_n B(n)$. Note that $E_*^{\{1,2\}}(B(n) \subset E_*^{\{1,2\}}(A))$ and $\mu(E_*^{\{1,2\}}(B(n))) \geq \mu(E_*^{\{1,2\}}(A(n)))$. Therefore, $\mu(E_*^{\{1,2\}}(A)) \geq \alpha$, as desired. If $\alpha < 1$, choose A such that $\mu(E_*^{\{1,2\}}(A)) = \alpha$ and let B be any nonmeasurable subset of $\Omega \setminus E_*^{\{1,2\}}(A)$. Note that $\mu(E_*^{\{1,2\}}(A \cup B)) > \mu(E_*^{\{1,2\}}(A)) = \alpha$, a contradiction. Hence, there exists A such that $\mu(E_*^{\{1,2\}}(A)) = 1$. Clearly, $A \in \mathcal{D}_\mu$.

Next, we will show that any diffuse set can be partitioned into two diffuse sets. Let $D \in \mathcal{D}_\mu$ and define $\Sigma_1 = \{E \cap D \mid E \in \mathcal{E}_\mu\}$ and $\mu_1(E \cap D) = \mu(E)$ for all $E \in \mathcal{E}_\mu$. Since $D \in \mathcal{D}_\mu$, it follows that when $E \cap D = E' \cap D$, E, E' differ by a set of measure 0. Hence, μ_1 is well-defined. It is easy to check that μ_1 is a prior on Σ_1. Then, repeating the previous argument yields a diffuse subset D_1 of D. Then, for any E such that $\mu(E) > 0$, we have $\mu_1(E \cap D) > 0$ and therefore $E \cap D_1 \neq \emptyset$. A symmetric argument yields $E \cap (D \setminus D_1) \neq \emptyset$. Hence, $D_1, D \setminus D_1$ are in $\mathcal{D}_\mu$. *Q.E.D.*

PROOF OF LEMMA 1: Lemma 1.2.1 (pp. 6–7) in van der Waart and Wellner (1996) establishes that every act f has an envelope. It remains to show that, for every interval act $\mathbf{f} \in \mathbf{F}$, there is an act $f \in \mathcal{F}$ such that $\mathbf{f} = [f]$. Since μ is a prior, Lemma A2 implies $\mathcal{D}_\mu$ is non-empty. Let $\mathbf{f} \in \mathbf{F}$ and $f = \mathbf{f}_1 D \mathbf{f}_2$ for $D \in \mathcal{D}_\mu$. We claim that $[f] = \mathbf{f}$. Note that $\mathbf{f}_1(\omega) \leq f(\omega) \leq \mathbf{f}_2(\omega)$ for all ω. For any real-valued function g on Ω, if there exists $E \in \mathcal{E}_\mu$ such that $\mu(E) > 0$ and $g(\omega) > \mathbf{f}_1(\omega)$ for all $\omega \in E$, then, since D is μ-diffuse, we have $g(\omega) > \mathbf{f}_1(\omega) = f(\omega)$ for some $\omega \in D \cap E$. Therefore, $\mathbf{g} \in \mathbf{F}$ and $\mathbf{g}_1(\omega) \leq f(\omega)$ for all ω implies $\mu\{\mathbf{g}_1 \leq \mathbf{f}_1\} = 1$. A symmetric argument yields $\mathbf{g} \in \mathbf{F}$ and $\mathbf{g}_2(\omega) \geq f(\omega)$ for all ω implies $\mu\{\mathbf{g}_2 \geq \mathbf{f}_2\} = 1$. *Q.E.D.*

APPENDIX B: THEOREM 1

B.1. *Outline of the Proof of Theorem 1*

If we restrict attention to ideal events, Axioms 1–6 yield a standard expected utility representation with a countably additive probability measure μ and a continuous utility index $v: X \to \mathbb{R}$. Fix any diffuse event D and, for $(x, y) \in I$, let $u(x, y) = v(z)$ such that $yDx \sim z$. Axioms 2 and 6 ensure that $z \in [x, y]$ exists and therefore u is well-defined. The proof of the theorem shows that W represents $\succeq^u_\mu$. For this, it is enough to show that $v(x^*) = W(f)$ implies $x^* \sim f$.

Consider any simple act f, let $\{x_1, \dots, x_n\}$ be the set of values that f takes, and assume without loss of generality that $x_i < x_{i+1}$. Consider the partition

$$\{A_i \mid A_i = f^{-1}(x_i) \text{ for } i = 1, \dots, n\}.$$

By Lemma A2, we can partition Ω into any finite number of diffuse events $D_1, \dots, D_l$. Since μ is non-atomic, given any $\alpha_1, \dots, \alpha_r > 0$ such that $\sum_i \alpha_i = 1$, we can also construct a partition of ideal events $E_1, \dots, E_r$ such that $\alpha_i = \mu(E_i)$ for all i. Moreover, we construct this partition so that, for all $(i, j) \in \{1, \dots, r\} \times \{1, \dots, l\}$, there is some $k \in \{1, \dots, n\}$ with $E_i \cap D_j \subset A_k$.

Let x, y be the minimal and maximal values of f on E_1. Let f_1 be an act that agrees with f on E_1^c, takes on the values x and y on E_1, and agrees with f when f is equal to x or y. That f_1 has the same envelope as f follows from the definition of a diffuse event. To see that f_1 is indifferent to f, consider the simplest case: $E_1 = \Omega$ and assume that $f = xD_1(zD_2y)$ for some diffuse partition D_1, D_2, D_3. We use monotonicity and uniform continuity (Axioms 2 and 6(ii)) to show that $f \geq g$ implies $f \succeq g$. It follows that $xD_1(zD_2y) \succeq xD_1 \cup D_2y$ and $xD_1y \succeq xD_1(zD_2y)$. By Axiom 3, $xD_1 \cup D_2y \sim xD_1y$ and therefore $xD_1 \cup D_2y \sim xD_1(zD_2y) \sim xD_1y$.

Then, by induction, f is indifferent to and has the same envelope as some act g that takes at most two values on each E_j and agrees with f whenever f takes its maximal or minimal value in E_j. Let y_j and x_j be these values, respectively. Then, it follows from the definition of an ideal event and Axiom 3 that g is indifferent to the act h such that $h(\omega) = z_j$ for all $\omega \in E_j$ for z_j such that $z_j \sim y_jDx_i$. Since h is measurable with respect to ideal events, $x^* \sim h \sim g \sim f$ for some x^* such that

$$v(x^*) = \sum_j v(z_j)\mu(E_j) = \sum_j u(x_j, y_j)\mu(E_j) = W(g) = W(f),$$

as desired. The extension to all acts uses Axiom 6 and follows familiar arguments.

B.2. *Proof of Theorem 1*

The proof is divided into a series of lemmas. It is understood that Axioms 1–6 hold throughout.

DEFINITION: A set E is left (right) ideal if $fEh \succeq gEh$ implies $fEh' \succeq gEh'$ ($hEf \succeq hEg$ implies $h'Ef \succeq h'Eg$). Let $\mathcal{E}^l$ and $\mathcal{E}^r$ be the collection of left and right ideal sets, respectively.

LEMMA B0: (i) $\mathcal{E}^r = \{E \mid E^c \in \mathcal{E}^l\}$. (ii) $\mathcal{E}^l \cap \mathcal{E}^r = \mathcal{E}$.

PROOF: Assertion (i) is obvious, as is the fact that $\mathcal{E}^l \cap \mathcal{E}^r \subset \mathcal{E}$. Suppose $E \in \mathcal{E}$ and assume $fEh \succeq gEh$. Let $f^* = fEh$ and $g^* = gEh$. Then, $f^*Eh = fEh \succeq gEh = g^*Eh$ and hence $f^*Eh \succeq g^*Eh$. Also, $hEf^* = h = hEg^*$ and hence $hEf^* \succeq hEg^*$, and therefore $f^*Eh' \succeq g^*Eh'$ and $h'Ef^* \succeq h'Eg^*$ since $E \in \mathcal{E}$. That is, $fEh' = f^*Eh' \succeq g^*Eh' = gEh'$ and hence $E \in \mathcal{E}^l$. A symmetric argument establishes that $E \in \mathcal{E}^r$ and therefore $\mathcal{E} = \mathcal{E}^l \cap \mathcal{E}^r$. *Q.E.D.*

LEMMA B1: (i) $f \geq g$ *implies* $f \succeq g$. (ii) $f \succ g$ *implies* $f \succ z \succ g$ *for some* $z \in X$. (iii) $f_n, g_n \in \mathcal{F}$, f_n *converges uniformly to* f, g_n *converges uniformly to* g, $f \succ g$ *implies* $f_n \succ g_n$ *for large* n. (iv) $f_n, g_n \in \mathcal{F}^e$, f_n *converges pointwise to* f, g_n *converges pointwise to* g, $f \succ g$ *implies* $f_n \succ g_n$ *for large* n. (v) *If* $E \in \mathcal{E}_+$ *and* $y > x$, *then* $yEh \succ xEh$ *for all* $h \in \mathcal{F}$. (vi) *If* $E \in \mathcal{E}_+$ *and* $f \in \mathcal{F}$, *then there exists a unique* $c_E(f) \in X$ *such that* $c_E(f)Ef \sim f$.

PROOF: To prove (i), let $f_n = \frac{1}{n}m + (\frac{n-1}{n})f$ and $g_n = \frac{1}{n}l + (\frac{n-1}{n})g$. Then, f_n converges to f uniformly and g_n converges to g uniformly. By Axiom 2, $f_n \succ g_n$. Then, by Axiom 6, $f \succeq g_n$, and applying Axiom 6 again yields $f \succeq g$ as desired.

To prove (ii), assume $f \succ g$ and let $y = \inf\{z \in X \mid z \succeq f\}$ and let $x = \sup\{z \in X \mid g \succeq z\}$. By (i) above, x and y are well-defined. Axiom 6 ensures that $y \sim f$ and $z \sim g$ and therefore $y \succ x$. Then, for $z = \frac{x+y}{2}$, we have $f \succ z \succ g$.

To prove (iii), let $f \succ g$ and apply (ii) three times to get z, y, x such that $f \succ z \succ y \succ x \succ g$. Axiom 6 ensures that $f_n \succ y$ and $y \succ g_n$ for all n large enough. Therefore, $f_n \succ g_n$ for all such n. An analogous argument proves (iv).

To prove (v), consider $E \in \mathcal{E}_+$, $h \in \mathcal{F}$, and $x < y$. Then, there exist f, g, h' such that $fEh' \succ gEh'$, which implies that $mEh \succeq fEh' \succ gEh' \succeq lEh'$ by part (i) above and hence $mEh' \succ lEh'$ by Axiom 1. Lemma B0(ii) implies $m \succ lEm$, which implies $y \succ xEy$ by Axiom 4, which, again by Lemma B0(ii), implies $yEh \succ xEh$ as desired.

Finally, let $z = \inf\{x \in X \mid xEf \succeq f\}$. By part (i), $mEf \succeq f$ and hence z is well-defined. Axiom 6 and part (v) of this lemma ensure that $zEf \sim f$ and also that $y \neq z$ implies $yEf \not\sim f$. Hence, $z = c_E(f)$. *Q.E.D.*

LEMMA B2: *The collection* $\mathcal{E}$ *is a* σ*-field*.

PROOF: Our proof relies on Theorem 1 in Gorman (1968). To state this theorem, let T be any finite nonempty set, $\mathcal{T}$ be the set of all subsets of T and consider compact intervals $Q_t \subset \mathbb{R}$ for $t \in T$. Let $Q = \bigtimes_{t \in T} Q_t$; i.e., the product of the Q_t's. For $q, \hat{q} \in Q$ and $\tau \in \mathcal{T}$, let $q^* = q\tau\hat{q}$ denote the element of Q such that $q^*_t = q_t$ for $t \in \tau$ and $q^*_t = \hat{q}_t$ for $t \in T \setminus \tau$.

Let $\succeq^*$ be a complete, transitive and continuous binary relation on Q. Call $\tau \in \mathcal{T}$ *separable* if $q\tau\hat{q} \succeq^* q'\tau\hat{q}$ implies $q\tau\bar{q} \succeq^* q'\tau\bar{q}$ for all $q, q', \hat{q}, \bar{q} \in Q$. Thus, the separability of τ is analogous to E being a left-ideal set. Call τ *essential* if there exists $q, q', \hat{q}$ such that $q\tau\hat{q} \succ q'\tau\hat{q}$; call τ strictly *strictly essential* if for all $\hat{q}$, there exists $q, q' \in Q$ such that $q\tau\hat{q} \succ^* q'\tau\hat{q}$. The following is a special case[18] of Gorman's Theorem 1:

THEOREM G: *If $\{t\}$ is essential for all $t \in T$, $\tau' \setminus \tau$ is strictly essential, τ, τ' are separable and $\tau \cap \tau' \neq \emptyset$, then $\tau \cup \tau'$, $\tau \setminus \tau'$, $\tau' \setminus \tau$ and $(\tau \setminus \tau') \cup (\tau' \setminus \tau)$ are all separable.*

First, we will show that $\mathcal{E}$ is a field.

FACT 1: (i) $\emptyset \in \mathcal{E}$, (ii) $E \in \mathcal{E}$ *implies* $E^c \in \mathcal{E}$ *and* (iii) *If* $E \in \mathcal{E}$ *and* $\hat{E}$ *is null, then* $E \cup \hat{E} \in \mathcal{E}$ *and* $E \setminus \hat{E} \in \mathcal{E}$.

Parts (i) and (ii) are obvious. To see why (iii) is true, note that if $f(E\cup\hat{E})h \succeq g(E\cup\hat{E})h$, then, since $\hat{E}$ is null, $(fEh')(E\cup\hat{E})h \sim f(E\cup\hat{E})h$ and $(gEh')(E\cup\hat{E})h \sim g(E\cup\hat{E})h$ and hence we have $(fEh')(E\cup\hat{E})h \succeq (gEh')(E\cup\hat{E})h$. Since, $E \in \mathcal{E}$, we conclude that $fEh' \succeq gEh'$ and hence $E\cup E' \in \mathcal{E}^l$. The arguments for establishing that $E \setminus \hat{E} \in \mathcal{E}^l$, $E \cup \hat{E} \in \mathcal{E}^r$ and $E \setminus \hat{E} \in \mathcal{E}^r$ are essentially identical and omitted. Hence, $E \cup \hat{E} \in \mathcal{E}^l \cap \mathcal{E}^r$ and $E \setminus \hat{E} \in \mathcal{E}^l \cap \mathcal{E}^r$, which by Lemma B0, proves that $E \cup \hat{E} \in \mathcal{E}$ and $E \setminus \hat{E} \in \mathcal{E}$.

FACT 2: *If* $E, \hat{E} \in \mathcal{E}^l$, *then* $E \cap \hat{E} \in \mathcal{E}^l$.

Suppose $E, \hat{E} \in \mathcal{E}^l$ and $fE \cap \hat{E}h \succeq gE \cap \hat{E}h$; that is $(fEh)\hat{E}h \succeq (gEh)\hat{E}h$. Since $\hat{E} \in \mathcal{E}$, we have $(fEh)\hat{E}h' \succeq (gEh)\hat{E}h'$; that is, $(f\hat{E}h')E(h\hat{E}h') \succeq (g\hat{E}h')E(h\hat{E}h')$. Since $E \in \mathcal{E}$, we have $fE \cap \hat{E}h' = (f\hat{E}h')E(h'\hat{E}h') \succeq (g\hat{E}h')E(h'\hat{E}h') = gE \cap \hat{E}h'$ and hence $E \cap \hat{E} \in \mathcal{E}^l$.

Given Fact 1(i) an 1(ii), to complete the proof that $\mathcal{E}$ is a field, we need only show that $E, \hat{E}$ implies $E \cup \hat{E} \in \mathcal{E}$. Take $E, \hat{E} \in \mathcal{E}$ and let $E_1 = E \setminus \hat{E}$, $E_2 = \hat{E} \setminus E$, $E_3 = \hat{E} \cap E$ and $E_4 = \Omega \setminus \{E \cup \hat{E}\}$. By Lemma B0, we need to show that $E \cup \hat{E} \in \mathcal{E}^l$ and $E^c \cap \hat{E}^c \in \mathcal{E}^l$; that is, we need to show that $E_4 \in \mathcal{E}^l$ and $E_1 \cup E_2 \cup E_3 \in \mathcal{E}^l$. Fact 1(ii) and Fact 2 imply $E_i \in \mathcal{E}^l$ for all $i = 1, 2, 3, 4$ and hence we have only $E_1 \cup E_2 \cup E_3 \in \mathcal{E}^l$ left to prove.

If E_i is null for $i = 1$ or 2, then $E_1 \cup E_2 \cup E_3 \in \mathcal{E}^l$ follows from Fact 1(iii) and Lemma B0. By Fact 1(i) and (ii), we have $\Omega \in \mathcal{E}$. Therefore, if E_4 is null, then since $E_1 \cup E_2 \cup E_3 = \Omega \setminus E_4$ and we get $E_1 \cup E_2 \cup E_3 \in \mathcal{E}^l$ again from Fact 1(iii) and Lemma B0. Hence, we assume E_1, E_2, and E_4 are nonnull. Furthermore, if E_3 is null, then $E_1 \in \mathcal{E}$ by Fact 1(iii) and hence we can, without loss of generality, replace E with E_1 and assume $E \cap \hat{E} = \emptyset$. Therefore, for the remainder of the proof, we make the following assumption:

[18]In Gorman's Theorem, Q_t's need not be subsets of $\mathbb{R}$.

ASSUMPTION 1: *E_1, E_2, E_4 are nonnull and E_3 is either empty or nonnull.*

Let (x_1, x_2, x_3, h) denote the act that yields x_i on $\omega \in E_i$ for $i = 1, 2, 3$ and $h(\omega)$ on $\omega \in E_4$.

CLAIM: *$(x_1, x_2, x_3, h) \succeq (y_1, y_2, y_3, h)$ implies $(x_1, x_2, x_3, h') \succeq (y_1, y_2, y_3, h')$.*

Suppose the Claim is false and without loss of generality, take x_i, y_i for $i = 1, 2, 3$ and h, h' such that $(x_1, x_2, x_3, h) \succeq (y_1, y_2, y_3, h)$ and $(y_1, y_2, y_3, h') \succ (x_1, x_2, x_3, h')$. Then, let $\succeq^*$ denote following binary relation on $X^3 \times [0, 1]$:

$$(x_1, x_2, x_3, \alpha) \succeq^* (y_1, y_2, y_3, \beta) \quad \text{if and only if}$$
$$\big(x_1, x_2, x_3, \alpha h + (1-\alpha)h'\big) \succeq \big(y_1, y_2, y_3, \beta h + (1-\beta)h'\big).$$

Let $T = \{1, 2, 3, 4\}$ if $E_3 \neq \emptyset$ and let $T = \{1, 2, 4\}$ otherwise. Then, let $\mathcal{T} = 2^T$. Hence, by hypothesis, $T \setminus \{4\}$ is not separable.

FACT 3: *$\succeq^*$ is complete, transitive and continuous and $\{t\}$ is strictly essential for $t \in T \setminus \{4\}$, is essential for $t = 4$.*

The completeness and transitivity of $\succeq$ follows from Axiom 1, continuity follows from Axiom 6(ii) and the boundedness of X. That $\{4\}$ is essential follows immediately from the hypothesis that $(y_1, y_2, y_3, h') \succ (x_1, x_2, x_3, h')$. Finally, to see that $\{t\}$ is strictly essential for $t \in T \setminus \{4\}$, note that since E_1 is nonnull, $fE_1h^* \succ gE_1h^*$ for all f, g, h^*. Then, by Lemma B1(i), $mE_1h^* \succ lE_1h^*$. Since $E_1 \in \mathcal{E}^l$, we conclude that $mE_1\hat{h} \succ lE_1\hat{h}$ for all $\hat{h}$; that is, $(m, z_2, z_3, \alpha) \succ (l, z_2, z_3, \alpha)$ for all z_2, z_3, α. The same argument applies to $\tau = 2$ and also to $\tau = 3$ whenever E_3 is nonempty (and hence nonnull by Assumption 1).

To prove the Claim, first consider the case in which $E_3 = \emptyset$. Then, note that $E^c = E_1 \cup E_4$ and since $E^c \in \mathcal{E}^l$, $\tau := \{2, 4\}$ is separable. By symmetry, so is $\tau' := \{1, 4\}$. Then, Fact 3 ensures that all of the conditions of Theorem G are met and therefore $\{1, 2\}$ is separable, yielding a contradiction and hence proving the Claim. Next, if $E_3 \neq \emptyset$, then set $\tau = \{1, 3\}$, $\tau' = \{2, 3\}$ and note again that all of the conditions of Theorem G are met and hence $\{1, 2, 3\}$ is separable, again yielding a contradiction that proves the Claim.

Then, by Lemma B1(vi), there is $x \in X$ such $(x(E_1 \cup E_3)f)(E_1 \cup E_2 \cup E_3)h^* \sim f(E_1 \cup E_2 \cup E_3)h^*$ and $y \in X$ such that $(x, y, y, h^*) \sim (x(E_1 \cup E_3)f)(E_1 \cup E_2 \cup E_3)h^*$ and therefore, $(x, y, y, h^*) \sim f(E_1 \cup E_2 \cup E_3)h^*$. Similarly, we can find $\hat{x}, \hat{y}$ such that $(\hat{x}, \hat{y}, \hat{y}, h^*) \sim g(E_1 \cup E_2 \cup E_3)h^*$. Hence, $(x, y, y, h^*) \succeq (\hat{x}, \hat{y}, \hat{y}, h^*)$. Then, from (iv) above, we get $(x, y, y, h') \succeq (\hat{x}, \hat{y}, \hat{y}, h')$. But since $E, \hat{E} \in \mathcal{E}^l$, $(x, y, y, h^*) \sim (x(E_1 \cup E_3)f)(E_1 \cup E_2 \cup E_3)h^*$ implies $(x, y, y, h) \sim (x(E_1 \cup E_3)f)(E_1 \cup E_2 \cup E_3)h$ and $(x(E_1 \cup E_3)f)(E_1 \cup E_2 \cup E_3)h^* \sim f(E_1 \cup E_2 \cup E_3)h^*$ implies $(x(E_1 \cup E_3)f)(E_1 \cup E_2 \cup E_3)h' \sim f(E_1 \cup E_2 \cup E_3)h'$ and therefore $f(E_1 \cup E_2 \cup E_3)h' \sim (x, y, y, h')$. A symmetric argument yields $g(E_1 \cup E_2 \cup E_3)h' \sim (x, y, y, h')$. Then, $(x, y, y, h') \succeq (\hat{x}, \hat{y}, \hat{y}, h')$ implies $f(E_1 \cup E_2 \cup E_3)h' \succeq g(E_1 \cup E_2 \cup E_3)h'$, proving that $E_1 \cup E_2 \cup E_3 \in \mathcal{E}^l$. This completes the proof that $\mathcal{E}$ is a field.

To prove that the field $\mathcal{E}$ is a σ-field, it is enough to show that if $E_i \in \mathcal{E}$ and $E_i \subset E_{i+1}$, then $\bigcup E_i \in \mathcal{E}$. Let $E_i \subset E_{i+1}$ for all i. Note that $\hat{f} E_i \hat{g}$ converges pointwise to $\hat{f} \bigcup E_i \hat{g}$ for all $\hat{f}, \hat{g} \in \mathcal{F}$. Hence, if $g \bigcup E_i h' \succ f \bigcup E_i h'$ or $h' \bigcup E_i g \succ h' \bigcup E_i f$ for some $f, g, h, h' \in \mathcal{F}^e$, by Lemma B1(iv) above, we have $gE_n h' \succ fE_n h'$ or $h'E_n g \succ h'E_n f$ for some n, proving that $E_i \in \mathcal{E}$ for all n implies $\bigcup_i E_i \in \mathcal{E}$. *Q.E.D.*

LEMMA B3: *There exists a finitely additive, convex-ranged probability measure μ on $\mathcal{E}$ and a function $v : \Omega \to \mathbb{R}$ such that* (i) *the function $V : \mathcal{F}^e \to \mathbb{R}$ defined by*

$$V(f) = \sum_{x \in X} v(x)\mu\big(f^{-1}(x)\big)$$

represents the restriction of $\succeq$ to $\mathcal{F}^e$ and (ii) *if $\mu\{f = g\} = 1$, then $f \sim g$.*

PROOF: Let $\succeq_e$ be the restriction of $\succeq$ to $\mathcal{F}^e$. By definition (of $\mathcal{E}$), $\succeq_e$ satisfies Savage's P2. Similarly, Axiom 1 is Savage's P1, Lemma B1(i) and (v) imply P3, Axiom 4 is P4, Axiom 2 implies P5, and Axiom 5 implies P6. Then, following Savage's proof yields the desired conclusion. This is true despite the fact that Savage's theorem assumes that the underlying σ-field is the set of all subsets of Ω; the arguments work for any σ-field. This proves (i). To prove (ii), note that, by hypothesis, there exists $E \in \mathcal{E}$ such that $\mu(E) = 1$ and $g = fEg$. But $m \sim mEl$ by part (i), and since $E \in \mathcal{E}$, we have $fEm \sim fEl$. Then, Lemma B1(i) yields $f = fEf \sim fEg$. *Q.E.D.*

LEMMA B4: *The probability measure μ on $\mathcal{E}$ is a prior.*

PROOF: To show that μ is countably additive, we need to prove that, given any sequence E_i such that $E_{i+1} \subset E_i$ for all i and $E^* := \bigcap_i E_i = \emptyset$, $\lim \mu(E_i) = 0$. Suppose $\lim \mu(E_i) > 0$. Then, convex-valuedness ensures the existence of E such that $\lim \mu(E_i) > \mu(E) > 0$. Hence, $\mu(E_i) > \mu(E)$ for all i; that is, $mE_i l \succ mEl$ for all i. But $mE_i l \in \mathcal{F}^e$ and converges pointwise to $mE^* l$. Hence, $mE^* l \succeq mEl \succ l$. Therefore, $\mu(E^*) > 0$, a contradiction.

To prove that μ is complete, we will assume $\mu(E) = 0$ and $A \subset E$, then show that this implies $fAh \sim h$ for all $f \in \mathcal{F}$. This means that $A \in \mathcal{E}$. By Lemma B1(i), $fAh \succeq lEh$. Since, $\mu(E) = 0$, Lemma B3 implies $lEm \sim m$, and since $E \in \mathcal{E}$, we conclude that $lEh \sim mEh$. But, $mEh \succeq h$ by Lemma B1(i), so we have $fAh \succeq h$. A symmetric argument yields $h \succeq fAh$ and, hence, $fAh \sim h$ as desired.

Since μ is convex-ranged, it is obviously non-atomic. Hence, μ is a prior. *Q.E.D.*

LEMMA B5: *The function v is strictly increasing and continuous.*

PROOF: That v is strictly increasing follows from $y \succ x$ whenever $y > x$. To prove continuity, assume, without loss of generality, that $v(m) = 1$ and $v(l) = 0$, and suppose $r = \lim v(x_n) < v(x)$ for some sequence x_n converging to x. Choose $E_r \in \mathcal{E}$ such that $\mu(E_r) = r$. By Lemma B3(i), $x \succ mE_rl \succeq x_n$ for n large. Therefore, $x \succ mE_rl \succeq \lim x_n = x$, a contradiction. Hence, $r \geq v(x)$. A symmetric argument yields $v(x) \geq r$, proving the continuity of v. *Q.E.D.*

LEMMA B6: (i) $\mathcal{D} = \mathcal{D}_\mu$. (ii) *For all $(x, y) \in X^2$, there is a unique $z \in X$ such that $yDx \sim z$ for all $D \in \mathcal{D}$.*

PROOF: (i) By construction, $\mathcal{E}_\mu = \mathcal{E}$ and $\mathcal{E}_+ = \{E \in \mathcal{E}_\mu \mid \mu(E) > 0\}$. Therefore, $\mu_*(D) = \mu_*(D^c) = 0$ implies $E \not\subset D$ and $E \not\subset D^c$ for all $E \in \mathcal{E}_+$ and, hence, $D \in \mathcal{D}$. Therefore, $\mathcal{D}_\mu \subset \mathcal{D}$. For the reverse inclusion, note that $D \in \mathcal{D}$ implies that $E \cap D \neq \emptyset$ and $E \cap D^c \neq \emptyset$ for all $E \in \mathcal{E}_+$. Since $\mathcal{E}_+ = \{E \in \mathcal{E}_\mu \mid \mu(E) > 0\}$, it follows that $\mu_*(D) = \mu_*(D^c) = 0$.

(ii) By Axiom 3, $xDy \sim xD'y$ for all $D, D' \in \mathcal{D}$. Hence, fix any $D \in \mathcal{D}$ and let $z = \sup\{w \in X \mid yDx \succeq w\}$. Since $yDx \succeq l$, by Lemma B1(i), z is well-defined. Then, Lemma B1(iii) rules out both $z \succ yDx$ and $yDx \succ z$. *Q.E.D.*

Lemma B6(i) and Lemma A2 imply $\mathcal{D} \neq \emptyset$. Then, Lemma B6(ii) ensures that the function $u : I \to \mathbb{R}$, below, is well-defined.

DEFINITION: For all $x \leq y$, let $u(x, y) = v(z)$ for $z \in X$ such that $yDx \sim z$ for $D \in \mathcal{D}$.

LEMMA B7: *Let $D_1, \ldots, D_n \in \mathcal{D}$ be a partition of Ω, $D_i \subset \{f = x_i\}$ for all i, and $y_{i+1} \geq y_i$ for $i \leq n - 1$. Then, $f \sim y_nDy_1$ for all $D \in \mathcal{D}$.*

PROOF: By Lemma B1(i), $y_n[D_2 \cup \cdots \cup D_n]y_1 \succeq f \succeq y_nD_ny_1$. By Axiom 3, $y_n[D_2 \cup \cdots \cup D_n]y_1 \sim y_nD_ny_1 \sim y_nDy_1$. *Q.E.D.*

LEMMA B8: *The function u is increasing and continuous.*

PROOF: Suppose $yDx \sim z$ and $\hat{y}D\hat{x} \sim \hat{z}$. If $\hat{y} > y$ and $\hat{x} > x$, then Axiom 2 implies $\hat{z} \succ z$, and applying Axiom 2 again yields $\hat{z} > z$ as desired. If $\hat{y} \geq y$ and $\hat{x} \geq x$, then by Lemma B1(i), $\hat{z} \succeq z$. Then, applying Axiom 2 again yields $\hat{z} \geq z$.

To prove continuity, assume $y_iDx_i \sim z_i$ for $i = 1, \ldots,$ and $\lim(x_i, y_i) = (x, y)$. Since X is compact, we can assume, without loss of generality, that z_i converges to some z. Then Lemma B1(iii) rules out $yDx \succ z$ and $z \succ yDx$ and establishes continuity. *Q.E.D.*

Define

$$W(f)=\int u[f]\,d\mu.$$

LEMMA B9: *The function W represents the restriction of $\succeq$ to $\mathcal{F}^o$.*

PROOF: Let $\{x_1,\dots,x_n\}$ be the range of f, let $A_i=f^{-1}(x_i)$, and let $\{E^J_*(f)\}$ be an ideal split of $\{A_i\}$. Lemma A2 implies that $\{E^J_*(f)\}$ exists and is unique up to measure zero. Let $\mathcal{N}^+(f)=\{J\mid\mu(E^J_*(f))>0 \text{ and } |J|>1\}$ and $\mathcal{H}^n=\{f\in\mathcal{F}^o\mid n=|\mathcal{N}^+(f)|\}$. The proof is by induction on $\mathcal{H}^n$. Note that, for $f\in\mathcal{H}^0$,

$$W(f)=\sum_{z\in X}v(z)\mu\big(f^{-1}(z)\big)=v(x)$$

for x such that $\mu\{x=f\}=1$. Hence, by Lemma B3(ii), W represents the restriction $\succeq$ to $\mathcal{H}^0$. Suppose W represents the restriction of $\succeq$ to $\mathcal{H}^n$ and choose $f\in\mathcal{H}^{n+1}$. Define h_f as follows: if $f\in\mathcal{H}^n$, then $h_f=f$; otherwise, choose $E^J_*(f)$ such that $|J|>1$ and $\mu(E^J_*(f))>0$, choose $D\in\mathcal{D}$, and define f^* as follows:

$$f^*(\omega)=\begin{cases} f(\omega), & \text{if } \omega\notin E^J_*(f),\\ \max f\big(E^J_*(f)\big), & \text{if } \omega\in D\cap E^J_*(f),\\ \min f\big(E^J_*(f)\big), & \text{if } \omega\in D^c\cap E^J_*(f).\end{cases}$$

By Lemma B7 and Axiom 3, $f^*\sim f$. Next, choose z such that $u(x,y)=v(z)$, and let $h_f(\omega)=f^*(\omega)$ for all $\omega\notin E^J_*(f)$ and $h_f(\omega)=z$ for all $\omega\in E^J_*(f)$. Again, Axiom 3 ensures that $h_f\sim f^*\sim f$. Note that $h_f\in\mathcal{H}^n$ and, by construction, $W(h_f)=W(f^*)$. By Lemma A1, $[f^*]=[f]$ and, therefore, $W(f^*)=W(f)$. Thus, $W(f)=W(h_f)$ for some $h_f\in\mathcal{H}^n$ such that $h_f\sim f$. Then, the induction hypothesis implies that W represents $\succeq$ on $\mathcal{H}^{n+1}$. *Q.E.D.*

LEMMA B10: *The function W represents $\succeq$.*

PROOF: Note that, for all f, there exists x_f such that $W(x_f)=u(x_f,x_f)=W(f)$. This follows from the fact that u is continuous and $u(m,m)\geq W(f)\geq u(l,l)$. Hence, by the intermediate value theorem, $u(x_f,x_f)=W(f)$ for some $x_f\in[l,m]$. The monotonicity of u ensures that this x_f is unique. Next, we show that $f\sim x_f$.

Without loss of generality, assume $l=0$ (if not, let $l^*=0$ and $m^*=m-l$ and identify each f with $f^*=f-l$, and apply all previous results to acts $\mathcal{F}^*=\{f-l\mid f\in\mathcal{F}\}$). Define, for any $x\geq 0$ and $\varepsilon>0$, $z^*(x,\varepsilon)=\min\{n\varepsilon\mid n=0,1,\dots \text{ such that } n\varepsilon\geq x\}$. Similarly, let $z_*(x,\varepsilon)=\max\{n\varepsilon\mid n=0,1,\dots \text{ such that } n\varepsilon\leq x\}$. Clearly,

(B.1) $$0\leq z^*(x,\varepsilon)-x\leq z^*(x,\varepsilon)-z_*(x,\varepsilon)\leq\varepsilon,$$

and the first two inequalities above are equalities if and only if x is a multiple of ε. Set $f^n(\omega) = z^*(f(\omega), m2^{-n})$ and $f_n(\omega) = z_*(f(\omega), m2^{-n})$ for all $n = 0, 1, \dots$. Equation (B.1), above, ensures that $f^n \geq f \geq f_n$ and f^n, f_n converge uniformly to f. Note also that $f^n, f_n \in \mathcal{F}^o$ with $f^n \downarrow f$. This implies that $[f^n] \downarrow [f]$ μ-almost surely (see van der Waart and Wellner (1996, p. 13)) and, therefore, $\int u[f^n]\,d\mu \to \int u[f]\,d\mu$.

Since $f^n \geq f$, we have $W(f^n) \geq W(f) = W(x_f)$ for all n. Since W represents the restriction of $\succeq$ to $\mathcal{F}^o$, we conclude that $f^n \succeq x_f$ for all n. Then, Axiom 6 implies $f \succeq x_f$. A symmetric argument with f_n replacing f^n yields $x_f \succeq f$ and, therefore, $x_f \sim f$ as desired.

To conclude the proof of the lemma, suppose $f \succeq g$; then $W(x_f) = W(f)$ and $W(x_g) = W(g)$ and $x_f \sim f \succeq g \sim x_g$. Since W represents the restriction of $\succeq$ to $\mathcal{F}^o$, we conclude that $W(x_f) \geq W(x_g)$ and, hence, $W(f) \geq W(g)$. Similarly, if $W(f) \geq W(g)$, we conclude $f \sim x_f \succeq x_g \sim g$ and, therefore, $f \succeq g$. *Q.E.D.*

Lemma B10 establishes sufficiency. To prove that the preference, $\succeq^u_\mu$, satisfies Axioms 1–6 for every prior μ and interval utility u, we first establish that $\mathcal{E}_\mu = \mathcal{E}$ and $\mathcal{D}_\mu = \mathcal{D}$.

LEMMA B11: *Let $\mathcal{E}, \mathcal{D}$ be the ideal and diffuse events for the preference $\succeq^u_\mu$. Then, $\mathcal{D} = \mathcal{D}_\mu$, $\mathcal{E} = \mathcal{E}_\mu$.*

PROOF: From the representation, it is immediate that $\mathcal{E}_\mu \subset \mathcal{E}$ which, in turn, implies $\mathcal{D} \subset \mathcal{D}_\mu$. Thus, to prove the lemma, it suffices to show that $\mathcal{E} \subset \mathcal{E}_\mu$. For $A \subset \Omega$, let $\{E^{\{1\}}_*, E^{\{2\}}_*, E^{\{1,2\}}_*\}$ be the ideal split of A, A^c. If $A \notin \mathcal{E}_\mu$, then $\mu(E^{\{1,2\}}_*) > 0$. Let $D \in \mathcal{D}_\mu$ and define

$$D_1 = \bigl(D \cap E^{\{1\}}_*\bigr) \cup \bigl(D \cap E^{\{2\}}_*\bigr) \cup \bigl(A \cap E^{\{1,2\}}_*\bigr),$$
$$D_2 = \bigl(D^c \cap E^{\{1\}}_*\bigr) \cup \bigl(D^c \cap E^{\{2\}}_*\bigr) \cup \bigl(A^c \cap E^{\{1,2\}}_*\bigr).$$

Note that $D_1, D_2 \in \mathcal{D}_\mu$. By Lemma A2, there are $D_{i1}, D_{i2} \in \mathcal{D}_\mu$ such that $D_{i1} \cap D_{i2} = \emptyset$ and $D_{i1} \cup D_{i2} = D_i$ for $i = 1, 2$. Let $B_{ij} = D_{ij} \cap E^{\{1,2\}}_*$ for $i = 1, 2$, $j = 1, 2$ and note that

$$A = E^{\{1\}}_* \cup B_{11} \cup B_{12},$$
$$A^c = E^{\{2\}}_* \cup B_{21} \cup B_{22}.$$

For $x_1 < x_2 < x_3 \in X$, let $g = x_1$ and let

$$f = \begin{cases} x_3, & \text{if } \omega \in B_{11}, \\ x_2, & \text{if } \omega \in B_{12}, \\ x_1, & \text{otherwise,} \end{cases} \qquad h = \begin{cases} x_3, & \text{if } \omega \in B_{21}, \\ x_1, & \text{if } \omega \in B_{22}, \\ x_1, & \text{otherwise,} \end{cases}$$

$$h' = \begin{cases} x_2, & \text{if } \omega \in B_{21}, \\ x_2, & \text{if } \omega \in B_{22}, \\ x_1, & \text{otherwise.} \end{cases}$$

Then, by Lemma A1,

$$W(fAh') = \mu(E_*^{\{1,2\}})u(x_2, x_3) + \mu(E_*^{\{1\}} \cup E_*^{\{2\}})u(x_1, x_1),$$
$$W(fAh) = \mu(E_*^{\{1,2\}})u(x_1, x_3) + \mu(E_*^{\{1\}} \cup E_*^{\{2\}})u(x_1, x_1),$$
$$W(gAh) = \mu(E_*^{\{1,2\}})u(x_1, x_3) + \mu(E_*^{\{1\}} \cup E_*^{\{2\}})u(x_1, x_1),$$
$$W(gAh') = \mu(E_*^{\{1,2\}})u(x_1, x_2) + \mu(E_*^{\{1\}} \cup E_*^{\{2\}})u(x_1, x_1).$$

Thus, $W(gAh) \geq W(fAh)$. Also, $hAf = x_1 = hAg$ and, hence, $W(hAg) \geq W(hAf)$. But, since u is increasing, $W(fAh') > W(gAh')$ and, therefore, A is not ideal. *Q.E.D.*

To complete the proof, note that μ is convex-ranged since it is non-atomic. Then, verifying Axioms 1, 2, and 5 involves nothing more than repeating familiar arguments from Savage's theorem. Given Lemma B11, Axioms 3 and 4 follow immediately from the representation. Note that, for any $f^n \in \mathcal{F}^e$, if $[f^n]$ converges pointwise to $[g]$, then $W(f^n)$ converges to $W(g)$. Hence, Axiom 6(i) follows from the fact that f^n converges to f pointwise implies $[f^n]$ converges pointwise to $[f]$, while Axiom 6(ii) follows from the fact that, for any f^n, if f^n converges to f uniformly, then $[f^n]$ converges pointwise to $[f]$.

APPENDIX C: PROOFS FOR SECTION 3

PROOF OF LEMMA 2: Let λ be an interval lottery and let $\{(x_1, y_2), \ldots, (x_n, y_n)\}$ be its support. Choose a partition $E_1, \ldots, E_n$ of Ω such that $\mu(E_i) = \lambda(x_i, y_i)$. This can be done since μ is non-atomic and, therefore, convex-valued. Let $\mathbf{f} \in \mathbf{F}$ be defined as $\mathbf{f}(\omega) = (x_i, y_i)$ for $\omega \in E_i$. By Lemma 1, there exists $f \in \mathcal{F}$ such that $[f] = \mathbf{f}$. That $\lambda_\mu^f = \lambda$ is immediate. *Q.E.D.*

PROOF OF LEMMA 3: Sufficiency is immediate. For necessity, first note that $v_{\bar{u}} = \alpha v_u + \beta$ for some $\alpha > 0$. This follows since two expected utility maximizers have the same preferences over simple lotteries only if their utility indices are the same, up to a positive affine transformation. For every $(x, y) \in I$, there exist unique $z, \bar{z}$ such that $v_u(z) = u(x, y)$ and $v_{\bar{u}}(\bar{z}) = \bar{u}(x, y)$. Let $D, \bar{D}$ be diffuse sets for $\mu, \bar{\mu}$, respectively; let $f = xDy$ and $\bar{f} = x\bar{D}y$, and note that $\lambda_\mu^f = \lambda_{\bar{\mu}}^{\bar{f}}$. Since $f \sim_\mu^u z$ and $f \sim_{\bar{\mu}}^{\bar{u}} \bar{z}$, it follows that $z = \bar{z}$. Since $v_{\bar{u}} = \alpha v_u + \beta$, it follows that $\bar{u} = \alpha u + \beta$. *Q.E.D.*

PROOF OF THEOREM 2: If $v_{\bar{u}} \circ v_u^{-1}$ is not concave, then familiar arguments ensure the existence of $x < z < y$ and $p \in (0, 1)$ such that $pv_{\bar{u}}(y) + (1 - p)v_{\bar{u}}(x) > v_{\bar{u}}(z)$ and $pv_u(y) + (1 - p)v_u(x) < v_u(z)$. Let λ be the interval lottery that yields (y, y) with probability p and (x, x) with probability $1 - p$. Lemma 2 implies that there are $f, \bar{f}$ such that $\lambda_\mu^f = \lambda_{\bar{\mu}}^{\bar{f}} = \lambda$. Since $\bar{f} \succ_{\bar{\mu}}^{\bar{u}} z$ and $z \succ_\mu^u f$, $\succeq_{\bar{\mu}}^{\bar{u}}$ is not more cautious than $\succeq_\mu^u$.

If $\sigma_{\bar{u}}^{xy} < \sigma_u^{xy}$, then choose σ strictly between these two numbers. Let λ be the interval lottery that yields (x, y) with probability 1. Lemma 2 implies that there are $f, \bar{f}$ such that $\lambda_\mu^f = \lambda_{\bar{\mu}}^{\bar{f}} = \lambda$. Since $\bar{f} \succ_{\bar{\mu}}^{\bar{u}} \sigma x + (1-\sigma)y$ and $\sigma x + (1-\sigma)y \succ_\mu^u f$, it follows that $\succeq_{\bar{\mu}}^{\bar{u}}$ is not more cautious than $\succeq_\mu^u$.

To prove sufficiency, assume $\sigma_{\bar{u}}^{xy} \geq \sigma_u^{xy}$ for all x, y and $v_{\bar{u}} \circ v_u^{-1}$ is concave. Let $f \in \mathcal{F}^o$ and let $\lambda = \lambda_\mu^f$ with support $\{(x_1, x_2), \dots, (x_n, y_n)\}$. For each (x_i, y_i), define $z_i = \sigma_{\bar{u}}^{x_i y_i} x_i + (1 - \sigma_{\bar{u}}^{x_i y_i}) y_i$. Let λ^* be the interval lottery with support $\{(z_1, z_1), \dots, (z_n, z_n)\}$ such that

$$\lambda^*(z_i, z_i) = \lambda(x_i, y_i).$$

By Lemma 2, there are $f^*, \bar{f}, \bar{f}^* \in \mathcal{F}^o$ such that $\lambda_\mu^{f^*} = \lambda^*$, $\lambda_{\bar{\mu}}^{\bar{f}} = \lambda$, and $\lambda_{\bar{\mu}}^{\bar{f}^*} = \lambda^*$. Note that $\bar{f} \sim_{\bar{\mu}}^{\bar{u}} \bar{f}^*$ by construction and $f \succeq_\mu^u f^*$ since $\sigma_{\bar{u}}^{xy} \geq \sigma_u^{xy}$. Since $v_{\bar{u}} \circ v_u^{-1}$ is concave, $v_u(z) \geq \sum_{i=1}^n v_u(z_i)\lambda^*(z_i, z_i)$ implies $v_{\bar{u}}(z) \geq \sum_{i=1}^n v_{\bar{u}}(z_i)\lambda^*(z_i, z_i)$. We conclude that $z \succeq_\mu^u f^*$ implies $z \succeq_{\bar{\mu}}^{\bar{u}} \bar{f}^* \sim_{\bar{\mu}}^{\bar{u}} \bar{f}$. Since $f \succeq_\mu^u f^*$, it follows that $z \succeq_\mu^u f$ implies $z \succeq_{\bar{\mu}}^{\bar{u}} \bar{f}$ as desired. *Q.E.D.*

PROOF OF COROLLARY 1: The proof is straightforward and therefore omitted. *Q.E.D.*

PROOF OF THEOREM 3: Let $f = xAy$ be a binary act with $x < y$. We first show that if $W = (u, \mu)$, then

$$(*) \qquad W(yAx) = \mu_*(A)u(y, y) + \mu_*(A^c)u(x, x) + (1 - \mu_*(A) - \mu_*(A^c))u(x, y).$$

Let $A_1 = A$, $A_2 = A^c$ and $\{E_*^{\{1\}}, E_*^{\{2\}}, E_*^{\{12\}}\}$ be the ideal split of A_1, A_2. Then, by Lemma A1 and the definition of an ideal split, it follows that

$$[f]_1 = x(E_*^{\{1\}} \cup E_*^{\{12\}})y,$$

$$[f]_2 = x(E_*^{\{1\}})y,$$

where $\mu(E_*^{\{i\}}) = \mu_*(A_i)$ for $i = 1, 2$. Applying the representation now proves the assertion.

Then, let $u(x, y) = x$ and $u^*(x, y) = y$. If $\mu_*(A) > \mu_*(B)$, the assertion above yields $yAx \succ^u yBx$; if $\mu_*(A^c) < \mu_*(B^c)$, the assertion above yields $yAx \succ^{u^*} yBx$. Hence, B dominates A implies $\mu_*(A) \leq \mu_*(B)$ and $\mu_*(A^c) \leq \mu_*(B^c)$. Conversely, if $\mu_*(A) \leq \mu_*(B)$, $\mu_*(A^c) \leq \mu_*(B^c)$, the assertion above ensures that $yBx \succ^{\hat{u}} yAx$ for all $\hat{u}$.

Suppose A is more uncertain than B and let u, u^*, W, W^* be as above. Suppose A, B are comparable so that $[\mu_*(B) - \mu_*(A)] \cdot [\mu_*(A^c) - \mu_*(B^c)] < 0$. Then choose z such that $x < z < y$ and

$$[\mu_*(B) - \mu_*(A)][y - z] + [\mu_*(A^c) - \mu_*(B^c)][z - x] = 0.$$

Let $t = \frac{y-z}{y-x}$ and $\hat{u}(x', y') = tx + (1-t)y$ for all $(x', y') \in I$. Note that $\succeq^u$ is more uncertainty averse than $\succeq^{u'}$ and, therefore, we must have

$$[\mu_*(B) - \mu_*(A)][y - x] \geq 0.$$

That is, we must have $\mu_*(B) - \mu_*(A) > 0$ and $\mu_*(A^c) - \mu_*(B^c) < 0$.

Conversely, assume $\mu_*(B) - \mu_*(A) > 0$, $\mu_*(A^c) - \mu_*(B^c) < 0$ and $\succeq^{u_1}$ is more uncertainty averse than $\succeq^{\hat{u}_2}$. Then, by Corollary 1, $v_{u_1} = a \cdot v_{\hat{u}_2} + b$ for some $a > 0$. Let $u_2 = a \cdot \hat{u}_1 + b$ and note that $v_{u_1} = v_{u_2}$ and $\succeq^{\hat{u}_2}_\mu = \succeq^{u_2}_\mu$. Let $v := v_{u_1} = v_{u_2}$. Then, by Corollary 1, for all $x < y$, we have $u_1(x, y) = v(tx + (1-t)y)$ and $u_2(x, y) = v(rx + (1-r)y)$ for $t \geq r$. Hence, $u_1(x, y) \leq u_2(x, y)$ for all $(x, y) \in I$.

If $yBx \succeq^{u_2}_\mu yAx$, then $\mu_*(B)v(y) + \mu_*(B^c)v(x) + (1 - \mu_*(B) - \mu_*(B^c)) \times u_2(x, y) \geq \mu_*(A)v(y) + \mu_*(B^c)v(x) + (1 - \mu_*(A) - \mu_*(A^c))u_2(x, y)$. That is,

$$\begin{aligned} &[\mu_*(B) - \mu_*(A)][v(y) - u_2(x, y)] \\ &\quad + [\mu_*(A^c) - \mu_*(B^c)][u_2(x, y) - v(x)] \geq 0. \end{aligned}$$

Since $\mu_*(B) - \mu_*(A) > 0$ $\mu_*(A^c) - \mu_*(B^c) < 0$ and $u_1(x, y) \leq u_2(x, y)$, the equation above still holds when u_2 is replaced with u_1. Therefore, $yBx \succeq^{u_1}_\mu yAx$, proving that A is more uncertain than B. *Q.E.D.*

APPENDIX D: PROOFS FOR SECTION 4

PROOF OF THEOREM 4: Let $W = (u, \mu)$, $S = \{1, \ldots, n\}$. Take any partition ρ. Let $\{E^J_*(\rho)\}$ be an ideal split of $\{\rho^{-1}(1), \ldots, \rho^{-1}(n)\}$. For any $a \subset \{1, \ldots, n\}$, $a \neq \emptyset$, define

$$\pi(a) = \mu(E^a_*(\rho)),$$

and note that $\pi \in \Pi$. Let $f = \phi \circ \rho$. For $\omega \in E^a_*(\rho)$, Lemma A1 implies that

$$\begin{aligned} [f]_1(\omega) &= \min\{f(\hat{\omega}) \mid \hat{\omega} \in E^a_*(\rho)\} = \min\{\phi(s) \mid s \in a\}, \\ [f]_2(\omega) &= \max\{f(\hat{\omega}) \mid \hat{\omega} \in E^a_*(\rho)\} = \max\{\phi(s) \mid s \in a\}. \end{aligned}$$

Then, Theorem 1 yields

$$W(f) = \sum_a u\Big(\min_{\omega \in E^a_*(\rho)} f(\omega), \max_{\omega \in E^a_*(\rho)} f(\omega)\Big)\mu\big(E^a_*(\rho)\big)$$
$$= \sum_a u\Big(\min_{s \in a} \phi(s), \max_{s \in a} \phi(s)\Big)\pi(s)$$

as desired.

For the converse, take any $\pi \in \Pi$. Choose a partition $\{E^a \mid a \subset S\}$ of Ω such that $E^a \in \mathcal{E}_\mu$ for all $a \subset S$ and $\mu(E^a) = \pi(a)$. For each a such that $|a| > 1$, let $\{D^a_i\}_{i \in a}$ be a partition of Ω into diffuse sets. Define ρ as follows: if $|a| > 1$, then $\rho(\omega) = i$ for $\omega \in E^a \cap D^a_i$; if $|a| = 1$, then, for all $\omega \in E^a$, $\rho(\omega) = s$ for s such that $\{s\} = a$. This construction ensures that $\{E^a \mid a \subset S\} = \{E^J_*(\rho)\}$ and, for $f = \phi \circ \rho$,

$$[f]_1(\omega) = \min\big\{\phi(s) \mid s \in a\big\},$$
$$[f]_2(\omega) = \max\big\{\phi(s) \mid s \in a\big\}.$$

It follows that $W(\phi \circ \rho) = \sum_a u(\min_{s \in a} \phi(s), \max_{s \in a} \phi(s))\pi(a)$ as desired.

Q.E.D.

PROOF OF THEOREM 5: Assume that $\kappa = \alpha\pi_* + (1-\alpha)\pi^*$. For any $\phi \in \Phi$, order $S = \{s_1, s_2, \dots, s_n\}$ so that $\phi(s_i) \geq \phi(s_{i+1})$ and let $a_0 = \emptyset$, $a_i = \{s_1, \dots, s_i\}$ for $i \geq 1$. Then, for the $V = (v, \kappa)$, we have

$$V(\phi) = \sum_{i=1}^n v\big(\phi(s_i)\big)\big[\kappa(a_i) - \kappa(a_{i-1})\big]$$
$$= \sum_{i=1}^n v\big(\phi(s_i)\big)$$
$$\times \big\{\alpha\big[\pi_*(a_i) - \pi_*(a_{i-1})\big] + (1-\alpha)\big[\pi^*(a_i) - \pi^*(a_{i-1})\big]\big\}$$
$$= \sum_{i=1}^n v\big(\phi(s_i)\big)\sum_{b \in \mathbf{A}_i} \alpha\pi(b) + \sum_{i=1}^n v\big(\phi(s_i)\big)\sum_{b \in \mathbf{B}_i}(1-\alpha)\pi(b),$$

where $\mathbf{A}_i = \{b \subset S \mid b \subset a_i, b \not\subset a_{i-1}\}$ and $\mathbf{B}_i = \{b \subset S \mid b \subset a^c_{i-1}, b \not\subset a^c_i\}$. Hence, we have

$$V(\phi) = \sum_{i=1}^n v\big(\phi(s_i)\big)\sum_{b \in \mathbf{A}_i} \alpha\pi(b) + \sum_{i=1}^n v\big(\phi(s_i)\big)\sum_{b \in \mathbf{B}_i}(1-\alpha)\pi(b)$$
$$= \sum_{i=1}^n \sum_{b \in \mathbf{A}_i} v\Big(\min_{s \in b} \phi(s)\Big)\alpha\pi(b)$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\sum_{b\in\mathbf{B}_i} v\Big(\max_{s\in b}\phi(s)\Big)(1-\alpha)\pi(b)\\
&=\sum_{b\in\mathcal{P}} v\Big(\min_{s\in b}\phi(s)\Big)\alpha\pi(b)+\sum_{b\in\mathcal{P}} v\Big(\max_{s\in b}\phi(s)\Big)(1-\alpha)\pi(b)\\
&=U(\phi),
\end{aligned}
$$

where $U=(u,\pi)$ for u such that $u(x,y)=\alpha v(x)+(1-\alpha)v(y)$ for all $(x,y)\in I$.

Define $\Delta^a=\{p\in\Delta^S \mid p(s)=0, s\notin a\}$. It is well-known (Strassen (1964)) that

$$
\Delta_\pi=\sum_a \pi(a)\Delta^a.
$$

For $V=(\alpha,\Delta_\pi,v)$, we have

$$
\begin{aligned}
V(\phi)&=\alpha\min_{\lambda\in\Delta_\pi}\sum_{s\in S} v\big(\phi(s)\big)\lambda(s)+(1-\alpha)\max_{\lambda\in\Delta_\pi}\sum_{s\in S} v\big(\phi(s)\big)\lambda(s)\\
&=\sum_a \pi(a)\\
&\quad\times\left(\alpha\min_{\lambda\in\Delta^a}\sum_{s\in S} v\big(\phi(s)\big)\lambda(s)+(1-\alpha)\max_{\lambda\in\Delta^a}\sum_{s\in S} v\big(\phi(s)\big)\lambda(s)\right)\\
&=\sum_{a\in\mathcal{P}}\pi(a)\Big(\alpha v\Big(\min_{s\in a}\phi(s)\Big)+(1-\alpha)v\Big(\max_{s\in a}\phi(s)\Big)\Big)=U(\phi),
\end{aligned}
$$

where $U=(u,\pi)$ for u such that $u(x,y)=\alpha v(x)+(1-\alpha)v(y)$ for all $(x,y)\in I$. *Q.E.D.*

APPENDIX E: PROOFS FOR SECTIONS 6 AND 7

PROOF OF THEOREM 6: Let $\mathcal{P}$ be the collection of non-empty subsets of S. Recall that there are k different configurations of ball colors in the urns and m different balls in each urn. For $b\in\mathcal{P}$, let

$$
\pi(b)=\begin{cases}\left(\dfrac{1}{m}\right)^k, & \text{if } |b_t|=1 \text{ for all } t,\\ 0, & \text{otherwise.}\end{cases}
$$

Note that π is indeed a probability on S. Then, for all $b \in \mathcal{P}$,

$$\pi_*(b) = \sum_{\substack{a \subset b \\ a \in \mathcal{P}}} \pi(a) = \frac{1}{m^k} \prod_{t=1}^{k} |b_t|, \tag{E.1}$$

$$\pi^*(b) = 1 - \frac{1}{m^k} \prod_{t=1}^{k} \big(m - |b_t|\big).$$

For $a \in \mathbf{A}$, $|a_1| = |a_t|$ for all $t = 1, \dots, k$ and, therefore, $\pi_*(a) = (\frac{|a_1|}{m})^k$ and $\pi^*(a) = 1 - (\frac{m-|a_1|}{m})^k$. Since $|a| \geq |a'|$ if and only if $|a_1| \geq |a'_1|$, part (i) follows.

When $|a| = |b|$, we have $\sum_t |a_t| = \sum_t |b_t|$. Furthermore, if $a \in \mathbf{A}$, $b \in \mathbf{B} \setminus \mathbf{A}$, then $|a_t| = |a_1|$ for all t, while $|b_t| \neq |b_1|$ for some t. Hence, equation (E1) implies that $\pi_*(a) > \pi_*(b)$ and $\pi^*(a) < \pi^*(b)$ whenever $a \in \mathbf{A}$, $b \in \mathbf{B} \setminus \mathbf{A}$ and $|a| = |b|$, proving (ii).

It follows from equation ($*$) in the proof of Theorem 3 that

$$U(yax) = \pi_*(a) + z\big(\pi^*(a) - \pi_*(a)\big).$$

Hence, $U(yax) - U(ybx) = (1-z)[\pi_*(a) - \pi_*(b)] + z[\pi^*(a) - \pi^*(b)]$. Let T be the set of all (a, b) such that $a \in \mathbf{A}$, $b \in \mathbf{B} \setminus \mathbf{A}$ and $|a| = |b|$, and define

$$z^* = \min_{(a,b) \in T} \frac{\pi_*(a) - \pi_*(b)}{\pi_*(a) - \pi_*(b) + \pi^*(b) - \pi^*(a)}.$$

By part (ii), $\pi_*(a) > \pi_*(b)$ and $\pi^*(a) < \pi^*(b)$ if $(a, b) \in T$. Since T is finite, it follows that $z^* \in (0, 1)$ is well-defined, proving part (iii). *Q.E.D.*

PROOF OF THEOREM 7: That separability precludes M-reversals is obvious. To conclude the proof, we will show that if there are no M-reversals, then u must be separable. Let $\phi = (x_1, x_2, x_3, x_4)$ and $\phi' = (x_1, x_4, x_3, x_2)$, with $x_1 \geq x_3 \geq x_2 \geq x_4$; then $\pi \in \Pi^m$ implies that

$$U(\phi) - U\big(\phi'\big) = \beta\big(u(x_2, x_1) - u(x_4, x_1) + u(x_4, x_3) - u(x_2, x_3)\big),$$

with $\beta = \pi(\{1, 2\}) = \pi(\{3, 4\}) > 0$. Thus, no M-reversals implies

$$u(z_1, y_1) + u(z_2, y_2) = u(z_1, y_2) + u(z_2, y_1)$$

whenever $(z_i, y_j) \in I$ for all $i = 1, 2$, $j = 1, 2$. Define $v_2(y) = u(l, y)$ and $v_1(z) = u(z, m) - u(l, m)$. Then, $v_1(z) + v_2(y) = u(z, m) - u(l, m) + u(l, y)$, and the displayed equation above ensures that $u(z, m) - u(l, m) = u(z, y) - u(l, y)$. Therefore, $v_1(z) + v_2(y) = u(z, y)$ for all $(z, y) \in I$, proving the separability of u. *Q.E.D.*

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Manuscript received March, 2010; final revision received August, 2013.