Self-Control and the Theory of Consumption†

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October 2000

Abstract

We present a model of temptation and self-control for infinite horizon consumption problems under uncertainty. We identify a tractable class of preferences called *Dynamic self-control (DSC)* preferences. These preferences are recursive, separable, and describe agents who are tempted by immediate consumption. We introduce measures comparing the preference for commitment and the self-control of DSC consumers and establish the following: In standard infinite-horizon economies equilibria exist but may be inefficient; in such equilibria, agents' steady state consumption is independent of their initial endowments and increasing in their self-control. In a representative agent economy, increasing the agents preference for commitment while keeping self-control constant increases the equity premium. Removing non-binding constraints may change equilibrium allocations and prices. Debt contracts with DSC agents can be sustained even if the only feasible punishment for default is the termination of the contract.

† We thank Pierre-Olivier Gourinchas and Per Krusell for their suggestions and advice. This research was supported by grants from the National Science Foundation and by the Sloan Foundation.

1. Introduction

In experiments, subjects exhibit a reversal of preferences when choosing between a smaller, earlier and a larger, later reward (Kirby and Herrnstein (1995)). The earlier reward is preferred when it offers an immediate payoff whereas the later reward is preferred when both rewards are received with delay. This phenomenon is referred to as *dynamic inconsistency* and has inspired theoretical work that modifies exponential discounting to allow for disproportionate discounting of the immediate future.[1]

This paper proposes an alternative approach to incorporate the experimental evidence. We extend our earlier analysis of self-control in two period choice problems (Gul and Pesendorfer (2000)) to an infinite horizon. Our goal is to capture the experimental evidence with tractable, dynamically consistent preferences and to apply the resulting model to the analysis of problems in macroeconomics and finance.

As an illustration, consider a consumption-savings problem. A consumer faces a constant interest rate r and a fixed wealth b. Each period, the consumer must decide how much to consume of his remaining wealth. Let $z(b)$ denote the corresponding choice problem and let c denote the current consumption choice. Our axioms imply that the consumer has preferences of the form:

$$W(z(b)) = \max_{c \in [0,b]} (u(c) + v(c) + \delta W(z(b'')) - v(b)$$

where u and v are von Neumann-Morgenstern utility functions and $b' = (b-c)(1+r)$ is the remaining wealth in the next period. These preferences represent an individual who, every period, is tempted to consume her endowment. Were she to do so, the term $v(c) - v(b)$ would drop out. When she consumes less than her endowment, she incurs the disutility of self-control, $v(c) - v(b)$. The utility function v represents "temptation", that is, the individual's urge for current consumption. Optimal behavior trades-off the temptation to consume with the long-run self-interest of the individual, represented by the discounted sum of u's. The main theoretical result of this paper (Theorem 2) is a representation theorem yielding the utility function W above.

[1] See, for example, Strotz (1955), Laibson (1997), O'Donague and Rabin (1998), Krusell and Smith (1999)

The preferences developed in this paper depend on what the individual consumes and on what she *could have consumed*. In a simple consumption-savings problem with no liquidity constraints, the maximal temptation is $v(b)$, the temptation utility of consuming all current wealth. More generally, the maximal temptation in each period depends on the set of possible consumption choices for that period. For example, if there are borrowing constraints or commitment opportunities, the term $v(\bar{c})$ where $\bar{c}$ is the total resources available for immediate consumption, replaces $v(b)$ in the utility function W.

To allow a direct dependence of preferences on opportunity sets we study preferences over *choice problems*. Building on work by Kreps and Porteus (1978), Epstein and Zin (1989) and Brandenburger and Dekel (1993) we develop the appropriate framework to study infinite horizon choice problems in section 2. With the set of choice problems as our domain, we define *Dynamic Self-Control (DSC)* preferences and derive a utility function for those preferences. The formula above is a special case of this derived utility function.

In the literature on dynamic inconsistency (see, for example, Laibson 1997), non-recursive preferences of the form $u(c_1) + \sum_{t=2}^{\infty} \beta\delta^{t-1}u(c_t)$ are specified for the initial time period. In addition, it is assumed that the preferences governing behavior in period t are given by $u(c_t) + \sum_{t'=t+1}^{\infty} \beta\delta^{t'-t}u(c_{t'})$. Therefore at time $t > 1$, the individual's period t preferences differ from her *conditional* preferences - the preference over continuation plans implied by her first period preferences and choices prior to period t. This preference change is taken as a primitive to derive the individual's desire for commitment.

By contrast, our approach takes as starting point preferences that may exhibit a desire for commitment. The description of a "consumption path" includes both the actual consumption and what the individual could have consumed in each period. On this extended space, preferences are recursive and the conditional preferences are the same as period t preferences. Our representation theorem shows that DSC preferences can be interpreted as describing an individual whose temptation utility v, interferes with her long-run self-interest (represented by the sum of discounted u's). The consumer uses self-control to mediate between her temptation utility and her long-run self-interest.

The recursive structure of our model allows us to apply standard techniques of dynamic programming to find optimal solutions. At the same time, the model is consistent with the type of preference reversal documented in the experimental literature. Consider a consumer who must decide between a smaller reward in period 1 and a larger reward in period 2. When opting for the later reward the consumer incurs a self-control cost since the earlier reward would lead to a larger consumption in the current period. Now consider the situation where the earlier reward is for period $t > 1$ and the later for period $t + 1$. Since the decision is taken in period 1, the choice does not affect the opportunity set for current consumption and the larger reward can be taken without incurring a self-control cost. Thus, when we consider the implied preferences over consumption alone, preferences appear non-recursive and consistent with the documented preference reversals.

To allow a broad set of applications to macroeconomics and financial economics, our framework is rich enough to accommodate the infinite horizon, stochastic dynamic programming problems central to these fields. Techniques developed in those areas can be applied to DSC preferences to explore how self-control and preference for commitment change the conclusions of standard (macro)economic models. In section 4.2 we analyze a deterministic exchange economy with DSC preferences. We give conditions under which a competitive equilibrium exists and find that in general, competitive equilibria are not Pareto efficient. Section 4.3. examines steady state equilibria and shows that DSC preferences allow us to predict the steady state distribution of wealth, independent of the initial wealth distribution. In section 4.4, we analyze a simple stochastic exchange economy and find that (under appropriate assumptions on u and v) increasing the preference for commitment increases the predicted premium of risky over safe assets. In section 4.4, we observe that in an economy with DSC agents, removing constraints that are non-binding given the original equilibrium allocations and prices, may change these allocations and prices. Finally, in section 4.5 incentive compatible debt contracts are explored. If the only punishment for default is exclusion from future borrowing, then standard preferences imply that there are no incentive compatible debt contracts (Bulow and Rogoff (1989)). In contrast, with DSC preferences, "natural" debt contracts turn out to be incentive compatible.

Section 2 below defines an infinite horizon consumption problem (IHCP). IHCPs serve as the domain for our preferences. Since section 2 is somewhat technical we start it with a non-technical summary after which readers who are less interested in the details may skip ahead to section 3.

2. Infinite Horizon Consumption Problems

Consider a consumer who must take an action at every period $t = 1, 2, \ldots$. Each action results in a consumption for that period and constrains future actions. The standard approach to this problem is to define preferences for the consumer over sequences of consumption realizations. This standard approach excludes preferences that depend not only on outcomes but also on what *could have been chosen*. Such a direct dependence on the opportunity set is natural in a context where individuals suffer from self-control problems.

To allow a dependence of preference on opportunity sets we make *choice problems* the domain of our preferences. Our treatment of dynamic choice problems is similar to the "descriptive approach" in Kreps and Porteus (1978) extended to an infinite horizon. This extension is closely related to work by Epstein and Zin (1989) and Brandenburger and Dekel (1993).

A one-period consumption problem is a set of consumption choices, representing the set of feasible choices for the consumer. Inductively, we can define a t-period consumption problem to be a set of choices, each of which yields a consumption for the current period and a $t-1$ period continuation problem. Choices by the consumer may yield random consumption and continuation problems. Hence, the t period consumption problem is a set of probability distributions where each realization yields a consumption today and a $t-1$ choice problem. An infinite horizon consumption problem (IHCP) is a *consistent* infinite product of finite consumption problems. Consistency requires that the $t-1$ period choice problem implied by the t period choice problem must be the same as the $t-1$ period choice problem. By C we denote the set of possible consumptions each period and by Z we denote the set of IHCPs.

It is often more convenient to think of infinite horizon choice problems in a recursive manner: each problem is a set of choices, each of which yields a consumption for the current period and an infinite horizon problem starting next period. Viewed this way, an IHCP is a set of probability distributions on $C \times Z$. Let $\Delta(C \times Z)$ denote the set of all probability distributions on $C \times Z$ and $\mathcal{K}(\Delta(C \times Z))$ denote the recursive IHCPs. Each $z \in \mathcal{K}(\Delta(C \times Z))$ is a (compact) set of probability distributions. In Theorem 1 we show that the function $f : Z \to \mathcal{K}(\Delta(C \times Z))$ which associates with each element of Z its recursive is a homeomorphisms.

We now begin the formal definition of IHCPs. Readers less interested in the details may skip ahead to section 3.

Let X be any metric space. The set $\mathcal{K}(X)$, of all non-empty compact subsets of X (endowed with the *Hausdorff* metric) is itself a metric space. If X is compact then so is $\mathcal{K}(X)$ (see Brown and Pearcy (1995) p. 222). Let $\Delta(X)$ denote the set of all measures on the Borel σ-algebra of X. We endow $\Delta(X)$ with the weak topology. If X is compact then $\Delta(X)$ is also compact and metrizable (with the Prohorov metric) (see Parthasarathy (1970)).

Given any sequence of metric spaces X_t, we endow $\times_{t=1}^{\infty} X_t$ with the product topology. This topology is also metrizable and $\times_{t=1}^{\infty} X_t$ is compact if each X_t is compact (Royden (1968) pp. 152, 166).

Let C denote the compact metric space of possible consumptions in each period. For example, think of C as a compact interval in $I\!R$. Let $Z_1 := \mathcal{K}(\Delta(C))$. A $z_1 \in Z_1$ is a one period consumption problem. Each choice $\mu_1 \in z_1$ yields a probability distribution on C. For $t > 1$, define Z_t inductively as $Z_t := \mathcal{K}(\Delta(C \times Z_{t-1}))$. Thus, a $z_t \in Z_t$ describes a t period consumption problem. An element $\mu_t \in z_t$ yields a measure on (C, Z_{t-1}), that is, consumption in the current period and $t-1$ period consumption problems.

Let $Z^* := \times_{t=1}^{\infty} Z_t$. The set of infinite horizon consumption problems (IHCP) are those elements of Z^* that are consistent, that is, for $z = \{z_t\}_{t=1}^{\infty} \in Z^*$ the $t-1$ period consumption problem implied by z_t is equal to z_{t-1}. To be more precise, let $G_1 : C \times Z_1 \to C$ be given by

$$G_1(c, z_1) := c$$

and let $F_1 : \Delta(C \times Z_1) \to \Delta(C)$ be defined as:

$$F_1(\mu_2)(A) := \mu_2(G_1^{-1}(A))$$

for A in the Borel σ-algebra of C. Clearly, $F_1(\mu_2)$ is the probability distribution over current consumption implied by $\mu_2 \in z_2$. Hence, $F_2(z_2) := \{F(\mu_2) \in Z_1 | \mu_2 \in z_2\}$ is the one period choice problem implied by z_2. Proceeding inductively, we define $G_t : C \times Z_t \to C \times Z_{t-1}$ by

$$G_t(c, z_t) := (c, F_{t-1}(z_t))$$

and $F_t : \Delta(C \times Z_t) \to \Delta(C \times Z_{t-1})$ by

$$F_t(\mu_{t+1})(A) := \mu_{t+1}(G_t^{-1}(A))$$

for A in the Borel σ-algebra of $C \times Z_{t-1}$. Then, $F(z_t)$ is the $t-1$ period choice problem implied by z_t. Finally, we define

$$Z := \{\{z_t\}_{t=1}^{\infty} \in Z^* \mid z_{t-1} = F_{t-1}(z_t) \forall t > 1\}$$

to be the set of all IHCP's. We prove in the appendix that Z is compact.

Note that we permit current consumption to be uncertain at the time the agent has to make her choice. However, in our applications, the period t consumption lottery will be degenerate at the time the period t choice has to be made.

As described, every IHCP is an infinite sequence of finite choice problems. Alternatively, an IHCP can also be viewed as a set of options, each of which results in a probability distribution over current consumption and IHCP's that describe the consumer's situation next period. Hence, the recursive view identifies an IHCP as an element of $\mathcal{K}(\Delta(C \times Z))$. Indeed, there is a natural mapping from Z to $\mathcal{K}(\Delta(C \times Z))$.

Theorem 1: *There exists a homeomorphism* $f : Z \to \mathcal{K}(\Delta(C \times Z))$.

Proof: See Appendix.

The mapping f is described in the appendix. To illustrate how f identifies recursive IHCPs with IHCPs, consider the problem $\{z_t\}_{t=1}^{\infty} \in Z$ and assume that each choice is deterministic, that is; each $\mu_t \in z_t$ implies a (certain) period 1 consumption, c_t, and a (certain) continuation problem z_{t-1}. Consider a sequence $\{\mu_t\}_{t=1}^{\infty}$ such that $\mu_t \in z_t$ for all t and let (c_t, z_{t-1}) denote the element of $C \times Z_{t-1}$ that occurs with probability 1 according to μ_t. Since $\{\mu_t\}_{t=1}^{\infty}$ is consistent ($\mu_{t-1} = F_t(\mu_t)$) it follows that $c_t = c_1$ for all t; that is, the period 1 consumption implied by μ_t is the same as the period 1 consumption implied by μ_1. Moreover, the sequence of continuation problems $z' := \{z_{t-1}\}_{t=2}^{\infty}$ is itself consistent. Hence, we can identify $\{\mu_t\}_{t=1}^{\infty}$ with the unique element in $\Delta(C \times Z)$ that puts probability 1 on (c, z'). Repeating this for every consistent element of z yields $f(z)$.

In the analysis below we view infinite horizon consumption problems as elements of $\mathcal{K}(\Delta(C \times Z))$; that is, we identify the set Z with $\mathcal{K}(\Delta(C \times Z))$. The homeomorphism f permits us to do so.

3. Dynamic Self-Control Preferences

The consumer has preferences over the set of infinite horizon consumption problems, Z. We use x, y or z to denote elements of Z. When there is no risk of confusion, we write Δ instead of $\Delta(C \times Z)$. We use μ, ν or η to denote elements of $x \subset \Delta(C \times Z)$. A lottery that yields the current consumption c and the continuation problem z with certainty is denoted (c, z).

For $\alpha \in [0, 1]$, let $\alpha\mu + (1-\alpha)\nu \in \Delta$ be the measure that assigns $\alpha\mu(A) + (1-\alpha)\nu(A)$ to each A in the Borel σ-algebra of $C \times Z$. Similarly, $\alpha x + (1-\alpha)y := \{\alpha\mu + (1-\alpha)\nu \mid \mu \in x, \nu \in y\}$ for $\alpha \in [0, 1]$. A preference is denoted by $\succeq$. We make following familiar assumptions on preferences.

Axiom 1: *(Preference Relation) $\succeq$ is a complete and transitive binary relation.*

Axiom 2: *(Continuity) The sets $\{x \mid x \succeq z\}$ and $\{x \mid z \succeq x\}$ are closed.*

We say that the function $W : Z \to \mathbb{R}$ represents the preference $\succeq$ when $x \succeq y$ iff $W(x) \geq W(y)$. Axioms 1 and 2 imply that $\succeq$ may be represented by a continuous function W.

A standard consumer evaluates a set of options by its best element. Adding options to the set can never make such a consumer worse off. Thus, if $x \succeq y$ then the best option in x is preferred to the best option in y and hence

$$x \succeq y \Rightarrow x \cup y \sim x \tag{$*$}$$

Axioms 1, 2 together with $(*)$ imply that there is utility function U such that $W(x) := \max_{\mu \in x} U(\mu)$ represents $\succeq$. Hence, Axioms 1, 2 and $(*)$ yield a standard consumer. By contrast, a consumer who is susceptible to temptation may prefer a smaller set of options to a larger set. That is, she may have a preference for commitment.

Definition: *The preference $\succeq$ has a preference for commitment at z if there is $x \subset z$ such that $x \succ z$. The preference $\succeq$ has a preference for commitment if $\succeq$ has a preference for commitment at some z.*

The agent may have a preference for commitment even when she does not succumb to temptation. Theorem 3 below shows that $x \succ x \cup y \succ y$ can be interpreted as a situation where the the agent chooses an alternative in x but suffers from the availability of the tempting options in y. Therefore, we say that the agent has self-control when $x \succ x \cup y \succ y$.

Definition: *The preference $\succeq$ has self-control at z if there are subsets x, y with $x \cup y = z$ and $x \succ z \succ y$. The preference $\succeq$ has self-control if $\succeq$ has self-control at some z.*

Axiom 3 allows a preference for commitment and self control.

Axiom 3: *(Set Betweenness) $x \succeq y$ implies $x \succeq x \cup y \succeq y$.*

Our objective is to characterize tractable preferences that satisfy Set Betweenness. To this end, we impose the following additional axioms.

A singleton set represents a situations where the agent has no choice. Hence, we refer to the restriction of $\succeq$ to singleton sets as the agent's commitment ranking. Axiom 4 says that the agent's commitment ranking satisfies von Neumann-Morgenstern's independence axiom.[2]

Axiom 4: *(Independence) $\{\mu\} \succ \{\nu\}$, $\alpha \in (0,1)$ implies $\{\alpha\mu + (1-\alpha)\eta\} \succ \{\alpha\nu + (1-\alpha)\eta\}$.*

Recall that (c, z) denotes a lottery that returns the consumption c in the current period and the continuation problem z. Axiom 5 requires that the correlation between the current consumption and the continuation problem does not affect preferences. The axiom considers two lotteries: $\mu = \frac{1}{2}(c, z) + \frac{1}{2}(c', z')$ returns either the consumption c together with the continuation problem z or the consumption c' together with the continuation

[2] Axiom 4 is weaker than the independence axiom in Dekel, Lipman and Rustichini (2000) and Gul and Pesendorfer (2000).

problem z'; $\nu = \frac{1}{2}(c, z') + \frac{1}{2}(c', z)$ returns either the consumption c together with the continuation problem z' or the consumption c' together with the continuation problem z. The axiom requires the agent to be indifferent between $\{\mu\}$ and $\{\nu\}$.

Axiom 5: *(Separability)* $\left\{\frac{1}{2}(c, z) + \frac{1}{2}(c', z')\right\} \sim \left\{\frac{1}{2}(c, z') + \frac{1}{2}(c', z)\right\}$.

Axiom 6 requires preferences to be stationary. Consider the lotteries, $(c, x), (c, y)$ each leading to the same consumption in the current period. The axiom requires that $\{(c, x)\}$ is preferred to $\{(c, y)\}$ if and only if the continuation problem x is preferred to the continuation problem y.

Axiom 6: *(Stationarity)* $\{(c, x)\} \succeq \{(c, y)\}$ *iff* $x \succeq y$.

Axiom 7 requires individuals to be indifferent to the timing of resolution of uncertainty. In standard models this indifference is implicit in the assumption that the domain of preferences is the set of lotteries over consumption paths. Our model follows Kreps and Porteus (1978) in that choice problems are the domain of preferences. This richer structure permits Kreps and Porteus to model agents who are not indifferent to the timing of resolution of uncertainty. In this paper, the richer domain is used to analyze temptation and self-control. To separate these issues, we rule out preference for early or late resolution of uncertainty.

Consider the lotteries $\mu = \alpha(c, x) + (1-\alpha)(c, y)$ and $\nu = (c, \alpha x + (1-\alpha)y)$. The lottery μ returns the consumption c together with the continuation problem x with probability α and the consumption c with the continuation problem y with probability $1 - \alpha$. By contrast, ν returns c together with the continuation problem $\alpha x + (1-\alpha)y$ with probability 1. Hence, μ resolves the uncertainty about x and y in the current period whereas ν resolves this uncertainty in the future. If $\{\mu\} \sim \{\nu\}$ then the agent is indifferent as to the timing of the resolution of uncertainty.[3]

Axiom 7: *(Indifference to Timing)* $\{\alpha(c, x) + (1-\alpha)(c, y)\} \sim_s \{(c, \alpha x + (1-\alpha)y)\}$.

Axiom 8 requires that two alternatives, ν, η, offer the same temptation if they have the same marginal distribution over current consumption. For any $\mu \in \Delta(C \times Z)$, μ^1

[3] To see the relation to the independence axiom, consider an extension of $\succeq$ to lotteries defined on Z. If this extended preference relation satisfies the independence axiom and is stationary (satisfies Axiom 6) then Axiom 7 follows.

denotes its marginal on the first coordinate (current consumption) and μ^2 its marginal on the second coordinate (the continuation problem).

Axiom 8: *(Temptation by Immediate Consumption) For $\mu, \nu \in \Delta$ suppose $\nu^1 = \eta^1$. If $\{\mu\} \succ \{\mu, \nu\} \succ \{\nu\}$ and $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$ then $\{\mu, \nu\} \sim \{\mu, \eta\}$.*

To understand Axiom 8, recall that $\{\mu\} \succ \{\mu, \nu\} \succ \{\nu\}$ represents a situation where the agent is tempted by ν but uses self-control and chooses μ. Similaly, $\{\mu\} \succ \{\mu, \eta\} \succ \{\eta\}$ means that the agent is tempted by η but chooses μ. Hence, the agent makes the same choice in both situations. If ν^1, η^1 then she experiences the same temptation in both situations and therefore is indifferent between the two situations; $\{\mu, \nu\} \sim \{\mu, \eta\}$.

Note that if the agent succumbs to temptation whenever she has a preference for commitent i.e. there exists no μ, ν satisfying the hypothesis of Axiom 8, the axiom has no bite. We call such preferences degenerate. In Theorem 2, we exclude such preferences.

Definition: *The preference $\succeq$ is nondegenerate if there exists x, y such that $y \subset x$ and $x \succ y$.*

Theorem 2 provides a recursive and separable representation of non-degenerate preferences that satisfy Axioms 1-8.

Theorem 2: *If the non-degenerate preference $\succeq$ satisfies Axioms $1-8$ then there is some $\delta \in (0, 1)$, continuous functions $u, v : C \to I\!R$ and a continuous function W that represents $\succeq$ such that*

$$W(z) := \max_{\mu \in z} \int (u(c) + v(c) + \delta W(z')) \, d\mu(c, z') - \max_{\nu \in z} \int v(c) d\nu(c, z')$$

for all $z \in Z$. Moreover, for any $\delta \in (0, 1)$, continuous $u, v : C \to I\!R$, the function W above is well-defined and the preference it represents satisfies Axioms $1 - 8$.

Proof: See Appendix.

We refer to nondegenerate preferences that satisfy Axioms $1-8$ as *dynamic self control* (DSC) preferences. If some W of the form given in Theorem 2 represents the preferences $\succeq$, we refer to the corresponding (u, v, δ) as a representation of $\succeq$ and sometimes as *the* preference $\succeq$.

Theorem 2 establishes that each (u, v, δ) corresponds to *unique* DSC preferences. This is in contrast to the approach taken in Laibson (1997) where the utility function alone cannot determine the individuals preferences over IHCPs. The problem that defines preferences over IHCPs in Laibson's approach is, in effect, an infinite-horizon game with multiple equilibria. Since typically a large number of outcomes can be supported as subgame prefect equilibria of such games, very little can be said about the individual's preferences over IHCPs, and hence about her behavior.

For the DSC preference (u, v, δ), define:

$$U(\mu) := \int u(c) + \delta W(x) d\mu$$
$$V(\mu) := \int v(c) d\mu^1$$

Note that $W(\{\mu\}) = U(\mu)$, hence U represents the agents commitment ranking. Also note that $x \subset y$ and $W(x) > W(y)$ implies $\max_{\mu \in y} V(\mu) > \max_{\mu \in x} V(\mu) >$, hence V represents the agents tempation ranking over the elements in Δ. Theorem 2 suggests that the agent facing the choice problem z chooses $\mu \in z$ that maximizes W. That is, she comprimizes between her commitment ranking and temptation. However, Theorem 2 asserts only that the agent assigns utility to IHCP's *as if* she were behaving in this manner. The difficulty stems from the fact that the domain of preferences is *choice problems* and not choices from those problems. This difficulty can be overcome by extending preferences to choices from z. In Gul and Pesendorfer (2000) we provide such an extension and give conditions that ensure that the agent indeed behaves as suggested by the representation. Here, we simply assume that the agent behaves in this way.

For any continuous fucntion F, let $\mathcal{C}(z, F) := \{\mu \in Z \,|\, f(\mu) \geq f(\nu) \text{ for all } \nu \in z\}$. Hence, the set $\mathcal{C}(z, U + V)$ denotes the agent's choices given the IHCP z. Theorem 3 below justifies our definition of self-control by showing its equivalence to the intuitive definition of self-control as the ability to resist temptation.

Theorem 3: *(Gul and Pesendorfer (2000)) The DSC preference (u, v, δ) has self-control at z iff $\mathcal{C}(z, V) \cap \mathcal{C}(z, U + V) = \emptyset$.*

Proof: See Theorem 2 of Gul and Pesendorfer 2000.

The next result establishes that the representation provided in Theorem 2 is unique upto a common affine transformation if $\succeq$ has a preference for commitment at some choice problem. Therefore, $\mathcal{C}(\cdot U + V)$, the choice behavior corresponding to any DSC is well-defined, that is depends only on $\succeq$ and not the particular representation (u, v, δ).

Definition: *The non-degenerate DSC preference $\succeq$ is regular if for some $y \subset x \in Z$, $y \succ x$.*

Thus, non-degenerate preference is regular if it has a preference for commitment at some choice problem.

Theorem 4: *Let $\succeq$ be a regular DSC preference with representation (u, v, δ). Then, (u', v', δ') also represents $\succeq$ if and only if $\delta = \delta'$ and there exist $\alpha > 0, \beta_u, \beta_v \in I\!R$ such that $u' = \alpha u + \beta_u$ and $v' = \alpha v + \beta_v$.*

4. Measures of Preference for Commitment and Self-Control

In this section, we introduce measures that allow us to compare the preference for commitment and the self-control of agents. These measures are based on similar concepts in Gul and Pesendorfer (2000). The versions here are weaker to facilitate the analysis of the applications considered in the next section.

To distinguish between differences in impatience and differences in preference for commitment (or self-control) we compare agents' behavior in restricted class of choice problems that involve no intertemporal trade-offs.

Definition: *z is intertemporally inconsequential (II) if, for every $\mu, \nu \in z$, $\mu^2 = \nu^2$. Let Z_{II} denote the set of all intertemporally inconsequential IHCP's.*

Thus, if z is an intertemporally inconsequential choice problem then every choice $\mu \in z$ leads to the same distribution over continuation problems.

Definition: *The preference $\succeq_1$ has more instantaneous preference for commitment than $\succeq_2$ if, for every $x \in \mathcal{K}_{II}$, $\succeq_2$ has preference for commitment at x implies $\succeq_1$ has preference for commitment at x. The preferences $\succeq_1, \succeq_2$ have the same instantaneous preference for commitment if $\succeq_1$ has more instantaneous preference commitment than $\succeq_2$ and $\succeq_2$ has more instantaneous preference commitment than $\succeq_1$.*

Theorem 5 characterizes when the preference $\succeq_1 = (u_1, v_1, \delta_1)$ has more instantaneous preference for commitment than $\succeq_2$. Consider the two element choice problems $x = \{\mu, \nu\}$ with $u_1(\mu^1) > u_1(\nu^1)$ and assume that x is in Z_{II}, that is, μ and ν lead to same distribution over continuation problems. If $\succeq_1$ has no preference for commitment at x then it must be that $v_1(\mu^1) \geq v_1(\nu^1)$. Now suppose that there are non-negative constants α, β such that

$$u_2 = \alpha u_1 + (1 - \alpha) v_1$$
$$v_2 = \beta u_1 + (1 - \beta) v_1$$

Clearly, it follows that $u_2(\mu^1) \geq u_2(\nu^1)$ and $v_2(\mu^1) \geq v_2(\nu^2)$. Therefore, if $\succeq_2 = (u_2, v_2, \delta_2)$ for some δ_2 then $\succeq_2$ has no preference for commitment at x. In addition, we may multiply v_2 with a positive constant γ without affecting the agent's preference for commitment. Hence, all preferences of the form $(u_2, \gamma v_2, \delta_2)$ for some $\gamma > 0$ have no preference for commitment at x. Theorem 5 demonstrates that this condition is both necessary and sufficient for $\succeq_1$ to have more preference for commitment than $\succeq_2$.

In Theorems 5 and 6 we require preferences to satisfy an stronger regularity condition: the agent has a preference for commitment at some instantaneously inconsequential choice problem y.

Definition: *The non-degenerate DSC preference $\succeq$ is instantaneously regular if for some $y \subset x \in Z_{II}$, $y \succ x$.*

Theorem 5: *Let $\succeq_1$, $\succeq_2$ be instantaneously regular DSC preferences and let (u_1, v_1, δ_1) be a representation of $\succeq_1$. Then, $\succeq_1$ has greater instantaneous preference for commitment than $\succeq_2$ if and only if there exists, u_2, v_2, δ_2 such that $(u_2, \gamma v_2, \delta_2)$ represents $\succeq_2$ and*

$$u_2 = \alpha u_1 + (1 - \alpha) v_1$$
$$v_2 = \beta u_1 + (1 - \beta) v_1$$

for some $\alpha, \beta \in [0, 1]$ and some $\gamma > 0$

Proof: See Appendix.

The following definition presents a comparative measure of self-control for instantaneous choice problems. This definition is based on an analogous definition in our earlier paper (Gul and Pesendorfer (2000)).

Definition: *The preference* $\succeq_1$ *has more instantaneous self-control than* $\succeq_2$ *if, for every* $x \in \mathcal{K}_{II}$, $\succeq_2$ *has self-control at* x *implies* $\succeq_1$ *has self-control at* x. *The preferences* $\succeq_1, \succeq_2$ *have the same instantaneous self-control if* $\succeq_1$ *has more instantaneous self-control than* $\succeq_2$ *and* $\succeq_2$ *has more instantaneous self-control than* $\succeq_1$.

Theorem 6 characterizes when the preference $\succeq_1 = (u_2, v_2, \delta_2)$ has more instantaneous self-control than $\succeq_2$. The intuition is analogous to the intuition for Theorem 5. Consider the two element choice problems $x = \{\mu, \nu\}$ with $u_1(\mu^1) + v_1(\mu^1) > u_1(\nu^1) + v_1(\nu^1)$ and assume that x is in Z_{II}. Then, μ is the optimal choice from x. If $\succeq_1$ has no self-control at x then it must be that μ is at least as tempting as ν and hence $v_1(\mu^1) \geq v_1(\nu^1)$. Now suppose that there are non-negative constants α, β such that

$$u_2 + v_2 = \alpha(u_1 + v_1) + (1 - \alpha)v_1$$
$$v_2 = \beta(u_1 + v_1) + (1 - \beta)v_1$$

Clearly, it follows that $u_2(\mu^1) + v_2(\mu^1) \geq u_2(\nu^1) + v_2(\mu^1)$ and $v_2(\mu^1) \geq v_2(\nu^2)$. Therefore, (u_2, v_2, δ_2) has no self-control at x. Observe that any preference of the form $(u_2 + (1 - \gamma)v_2, \gamma v_2, \delta_2)$ has self-control if and only if (u_2, v_2, δ_2) has self-control. Theorem 6 demonstrates that $\succeq_1$ has more self-control than $\succeq_2$ if and only if $\succeq_2 = (u_2 + (1 - \gamma)v_2, \gamma v_2, \delta_2)$ for some $\gamma > 0$.

Theorem 6: *Let* $\succeq_1$, $\succeq_2$ *be instantaneously regular DSC preferences and let* (u_1, v_1, δ_1) *be a representation of* $\succeq_1$. *Then,* $\succeq_1$ *has more instantaneous self-control than* $\succeq_2$ *if and only if there exist* u_2, v_2, δ_2 *such that* $(u_2 + (1 - \gamma)v_2, \gamma v_2)$ *represents* $\succeq_2$ *and*

$$u_2 + v_2 = \alpha(u_1 + v_1) + (1 - \alpha)v_1$$
$$v_2 = \beta(u_1 + v_1) + (1 - \beta)v_1$$

for some $\alpha, \beta \in [0, 1], \gamma > 0$.

Proof: See Appendix.

The following corollary analyzes situations where either agents have the same level of preference for commitment and differ with respect to their self-control or agents have the same level of self-control and differ with respect to their preference for commitment. Since we only make instantaneous comparisons in this paper, henceforth, without risk of confusion we say "preference for commitment" and "self-control" rather than "instantaneous preference for commitment" and "instantaneous self-control".

Corollary 7: *Let $\succeq_1, \succeq_2$ be instantaneously regular DSC preferences and let (u_1, v_1, δ_1) be a representation of $\succeq_1$. Then,*

(i) the preferences $\succeq_1, \succeq_2$ have the same preference for commitment and $\succeq_1$ has more self-control than $\succeq_2$ if and only if there exist some $\delta_2 \in (0,1)$, $\gamma \geq 1$ such that $(u_1, \gamma v_1, \delta_2)$ is a representation of $\succeq_2$;

(ii) the preferences $\succeq_1, \succeq_2$ have the same self-control and $\succeq_2$ has more preference for commitment than $\succeq_1$ if and only if there exist some $\delta_2 \in (0,1)$ and $\gamma \geq 1$ such that $(u_1 + (1-\gamma)v_1, \gamma v_1, \delta_2)$ is a representation of $\succeq_2$.

Proof: See Appendix.

Part (i) of the Corollary says that keeping u constant and changing v to γv for some $\gamma > 1$ is equivalent to a decrease in self-control without changing preference for commitment. Part (ii) of the Corollary says that keeping $u + v$ constant and changing v to γv for some $\gamma > 1$ is equivalent to increasing preference for commitment without changing self-control.

5. Applications

This section contains examples of choice problems and general equilibrium models that incorporate agents with DSC preferences.

5.1 Preference Reversal

Experimental evidence on time preference has been the main motivation for research on preference for commitment. In a typical experiment a subject is asked to choose between a smaller, earlier and a larger, later reward. Subjects tend to reverse their choices from the smaller, earlier reward to the larger, later reward as the delay to both rewards increases (see, for example, Kirby and Herrnstein (1995)). Such a preference reversal is inconsistent with standard, exponential discounting. The following example demonstrates that DSC preferences are consistent with this experimental evidence.

Consider an agent whose consumption path is $\{c_0, ..., c_0, ...\}$. In period 1, the experimenter asks her to choose between the reward α in period τ and the reward β in period $\tau + 1$. The reward must be consumed in the period it is received. If the agent chooses the earlier reward her consumption path is $\{c_t^\alpha\}$ where $c_t^\alpha = c_0$ if $t \neq \tau$ and $c_t^\alpha = c_0 + \alpha$ if $t = \tau$. If the agent chooses the later reward her consumption path is $\{c_t^\beta\}$ where $c_t^\beta = c_0$ if $t \neq \tau + 1$ and $c_t^\beta = c_0 + \beta$ if $t = \tau + 1$.

The consumer prefers the later reward if

$$u(c_1^\beta) + v(c_1^\beta) - \max\{v(c_1^\beta), v(c_1^\alpha)\} + \sum_{t>1} \delta^{t-1} u(c_t^\beta) \geq u(c_1^\alpha) + v(c_1^\alpha) - \max\{v(c_1^\beta), v(c_1^\alpha)\} + \sum_{t>1} \delta^{t-1} u(c_t^\alpha)$$

Hence, for $\tau > 1$, the consumer prefers the later reward if

$$u(c_0) + \delta u(c_0 + \beta) \geq u(c_0 + \alpha) + \delta u(c_0)$$

For $\tau = 1$, the agent prefers the later reward if

$$u(c_0) + v(c_0) - v(c_0 + \alpha) + \delta u(c_0 + \beta) \geq u(c_0 + \alpha) + \delta u(c_0)$$

To forgo the reward in period 1, she incurs a self-control penalty $v(c_0 + \alpha) - v(c_0) > 0$. Hence, she is more eager to receive the reward a day earlier if she can receive it in period 1. If the rewards are such that

$$u(c_0) + \delta u(c_0 + \beta) > u(c_0 + \alpha) + \delta u(c_0) > u(c_0) + v(c_0) - v(c_0 + \alpha) + \delta u(c_0 + \beta)$$

the agent chooses the earlier reward for $\tau = 1$ and the later reward for $\tau > 1$.

If we only observe the consumer's choice over consumption paths, her behavior appears to be inconsistent with a recursive preference. She prefers $\{c_0 + \alpha, c_0, ...\}$ to $\{c_0, c_0 + \beta, c_0, ...\}$ but she prefers $\{c_0, c_0 + \alpha, c_0, ...\}$ to $\{c_0, c_0, c_0 + \beta, c_0, ...\}$. However, for an agent with DSC preferences, specifying only the implied consumption streams is not an adequate description of choice problem. Her decision also depends on the maximal feasible consumption in the decision periods. When faced with a choice between α in period 1 and

β in period 2, the maximal feasible consumption in the decision period (period 1) is $c_0+\alpha$. On the hand, when facing a choice between receiving α in period 2 and β in period 3 the maximal consumption in the decision period (period 1) is independent of her choice and equal to c_0. Hence, the consumer makes a different choice for $\tau = 1$ than for $\tau = 2$ because the two problems imply different temptations in the decision period.

5.2 Deterministic Exchange Economy

In this section, we analyze Walrasian equilibria of a deterministic, infinite horizon, exchange economy and show that those equilibria may be inefficient. There are n households, $i \in \{1, ..., n\}$ and L physical goods. Each household i is characterized by DSC preferences (u_i, v_i, δ_i) and an endowment $\omega_i = (\omega_{i1}, \ldots, \omega_{it}, \ldots)$. We assume that endowments and consumption in each period are in the compact set C where $C := \{c_t \in I\!R^L_+ \mid c_{tl} \leq k\}$. In addition, we impose the following restrictions on the curvature of u and v.

Assumption 1: *u_i and v_i are strictly increasing; u_i is concave; v_i is convex; $w_i := u_i + v_i$ is strictly concave.*

Finally, we assume that all endowments are bounded away from zero.

Assumption 2: *There is an $\varepsilon > 0$ such that $\omega_{itl} > \varepsilon, \forall i, \forall t, \forall l$.*

The sequence $p = (p_1, \ldots, p_t, \ldots) \in (I\!R^L_+)^\infty$ denotes the period 1 prices of consumption. A consumer with wealth $b \in I\!R_+$ faces the choice problem $x(p, b)$ which is defined recursively as follows:

$$x(p, b) = \{(c_1, x(p', b')) \; | p_1 c_1 + b' = b, c_1 \in C\}$$

where $p' = (p_2, \ldots, p_t, \ldots) \in (R^L_+)^\infty$. Thus, in the choice problem $x(p, b)$ the consumer chooses a consumption for the current period. This consumption leads to a continuation wealth $b' = b - p_1 c$ and the continuation problem $x(p', b')$. It is straightforward to verify that $x(p, b) \in Z$ for all $p \in (I\!R^L_+)^\infty, b \in I\!R_+$.

The choice problem of consumer i with wealth b is given by:

$$W_i(x(p, b)) = \max_{x(p,b)} \{u_i(c_1) + v_i(c_1) + \delta W_i(p', x(p', b'))\} - \max_{x(p,b)} v_i(c_1) \qquad (1)$$

Solving the dynamic programming problem (1) allows us to determine recursively, an optimal sequence of consumptions $\mathbf{c}_i = (c_1, \ldots, c_t, \ldots)$. We say that $\mathbf{c}_i$ is optimal for (u_i, v_i, δ) at prices p, wealth b if $\mathbf{c}_i$ can be obtained from a solution to (1). We use $\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_n)$ to denote an allocation for the economy.

Definition: *The pair $(p, \mathbf{c})$ is an equilibrium for the economy $((u_i, v_i, \delta), \omega_i)_{i=1}^n$ if for all i, $\mathbf{c}_i$ is optimal for $((u_i, v_i, \delta), \omega_i)$ at prices p, wealth $p\omega_i$ and $\sum_{i=1}^n c_i = \sum_{i=1}^n \omega_i$.*

Proposition 1: *Every economy satisfying Assumptions 1 and 2 has an equilibrium.*

Proof: see Appendix.

We now examine the welfare properties of equilibria. Recall that a consumer's utility depends not only on her consumption but also on the possible consumption vectors each period. Therefore, the definition of admissible interventions for a social planner and hence Pareto efficiency, must specify not only the feasible allocations of consumption but also the ways in which the social planner can restrict the set of feasible choices for consumers. If, for example, the social planner can impose arbitrary restrictions on choice sets then he can improve welfare simply by restricting the consumers' choice sets to the singleton set containing only the equilibrium allocation. We assume that the consumers' choice problems must be of the form $x(p, b)$, that is, the planner may reallocate resources or change prices but cannot put additional restrictions on the feasible choices. Further, the allocation must be incentive compatible, that is, each consumer maximizes her utility on the set of feasible choices. Hence, we permit a somewhat limited set of interventions for the planner which leads to a weak notion of Pareto efficiency. We will show that competitive equilibria with a representative DSC agent may fail even this weak notion of efficiency.

Definition: *The pair $(p, \mathbf{c})$ is admissible if for all i, $\mathbf{c}_i$ is optimal at prices p, wealth $p\mathbf{c}_i$ and $\sum_{i=1}^n \mathbf{c}_i \leq \sum_{i=1}^n \omega_i$. An admissible pair $(p, \mathbf{c})$ is Pareto optimal if there is no admissible $(p', \mathbf{c}')$ with $W_i(x(p', p'\mathbf{c}_i')) > W_i(x(p, p\mathbf{c}_i))$ for all i.*

The following example demonstrates that a competitive equilibrium may not be Pareto optimal. In this example, there is a unique equilibrium allocation but multiple equilibrium prices. Only good 1 is tempting, that is, v only depends on good 1. The competitive equilibrium with the highest price of good 1 Pareto dominates all other equilibria because it has the lowest cost of self-control. Hence, in this example equilibria are Pareto ranked.

Example: Consider the economy with two physical goods and a representative household. Preferences are $u(c_{t1}, c_{t2}) = \min\{c_{t1}, c_{t2}\}$ and $v(c_{t1}, c_{t2}) = c_{t1}$, $\delta \in [0, 1)$. The endowment is $(1, 1)$ in every period. Then, for every $\alpha \in [0, 1)$ and $\beta = \frac{\delta(1-\alpha)}{2}$, the price vector

$$(p_{t1}, p_{t2}) = \left(\beta^{t-1}, \alpha\beta^{t-1}\right)$$

is an equilibrium price. Therefore, the individual's equilibrium wealth (in terms of $c_{1,1}$) is $\frac{2+2\alpha}{2-\delta(1-\alpha)} \in [1, 2)$. The equilibrium utility is $\frac{1}{1-\delta}\left[2 - \frac{2+2\alpha}{2-\delta(1-\alpha)}\right]$ which is decreasing in α. Hence, equilibria are Pareto ranked: the lower the price of good 2, the better off the individual.

Observe that our definition of Pareto efficiency does not allow the planner to provide the consumers with commitment opportunities. Nevertheless, competitive equilibria are not Pareto efficient.

5.3 Time Preference and Steady States

DSC preferences have the feature that the rate of time preference depends on future wealth. For an economy with one physical good and a differentiable utility function given by $\sum \delta^{t-1} u(c_t)$, the rate of time preference is defined by $\frac{u'(c_t)}{\delta u'(c_{t+1})} - 1$. To define an analogous notion for DSC preferences note that when the consumer transfers resources from the current period t to period $t+1$ she affects consumption in period t, consumption in period $t+1$ and the maximally feasible consumption in period $t+1$. Let $\bar{c}_t$ be the maximally feasible consumption in period t. Then, the rate of time preference for the DSC preferences (u, v, δ) is

$$\frac{u'(c_t) + v'(c_t)}{\delta(u'(c_{t+1}) + v'(c_{t+1}) - v'(\bar{c}_{t+1}))} - 1$$

where $\bar{c}_{t+1}$ depends on the consumers wealth in period $t+1$.

DSC preferences have similar implications for time preference as the utility functions introduced by Uzawa (1968) and Epstein and Hynes (1983). In their model, the marginal rate of substitution between periods t and $t+1$ may depend on the consumption in all periods $t' \geq t$ and hence implicitly on the wealth in period $t+1$.

Let $L = 1$ and assume u, v are continuously differentiable functions satisfying Assumption 1. Fix $\delta \in (0,1)$ and $\lambda_i > 0$ for $i = 1, \ldots, n$. Let $\mathbf{E}$ denote the set of all economies $\mathcal{E}$ with aggregate endowment $\bar{\omega}$ in every period, individual endowments satisfying Assumption 2 and consumer preferences given by $(u, \lambda_i v, \delta)_{i=1}^n$. Thus, the economies $\mathcal{E}, \mathcal{E}' \in \mathbf{E}$ differ only in the distribution of initial endowments. By Corollary 7, consumers have the same preference for commitment and can be ranked according to their self-control: if $\lambda_i > \lambda_j$ then j has more self-control than i.

The pair $(p, \mathbf{c})$ is an equilibrium of $\mathbf{E}$ if $(p, \mathbf{c})$ is an equilibrium for some $\mathcal{E} \in \mathbf{E}$. The pair $(p, \mathbf{c})$ is a steady state equilibrium of $\mathbf{E}$ if $(p, \mathbf{c})$ is an equilibrium of $\mathbf{E}$, and if for all t, t', $\mathbf{c}_{it} = \mathbf{c}_{it'}$ and $\frac{p_t}{p_{t+1}} = \frac{p_{t'}}{p_{t'+1}}$.

For consumers with standard concave utility functions of the form $\sum \delta^{t-1} u_i$, every vector of constant consumption paths $(\mathbf{c}_1, ..., \mathbf{c}_n)$ such that $\sum_i \mathbf{c}_{it} = \bar{\omega}$ together with the prices $p_t = \delta^{t-1}$ is a steady state equilibrium. Thus, in a standard economy the aggregate endowment and preferences alone cannot determine the equilibrium distribution of wealth and consumption. By contrast, in an exchange economy with self-control preferences there is a unique steady state equilibrium. Thus, aggregate endowment and the distribution of preferences uniquely determine the steady state distribution of wealth.

Proposition 2: *Suppose u and v are differentiable and $\lim_{c_t \to 0}(u'(c_t) + v'(c_t)) \to \infty$. Then $\mathbf{E}$ has a unique steady state equilibrium $(p, \mathbf{c})$. Moreover, $\mathbf{c}_{it} > \mathbf{c}_{jt}$ iff $\lambda_i < \lambda_j$, that is, agent i consumes more than agent j if and only if i has more self-control than j.*

Proof: See Appendix.

To illustrate the steady state equilibrium for a simple example, let $v(x) = x$ and $u(x) = \ln x$.[4] Setting $\gamma_i = 1/\lambda_i$, the steady state satisfies

$$\mathbf{c}_{it} = \frac{\bar{\omega}\gamma_i}{\sum_i \gamma_i}$$

and

$$\frac{p_t}{p_{t+1}} = \frac{\bar{\omega} + \sum_i \gamma_i}{\delta \sum_i \gamma_i}$$

[4] To ensure that the economy satisfies Assumption 1, C and the λ_i's have to be chosen appropriately.

Hence, a mean preserving spread in the γ_i's will leave the interest rate unchanged but result in greater inequality in the steady state. On the other hand, an increase in $\bar{\omega}$, the aggregate endowment, will leave dispersion unchanged but lead to an increase in the steady state interest rate.

5.4 Stochastic, Representative Agent Economy:

In this subsection, we analyze a simple example of a stochastic representative agent economy (Lucas (1978)). There is one productive asset which yields a dividend in each period. Let d denote the random variable that describes dividends and let D denote a realization. We assume that d is identically and independently distributed across periods, with mean 1, variance σ^2 and support $[D_l, D_h], 0 < D_l < D_h$.

We will characterize the price of this and other assets in a stationary competitive equilibrium. Such an equilibrium is characterized by a price function $p : [D_l, D_h] \to \mathbb{R}_+$ that describes the price of the productive asset as a function of the current period dividend.

Let s denote the asset holdings of the representative consumer. The consumer observes the realization of dividends in the current period and chooses a level of consumption $c_1 \in C = [0, k]$, where $D_h \leq k$. This consumption implies an asset holding for the current period and a distribution over continuation problems. Formally, the consumer's choice problem $y(p, s, D)$ is defined recursively as follows:

$$y(p, s, D) = \left\{ (c_1, y(p, s', d)) \mid s' = s + \frac{Ds - c_1}{p(D)},\ c_1 \in C \right\}$$

where $(c_1, y(p, s', d))$ denotes the measure that yields the current consumption c_1 with certainty and the distribution of continuation problems induced by the random variable d.

The consumer's utility function is (u, v, δ). The consumer chooses consumption (and the corresponding distribution over continuation problems) to maximize

$$W(y(p, s, D)) = \max_{y(p,s,D)} E\left\{ u(c_1) + v(c_1) + \delta W(y(p, s', d)) \right\} - \max_{y(p,s,D)} v(c_1) \tag{2}$$

We assume that u and v are continuously differentiable, $w := u + v$ is strictly concave, and increasing, u is bounded above and v is convex. Under these assumptions, the dynamic programming problem (2) has a unique solution (see Stokey and Lucas, Theorem 9.8.). In a competitive equilibrium $s_t = 1$ and $c_t = D_t$. Substituting this into the necessary first order conditions for an optimal solution to (2) yields

$$w'(D)p(D) = \delta E\left\{\left(w'(d) - v'(p(d) + d)\right)(p(d) + d)\right\}$$

Solving for the equilibrium price function we get

$$p(D) = \frac{\kappa}{w'(D)}$$

where $\kappa := \delta E\{(w'(d) - v'(p(d) + d))(p(d) + d)\}$.

Consider the following special case: $w(c_t) = \ln c_t$, $v(c_t) = \lambda c_t^2/2$ with $\lambda < 1/D_h^2$. Note that by Corollary 1 an increase in λ in this example corresponds to an increase in the preference for commitment while keeping the agent's self-control constant. For this example we compare the equilibrium prices of two assets that are traded in period t, pay dividend in period $t+1$ and pay no dividends thereafter. The first asset is a risk-free asset that pays 1 unit of consumption with certainty in the next period and the second asset is the "market" asset that pays the random variable d in the next period. Let p_f, p_m denote the equilibrium prices of the risk-free and the market assets. Straightforward calculations show that

$$p_f(D) - p_m(D) = p_f(D)\frac{h - 1 + \lambda(\kappa + 1)\sigma^2}{h - \lambda}$$

where $h := E\left(\frac{1}{d}\right)$. The standard equity premium for this type of economy (i.e. when $\lambda = 0$) is $p_f(D)(h-1)/h$. When $\lambda > 0$ the equity premium is higher due to the cost of self-control. Moreover, $\lambda(\kappa + 1)$ is increasing in λ and hence the equity premium per dollars worth of the risk-free asset, is increasing in λ. Thus, increasing the representative agents preference for commitment while keeping her self-control constant, increases the equity premium.

Just as standard agents, DSC consumers care about uncertainty in their consumption. In addition, DSC consumers also care about uncertainty in the maximal feasible consumption. This latter uncertainty affects the future cost of self-control. In our example, self-control adds to the equity premium because temptation is risk loving. This "urge to gamble" makes the agent averse to situations where the maximal feasible consumption next period is uncertain.

5.5 Borrowing constraints

In this section, we consider the following example of a deterministic exchange economy with one physical good. There are two consumers with identical preferences $(u, \lambda v, \delta)$ where $u(c_t) + v(c_t) = c, v(c_t) = \lambda c_t^2$. The aggregate endowment is 3 every period and

$$\omega_{1t} = 3 - \omega_{2t} = \begin{cases} 2 & \text{if } t \text{ odd} \\ 1 & \text{if } t \text{ even} \end{cases}$$

By Proposition 2, this economy has a unique steady state equilibrium. Moreover, in a steady state both agents must consume $\frac{3}{2}$ in every period.

Now assume that consumers face a borrowing constraint. In particular, the maximum amount the agent can borrow is next period's endowment. Let s_i denote agent i's savings at the start of the current period. Hence, $s_i < 0$ means that the agent i owes $-s_i$. We define the choice problem faced by agents in this economy as follows:

$$z(p, \omega_i, s_i) = \left\{ (c_1, z(p', \omega_i', s_i')) \mid s_i' = \frac{(\omega_{i1} + s_i - c_1)p_1}{p_2}, \; -s_i' \leq \omega_{i2}, \; c_1 \in C \right\}$$

where $p' = (p_2, \ldots, p_t, \ldots) \in R_+^\infty$, $\omega_i' = (\omega_{i2}, \ldots, \omega_{it}, \ldots) \in C^\infty$.

Therefore, the consumer's problem is given by

$$W_i(z(p, \omega_i, s_i)) = \max_{z(p,\omega_i,s_i)} \{c_1 + \delta W_i(z(p', \omega_i', s_i'))\} - \max_{z(p,\omega_i,s_i)} \lambda c_1^2$$

For $\frac{p_t}{p_{t+1}} = \frac{1}{\delta(1-3\lambda)}$ for all t, the optimal solution to the maximization problem above is for both agents to set $s_i = 0$ every period. Hence, we have an equilibrium in which agents choose not to smooth income, even though the borrowing constraint is not binding. The interest rate $r = \frac{p_t}{p_{t+1}} - 1$ in the above equilibrium with the borrowing constraint is lower than the steady state interest rate when there is no borrowing constraint. Hence, removing the non-binding constraint raises interest rates and results in borrowing.

In the equilibrium above, consumption tracks income because borrowing opportunities are much greater in low income periods than in high income periods. The individual refrains from shifting funds to low income periods to avoid increasing the cost of self-control. With a different borrowing constraint one could obtain the opposite result: that is, consumption could be larger in periods where the individual has low endowment. For example, if the individual may borrow 1 unit independent of his future endowment, then equilibrium consumption in periods of high endowment could be lower than in periods of low endowment.

5.6 Sustainable Debt

This subsection examines incentive compatible debt contracts for an individual with DSC preferences. We consider an environment where the only punishment for default is exclusion from future borrowing. This restriction on feasible punishments after default is particularly relevant in the case of sovereign debt. After default, the agent may still save funds at the market interest rate. With standard preferences (i.e. no preference for commitment), Bulow and Rogoff (1989) show that in this environment there is no incentive compatible contract that allows the individual to borrow.

We assume that there are no investment opportunities that offer commitment. Thus, the consumer always has the option of exchanging her savings for current consumption. This assumption may be justified by allowing *collateralized* loans after default. To see how this works, suppose that in period 1 the consumer invests in a contract that offers a return of 1 unit of consumption in period 3. In period 2 the consumer is able to use that contract as collateral for a loan on current consumption. Under this hypothesis there are incentive compatible contracts that allow individuals with DSC preferences to borrow. In the following we provide an example of such a contract.

Let $\omega = (0, 0, 4, 0, 0, 4, 0, 0, 4 \ldots)$ be the agents endowment. Her utility function is (u, v, δ) with $v(c_t) = \lambda c_t$. The agent borrows and lends at a fixed interest rate r.

A generic debt contract is denoted $(\beta_1, \beta_2, \beta_3)$ with the understanding that β_j is the required outstanding balance at the end of any period $3t + j$. If, at the end of any period, the agent's balance is not at the required level, then she is excluded from borrowing in all future periods. The individual may always invest funds at a rate $r > 0$ - even after default.

The borrowing constraints $(\beta_1, \beta_2, \beta_3)$ with $\beta_1 \geq 0, \beta_2 \geq 0, \beta_3 \geq 0$ are incentive compatible if for any feasible debt level in period t, there is an optimal plan in which the consumer does not default. Simple calculations establish

$$\beta_1 = (1+r)\beta_3 + c_{3n+1}$$

$$\beta_2 = (1+r)\beta_1 + c_{3n+2}$$

$$\beta_3 = (1+r)\beta_2 + c_{3n} - 4$$

for $n = 1, 2, \ldots$Since consumption is nonnegative, the above equations imply that $\beta_2 \geq \beta_1 \geq \beta_1$. Hence, the maximal level of debt is β_2 and borrowing occurs if and only if $\beta_2 > 0$. Finally, the above three equations imply

$$\frac{c_{3n+1}}{(1+r)^2} + \frac{c_{3n+2}}{(1+r)} + c_{3n} = 4 + [(1+r) - \frac{1}{(1+r)^2}]\beta_2 \qquad (*)$$

for all $n \geq 1$. First, consider the case where $\lambda = 0$. The agent has outstanding debt $\beta_2(1+r)$ at the beginning of period 3. If $\beta_2 > 0$ then she has an incentive to default. To see this, suppose, instead of repaying the debt the individual "deposits" $\frac{c_{3n+1}}{1+r} + \frac{c_{3n+2}}{(1+r)^2}$ into a savings account. By $(*)$ this is feasible and yields strictly more than c_3 units of consumption for period 3 whenever $\beta_2 > 0$. This argument is a special case of the argument given by Bulow and Rogoff (1989) to demonstrate that without direct penalties there can be no borrowing without collateral.

By constrast, for $\lambda > 0$ incentive compatible borrowing is possible. Suppose

$$u(c_t) = \begin{cases} 2c_t & \text{if } c_t \leq 1 \\ 1 + c_t & \text{if } c_t > 1 \end{cases}$$

Assume that δ is close enough to 1, so that $\alpha := \frac{1-\delta^2}{\delta^3} + \frac{1-\delta^4}{\delta^4} < \frac{2\delta^2-1}{2\delta^2+1}$. Let $\underline{\lambda} := \frac{\alpha}{1-\alpha}$ and $\bar{\lambda} := \frac{2\delta^2-1}{1+\delta}$.

Proposition 3: *If $\lambda \in (\underline{\lambda}, \bar{\lambda})$ then for all $r \leq \frac{1-\delta}{\delta}$, the debt contract $\beta_1 = 1, \beta_2 = 2 + r, \beta_3 = 0$ is incentive compatible.*

Proof: First, consider the optimal program for an individual who cannot borrow starting in period 3 (the period when he will be tempted to default): Note that it cannot be optimal to consume more than 1 in periods other than $3t$. To see this recall that $(1+r)\delta \leq 1$ and consumption beyond 1 has a marginal utility of 1 in each period. Hence avoiding a self-control penalty makes it optimal to consume less than or equal to one in all periods but $3t$. On the other hand, consuming less than 1 in any period cannot be optimal: suppose that the individual consumes less than 1 in period 5. The marginal utility of consumption in period 5 is 2, therefore, at the margin, saving until period 5 will increase utility by at least $2\delta^2$. On the other hand, by doing so, the agent will at the margin, forego 1 util due to less consumption and $\lambda(1+\delta)$ utils of self-control costs in periods 3 and 4. Since $\lambda < \bar{\lambda}$, increasing period 5 consumption to 1 increases payoff. A similar argument holds for period 4 and hence the individual will consume exactly one unit in periods $3t+1, 3t+2, t \geq 1$. Therefore, the optimal utility starting from any period $3t$ is:

$$W_d^3 = \frac{1}{1-\delta^3}\left[5 - \frac{1+\lambda}{1+r} - \frac{1+\lambda}{(1+r)^2} + \left(2 - \frac{\lambda}{1+r}\right)\delta + 2\delta^2\right]$$

Now, consider the consumer who does not default. In period $3t+2$, the continuation utility of the plan is

$$W_p^3 = \frac{1}{1-\delta^3}\left[5 - (1+\lambda)(1+r) - (1+\lambda)(1+r)^2 + 2\delta + 2\delta^2\right]$$

Straightforward calculations, using the facts $r \leq \frac{1-\delta}{\delta}$ and $\lambda > \underline{\lambda}$ establish that $W_p^3 > W_d^3$. Therefore, the individual has no incentive to default in any period $3t+2$. But this is the period with the highest incentive to default. Hence, the debt contract is incentive compatible. □

The principle extends to other utility functions: the advantage of a borrowing program over a savings program is the reduction in self-control costs in periods $3t+1$ provided by the former. This saving results from the fact that $(1+r)\beta_1 < \beta_2$; an individual who enters period 5 with a debt $(1+r)\beta_1$ is extended additional credit equal to $\beta_2 - (1+r)\beta_1$. These funds are not available to the agent in period 4 and therefore she does not suffer the self-control costs associated with "transferring" them to period 5. Note that unlike savings, "credit-worthiness" is not an asset that can be used as collateral. Thus, the commitment offered by the debt contract cannot be undone in the open market.

6. Conclusion

The starting point of the literature on dynamic inconsistency is a non-recursive preference $\succeq_1$ over consumption streams $\{c_t\}_{t=1}^{\infty}$. Then, it is assumed that the preference $\succeq_\tau$ over consumption streams starting at time τ is the same as $\succeq_1$. Since $\succeq_1$ is not recursive this implies that the conditional preferences induced by $\succeq_1$ on consumption streams starting at date τ is not the same as $\succeq_\tau$. That is, there exists some $\{c_1, \ldots, c_{\tau-1}, c_\tau, , \ldots\}$ and $\{c_1, \ldots, c_{\tau-1}, \hat{c}_\tau, \ldots\}$ such that

$$\{c_1, \ldots, c_{\tau-1}, c_\tau, \ldots\} \succeq_1 \{c_1, \ldots, c_{\tau-1}, \hat{c}_\tau, \ldots\}$$
$$\text{and } \{\hat{c}_\tau, \ldots\} \succ_\tau \{c_\tau, \ldots\}$$

This "reversal" of preference is called dynamic inconsistency and the resulting preference for commitment is its significant behavioral implication.

In contrast, the approach of Gul and Pesendorfer (2000) and the current paper is to take a single preference, not over consumption, but over a class of choice problems and to permit a strict preference for a smaller set of options. In our approach, this preference for commitment does not arise from a change in preference but form a desire to avoid temptation. In this section, we focus on the connections between these two approaches. We refer to the first as the preference reversal approach and to ours as the preference for commitment approach.

The goal of any economic application is to relate parameters of preferences (demand elasticities, measures of risk aversion, etc.) to the chosen (random) consumption sequences in specific choice problems (utility maximization subject to budget constraints) and then to the equilibrium values of the parameters that define those choice problems (prices and wealth). The two approaches achieve this goal in different ways. In the preference reversal approach, it is postulated that at each τ the agent behaves in a manner that maximizes $\succeq_\tau$ given the predicted behavior of her subsequent selves. For finite horizon choice problems with a finite set of choice at each τ, this specification, together with a rule that describes how the agent resolves ties, establishes an unambiguous relationship between preferences parameters and predicted consumption paths. In other situations, technical and conceptual difficulties are dealt with in a game-theoretic manner. Hence, the agent at each time period

τ is treated as a different "player" and the predicted consumption paths are determined as the subgame perfect Nash outcomes of this game.

In the preference for commitment approach agents preference relation over a class of choice problems much larger than the structured class relevant for the particular application is taken as primitive. Thus, a very general "indirect utility function" that not only permits comparison of budget sets but also arbitrary compact sets is specified. Then, a revealed-preference criterion is used to relate this utility function over choice problems to the agent's choice over consumption plans (i.e., to determine the direct utility function).

A consequence of the preference reversal approach is that a given choice problem may not have a unique payoff associated with it. From a game-theoretic perspective, this is not surprising; we almost never expect all equilibria of a given game to yield the same payoff for a particular agent. In a multi-person context, subgame perfect Nash equilibrium is meant to capture a rest point of the player's expectations and strategizing. Then, the multiplicity of such equilibria is a reflection of the fact that no single person controls the underling forces that might lead to the particular rest point. Nevertheless, multiplicity has lead to some concerns. Various notions of renegotiation-proofness have emerged as an expression of these concerns.[5]

But this multiplicity is more difficult to understand within the context of a single person choice problem, even if the person in question is dynamically inconsistent. In that case, the game-theoretic argument for multiplicity loses much of its force since it should be straightforward to re-negotiate one's self out of an unattractive continuation equilibrium. And, foresight of this renegotiation would lead to the unraveling of the original plan. More generally, the notion of (subgame perfect) Nash equilibrium is a tool for the analysis of *non-cooperative* behavior, and its appropriateness often rests on the implicit assumption of "independent" behavior and absence of communication. Therefore, analyzing the interaction between the agent at time τ and her slightly modified self at time $\tau + 1$ as the Nash equilibrium of a game may not be appropriate.

Here, the distinction between finite and infinite horizon is important. In a finite problem, subgame perfection is little more than dynamic programming. In the final stage, the

[5] See Kotcherlakota (1996).

behavior is obvious, which makes the predicted behavior in the penultimate stage compelling and so forth. In an infinite horizon problem, even with perfect information, subgame perfect Nash equilibrium entails much more than correct anticipation of obvious behavior. In this case, the type of "bootstrapping" that is part and parcel of non-cooperative game theory is the essential ingredient of the prediction.[6]

Most of the work in this literature tends to ignore the observations offered by Kocherlakota (1996) and the unresolved difficulties associated with the interpreting the multiplicity of "equilibria" in a single-person problem. Within the preference reversal approach, the only formulation of self-control entails using this multiplicity to construct equilibria in which the decision-maker sustains a desirable plan by threatening *herself* with even less desirable behavior after a deviation.

As demonstrated by Theorem 2, multiplicity does not arise in the preference for commitment approach. Every (u, v, δ) corresponds to a unique preference over choice problems. Hence, all optimal plans yield the same payoff.

7. Appendix

7.1 Proof of Theorem 1

Definition: *Let X be a compact metric space. Let $z^n \in \mathcal{K}(X)$ for all n. The closed limit inferior of the sequence z^n (denoted $\underline{L}z^n$) is the set of all $x \in X$ such that $x = \lim x^n$ for some sequence such that $x^n \in z^n$ for every n. The closed limit superior of z^n (denoted $\bar{L}z^n$) is the set of all $x \in X$ such that $x = \lim x_{n_j}$ for some sequence such that $x_{n_j} \in z_{n_j}$ for every n_j. The set Lz^n is the pointwise limit of z^n if $Lz^n = \underline{L}z^n = \bar{L}z^n$.*

Lemma 1: *Let X be a compact metric space. The sequence $z^n \in \mathcal{K}(X)$ converges to z iff $z = Lz^n$.*

Proof: The lemma follows from exercise X in (Brown and Pearcy (1995), p.132) □

Lemma 2: *Let X, Y be compact metric spaces and $g : X \to Y$ a continuous function. Then, $\bar{g} : \mathcal{K}(X) \to \mathcal{K}(Y)$ defined by $\bar{g}(z) = \{g(x) \,|\, x \in z\}$ is also continuous.*

[6] For a related critique of the use of Nash equilibrium to model a (different) departure from fully rational behavior, see Piccione and Rubinstein (1997)

Proof: The lemma follows from exercise X in (Brown and Pearcy (1995), p.222). ☐

Definition: *Let* $\Upsilon_1 := M_1$ *and for* $t > 1$ *let* $\Upsilon_t := \Delta(C, \ldots, Z_{t-1})$. *The sequence of probability measures* $\{\hat{\mu}_t\} \in \times_{k=0}^{\infty} \Upsilon_t$ *is consistent if* $\text{marg}_{C,\ldots,Z_{t-1}} \hat{\mu}_{t+1} = \hat{\mu}_t$ *for all* $t \geq 1$. *Let* Υ^c *denote all the set of all consistent sequences in* $\times_{t=0}^{\infty} \Upsilon_t$.

Lemma 3: *For every* $\{\hat{\mu}_t\} \in \Upsilon^c$ *there exists a unique* $\mu \in \Delta(C \times Z^*)$ *such that* $\text{marg}_C \mu = \hat{\mu}_1$ *and* $\text{marg}_{C,\ldots,Z_t} \mu = \hat{\mu}_t$ *for all* $t \geq 1$. *The mapping* $\psi : \Upsilon^c \to \Delta(C \times Z^*)$ *that associates this* μ *with the corresponding* $\{\hat{\mu}_t\}$ *is a homeomorphism.*

Proof: The first assertion is Kolmogorov's Existence Theorem [Dellacherie and Meyer (1978), p. 73]. Since every compact space is complete and separable, the second assertion follows from Lemma 1 in Brandenburger and Dekel (1993).

Definition: Let $D_t := \{(z_1, \ldots, z_t) \in \times_{n=1}^{t} Z_t \,|\, z_k = F_k(z_{k+1}) \, \forall k = 1, \ldots, t-1\}$, $M^d := \{\{\mu_t\} \in \times_{t=1}^{\infty} \Delta(C \times Z_{t-1}) \,|\, F_t(\mu_{t+1}) = \mu_t \, \forall t \geq 1\}$ and $\Upsilon^d := \{\{\hat{\mu}_t\} \in \Upsilon^c \,|\, \hat{\mu}_{t+1}(C \times D_t) = 1 \forall t \geq 1\}$.

Lemma 4: *For every* $\{\mu_t\} \in M^d$ *there exists a unique* $\{\hat{\mu}_t\} \in \Upsilon^d$ *such that* $\hat{\mu}_1 = \mu_1$ *and* $\text{marg}_{C,Z_{t-1}} \hat{\mu}_t = \mu_t$ *for all* $t \geq 2$. *The mapping* $\phi : M^d \to \Upsilon^d$ *that associates this* $\{\mu_t\}$ *with the corresponding* $\{\hat{\mu}_t\}$ *is a homeomorphism.*

Proof: Let m_0 be the identity function on C and let m_1 be the identity function on $C \times Z_1$. For $t \geq 2$, define $m_t : C \times Z_t \to \times_{k=0}^{t} Z_k$ as follows: $m_t(c, z_t) = (\hat{c}, \ldots \hat{z}_t)$ iff $\hat{c} = c, \hat{z}_t = z_t$ and $\hat{z}_{k-1} = F_{k-1}(\hat{z}_k)$ for all $k = 2, \ldots, t$. Note that m_t is one-to-one for all t. Also, $m_t(C \times Z_t) = C \times D_t$. Let π_0 and π_1 be the identity mappings on C and $C \times Z_1$ respectively. For $t \geq 2$, let $\pi_t(c, \ldots, z_t) = (c, z_t)$ for all $(c, \ldots, z_t) \in C \times D_t$. Clearly, π_t is continuous for all t. Since $C \times D_t$ is a compact set π_t can be extended to all of $\times_{k=0}^{t} Z_k$ in a continuous manner. Hence, π_t is continuous function on $\times_{k=0}^{t} Z_k$ and its restriction to $C \times D_t$ is the inverse of $m_t : C \times Z_t \to m_t(C \times Z_t)$. Since F_t is continuous for all t, so is m_t.

Next, for any $\{\mu_t\} \in M^d$, define $\{\hat{\mu}_t\}$ by $\hat{\mu}_t(A) := \mu_t(m^{-1}(A))$ for every $A \in \mathcal{B}(\times_{k=0}^{t} Z_k)$. Clearly, $\{\hat{\mu}_t\}$ defined in this manner is the element in Υ^d such that $\hat{\mu}_1 = \mu_1$ and $\text{marg}_{C,Z_{t-1}} \hat{\mu}_t = \mu_t$ for all $t \geq 1$. Define $\phi(\{\mu_t\})$ to be this unique $\{\hat{\mu}_t\}$ and note that ϕ is one-to-one. Pick any $\{\hat{\mu}_t\}$ in Υ^d. Define $\{\mu_t\}$ as follows $\mu_1 = \hat{\mu}_1$ and $\mu_t(A) := \hat{\mu}_t(\pi_{t-1}^{-1}(A))$ for all $A \in \mathcal{B}(C \times Z_{t-1})$. Note that $\phi(\{\mu_t\}) = \hat{\mu}_t$ hence, ϕ is a bijection. Observe that the t'th element of $\phi(\{\mu_t\})$ depends only on μ_t. Hence, without risk of confusion we write $\phi_t(\mu_t)$ to denote this element. Note that for any continuous real-valued function $\hat{h}$ on $\times_{k=0}^{t} Z_k$ and h on $C \times Z_t$, $\int \hat{h} d\phi_t(\mu_t) = \int \hat{h} \circ m_t d\mu_t$ and $\int h d\phi_t^{-1}(\hat{\mu}_t) = \int h \circ \pi_t d\hat{\mu}_t$. Hence, the continuity of ϕ and ϕ^{-1} follows from the continuity of m_t and π_t for all t. $\square$

Lemma 5: $\psi(\Upsilon^d) = \{\mu \in \Delta(C \times Z^*) | \mu(C \times Z) = 1\}$.

Proof: Let $\Gamma_t = C \times D_t \times_{k=t+1}^{\infty} Z_k$ for all $t \geq 1$ and $\mu = \psi(\{\hat{\mu}_t\})$. Observe that $\mu(\Gamma_t) = \hat{\mu}_t(C \times D_t) = 1 \forall t$ if $\{\hat{\mu}_t\} \in \Upsilon^d$. Hence $\mu(C \times Z) = \mu(\cap_{t \geq 1} \Gamma_t) = \lim \mu(\Gamma_t) = 1$. Conversely, if $\mu(C \times Z) = 1$ then $\mu(\Gamma_t) = 1 \forall t$ and hence there is a corresponding $\{\hat{\mu}_t\} \in \Upsilon^d$. $\square$

Lemma 6: *Let $\xi(z) := \{\{\mu_t\} \in M^d \,|\, \mu_t \in z_t \forall t \geq 1\}$. Then $\xi : Z \to \mathcal{K}(M^d)$ is a homeomorphism and $\{\mu_t\} \in \xi(z)$ iff $\mu_t \in z_t$ for all t.*

Proof:

Step 1: Let $z \in Z$, $\mu_t \in z_t$. Then, there exists $\{\nu_t\} \in \xi(z)$ such that $\nu_t = \mu_t$.

Proof of Step 1: Let $\nu_t = \mu_t$. For $k = 1, \ldots, t-1$, define ν_{t-k} inductively as $\nu_{t-k} := F_{t-k+1}(\mu_{t-k+1})$. Similarly, define ν_{t+k}, for $k \geq 1$ inductively by picking any $\nu_{t+k} \in F_{t+k}^{-1}(x_{t+k-1}) \cap z_{t+k}$. The $\{\nu_t\}$ constructed in this fashion has all of the desired properties. $\square$

By Step 1, $\xi(z) \neq \emptyset$. To see that $\xi(z)$ is compact, take any sequence $\{\mu_t^n\} \in M^d$. We can use the diagonal method to find a subsequence $\{\mu_t^{n_j}\}$ such that $\mu_t^{n_j}$ converges to some μ_t for every t. Since each z_t is compact, $\mu_t \in z_t$ for all t and hence $\{\mu_t\}$ is in $\xi(z)$. Therefore $\xi(z)$ is compact. Suppose $z \neq z'$ for some $z, z' \in Z$. Without loss of generality, assume there exists some t such that $\mu_t \in z_t \backslash z_t'$ for some μ_t. By Step 1, we obtain $\{\nu_t\} \in \xi(z)$ such that $\nu_t = \mu_t$. Clearly, $\{\nu_t\} \in \xi(z) \backslash \xi(z')$. Therefore, ξ is one-to-one. Take any $\bar{z} \in \mathcal{K}(M^d)$. Define $z_t := \{\mu_t \,|\, x_t = \mu_t$ for some $x \in \bar{z}\}$. Let $z = (z_1, z_2, \ldots)$. Then, $z \in Z$ and $\xi(z) = \bar{z}$. So, ξ is onto.

To prove that ξ is continuous, let z^n converge to z. By Lemma 1, this is equivalent to $Lz_t^n = z_t$ for all t. We need to show that $\xi(z^n)$ converges to $\xi(z)$. Take any convergent sequence $\{\nu_t^n\}$ such that $\{\nu_t^n\} \in \xi(z^n)$ for all n. Then, $\lim \nu_t^n \in z_t$ for all t and therefore

$\lim\{\nu_t^n\} \in \xi(z)$. Let $\{\nu_t\} \in \xi(z)$. Since $Lz_t^n = z_t$, there exists μ_t^n converging to ν_t such that $\mu_t^n \in z_t^n$ for all n. By step 1, for each μ_t^n we can construct $\{\nu_t^n\}(t) \in M^d$ such that $\nu_t^n(t) = \mu_t^n$. Since F_t is continuous for all t, $\nu_k^n(t)$ converges to ν_k for all $k \leq t$. Consequently, $\{\nu_t^n\}(n)$ converges to $\{\nu_t\}$. Hence, $L\xi(z^n) = \xi(z)$. Again by Lemma 1, this implies $\xi(z^n)$ converges to $\xi(z)$ and hence ξ is continuous.

To see that ξ^{-1} is continuous, assume $L\bar{z}^n = \bar{z}$, $\xi(z^n) = \bar{z}^n$ for all n and $\bar{z} = \xi(z)$. We need to show that $Lz_t^n = z_t$ for $z^n = \xi^{-1}(\bar{z}^n)$ and $z = \xi^{-1}(\bar{z})$. For any t, let μ_t^n be any convergent sequence in $\Delta(C \times Z_{t-1})$ such that $\mu_t^n \in z_t^n$ for all n. By Step 1, there exists $\{\nu_t^n\} \in \bar{z}^n$ such that $\nu_t^n = \mu_t^n$ for all n. Take any convergent subseqence of $\{\nu_t^n\}$ and let $\{\nu_t\} \in \bar{z}$ be the limit of this subsequence. Hence, $\lim \mu_t^n = \nu_t \in z_t$. Next, take any $\mu_t \in z_t$. Again, by Step 1, there exists $\{\nu_t\} \in \bar{z}$ such that $\nu_t = \mu_t$. Since, $L\bar{z}^n = \bar{z}$, there exists $\{\nu_t^n\} \in \bar{z}^n$ such that $\{\nu_t^n\}$ converges to $\{\nu_t\}$. Hence, $\nu_t^n \in z_t^n$ converges to μ_t. Therefore, $L\bar{z}_t^n = \bar{z}_t$ for all t and hence $L\bar{z}^n = \bar{z}$ as desired. □

To conclude the proof of Theorem 1, note that by Lemmas $3-5$, $\psi \circ \phi$ is a homeomorphism from M^d to $\Delta(C \times Z)$. Hence, by Lemma 2, the function $\zeta : \mathcal{K}(M^d) \to \mathcal{K}(\Delta(C \times Z))$ defined by $\zeta(A) := \psi \circ \phi(A)$ for all $A \in \mathcal{K}(M^d)$ is also a homeomorphism. Then, by Lemma 6, $\zeta \circ \xi$ is the desired homeomorphism from Z to $\mathcal{K}(\Delta(C \times Z))$. □

7.2 Proof of Theorem 3

Lemma 7 (A Fixed-Point Theorem): *If B is a closed subset of a Banach space with norm $\|\cdot\|$ and $T : B \to B$ is a contraction mapping (i.e., for some integer m and scalar $\alpha \in (0,1)$, $\|T^m(W) - T^m(W')\| \leq \alpha \|W - W'\|$ for all $W, W' \in B$), then there is a unique $W^* \in B$ such that $T(W^*) = W^*$.*

Proof: See [Bertsekas and Shreve (1978), p. 55] who note that the theorem in Ortega and Rheinholt (1970) can be generalized to Banach spaces.

Lemma 8: *Let $u : C \to I\!R$, $v : C \to I\!R$ be continuous and $\delta \in (0,1)$. There is a unique continuous function $W : Z \to I\!R$ such that*

$$W(z) = \max_{\mu \in z} \left\{ \int (u(c) + v(c) + \delta W(z'))\, d\mu(c, z') \right\} - \max_{\nu \in z} \int v(c) d\nu(c, z') \qquad (1)$$

for all $z \in Z$.

Proof: Let $\mathcal{W}_b$ be the Banach space of all real-valued, bounded functions on Z (endowed with the sup norm). The operator $T : \mathcal{W}_b \to \mathcal{W}_b$, where

$$TW(z) = \max_{\mu \in z} \left\{ \int (u(c) + v(c) + \delta W(z')) \, d\mu(c, z') \right\} - \max_{\nu \in z} \int v(c) d\nu(c, z')$$

is well-defined and is a contraction mapping. Hence, by Lemma 7, there exists a unique W such that $T(W) = W$. Hence, W satisfies (1). To prove that W is continuous, repeat the above argument for the subspace $\mathcal{W}_c \subset \mathcal{W}_b$ of all continuous, real valued functions on $\mathcal{K}$. Note that $T(\mathcal{W}_b) \subset \mathcal{W}_b$. Hence, again by Lemma 7, T has a fixed point $W^* \in \mathcal{W}_c$. Since W is the unique fixed-point of T in $\mathcal{W}_b$, we have $W = W^* \in \mathcal{W}_c$. □

By Lemma 8, for any continuous u, v, δ, there exists a unique continuous W that satisfies (1). It is straightforward to verify that Axioms $1 - 8$ hold for any binary relation represented by a continuous function W that satisfies (1).

In the remainder of the proof we show that if $\succeq$ is nondegenerate and satisfies Axioms $1 - 8$ then the desired representation exists. It is easy to show that if $\succeq$ satisfies Axioms 4, 6 and 7 then it also satisfies the following stronger version of the independence axiom:

Axiom 4^***:** $x \succ y$, $\alpha \in (0, 1)$ *implies* $\alpha x + (1 - \alpha) z \succ \alpha y + (1 - \alpha) z$.

Theorem 1 of Gul and Pesendorfer (2000) establishes that $\succeq$ satisfies Axioms 1-3 and 4^* if and only if there exist linear and continuous functions U, V such that the function W defined by

$$W(z) := \max_{\mu \in z} \{U(\mu) + V(\mu)\} - \max_{\nu \in z} V(\nu) \tag{2}$$

represents $\succeq$. To complete the proof we will show that there exists continuous functions u, v and $\delta \in (0, 1)$ such that for all $z \in Z$,

$$W(z) = \max_{\mu \in z} \int (u(c) + v(c) + \delta W(z')) \, d\mu(c, z') - \max_{\nu \in z} \int v(c) d\nu(c, z')$$

$$U(\mu) = \int u(c) + \delta W(x) d\mu$$

$$V(\mu) = \int v(c) d\mu^1$$

Lemma 9: *There exists a continuous* $u : C \to I\!R$, $\delta \in (0,1), \gamma \in I\!R$ *such that* $U(\nu) = \int (u(c) + \delta W(z))d\nu(c,z) + \gamma$ *for all* $\nu \in \Delta(C \times Z)$.

Proof:

Step 1: There are continuous $u : C \to I\!R$, $\bar{W} : Z \to I\!R$ such that $U(\nu) = \int (u(c) + \bar{W}(z))d\nu(c,z)$ for all $\nu \in \Delta$.

Proof: Since U is linear and continuous there exists a continuous $\bar{u} : C \times Z \to I\!R$ such that $U(\mu) = \int \bar{u}(c,z)d\mu(c,z)$. By Axiom 5, $U(.5(\bar{c},\bar{z}) + .5(c,z)) = U(.5(\bar{c},z) + .5(c,\bar{z}))$. Therefore,

$$\bar{u}(c,z) = \bar{u}(\bar{c},z) - \bar{u}(\bar{c},\bar{z}) + \bar{u}(c,\bar{z}) \tag{3}$$

Then,

$$\begin{aligned} U(\nu) &= \int \bar{u}(c,z)d\nu(c,z) \\ &= \int \bar{u}(\bar{c},z)d\nu(c,z) - \int \bar{u}(\bar{c},\bar{z})d\nu(c,z) + \int \bar{u}(c,\bar{z})d\nu(c,z) \end{aligned}$$

Setting $u := \bar{u}(\cdot,\bar{z}) - \bar{u}(\bar{c},\bar{z})$ and $\bar{W} := \bar{u}(\bar{c},\cdot)$ gives the desired result.

Step 2: There exists some $\delta > 0, \gamma \in I\!R$ such that $\bar{W}(z) = \delta W(z) + \gamma$ for all $z \in Z$.

Proof: Define $K := \max_Z W(z), k := \min_Z W(z), \bar{K} := \max_Z \bar{W}(z), \bar{k} := \min_Z \bar{W}(z)$. Since U is not constant, it follows from (2) that W is not constant. Axioms 6 implies that $\bar{W}(x) \geq \bar{W}(y)$ iff $W(x) \geq W(y)$. By non-degeneracy $W, \bar{W}$ are not constant. Hence $\bar{K} > \bar{k}, K > k$. To establish the desired conclusion we will show that

$$\bar{W}(z) := \frac{\bar{K}k - K\bar{k}}{(K-k)(\bar{K}-\bar{k})} + \frac{\bar{K}-\bar{k}}{K-k}W(z) \tag{4}$$

for all $z \in Z$. For any $z \in Z$ there exists a unique α such that

$$\bar{W}(z) = \alpha\bar{K} + (1-\alpha)\bar{k} \tag{5}$$

Let z^* maximize $\bar{W}$ and z_* minimize it. By (2), the linearity of U and the that $\bar{W}(x) \geq \bar{W}(y)$ iff $W(x) \geq W(y)$,

$$W(\{(\bar{c},z)\}) = W(\{\alpha(\bar{c},z^*) + (1-\alpha)(\bar{c},z_*)\})$$

Applying Axiom 7 yields

$$W(\{(\bar{c}, z)\}) = W(\{(\bar{c}, \alpha z^* + (1-\alpha)z_*)\})$$

Apply Axiom 6 to get

$$W(z) = W(\alpha z^* + (1-\alpha)z_*)$$

Linearity of W together with the fact that $\bar{W}(x) \geq \bar{W}(y)$ iff $W(x) \geq W(y)$ implies

$$W(z) = \alpha K + (1-\alpha)k \tag{6}$$

Solving (6) for α, substituting the result into (5) and re-arranging terms then yields (4) and proves step 2.

Step 3: $\delta < 1$ in the representation of Step 2.

Proof: Let z^c be the unique $z \in Z$ with the property that $z^c = \{(c, z^c)\}$. Let z be such that $W(z) \neq W(z^c)$. Let $y^1 = \{(c, z)\}$ and define y^n inductively as $y^n = \{(c, y^{n-1})\}$. Then y^n converges to z^c. Hence, by continuity $W(y^n) - W(z^c)$ must converge to zero. But

$$W(y^n) - W(z^c) = \delta^n (W(z) - W(z^c))$$

Since $W(z) - W(z^c) \neq 0$ it follows that $\delta < 1$. □

Let $U' = U - \frac{\gamma}{1-\delta}$ and $W' = W - \frac{\gamma}{1-\delta}$. Then, W', U' are continuous and linear with

$$W'(z) := \max_{\mu \in z}\{U'(\mu) + V(\mu)\} - \max_{\nu \in z} V(\nu)$$

Moreover, W' represents $\succeq$ and

$$U'(\nu) = \int (u(c) + \delta W'(z)) d\nu(c, z)$$

Therefore, without loss of generality, we can set $\gamma = 0$ in Lemma 9.

Lemma 10: *Assume there exists $\bar{\mu}$ and $\underline{\mu}$ such that $U(\bar{\mu}) + V(\bar{\mu}) - U(\underline{\mu}) - V(\underline{\mu}) > 0 > V(\bar{\mu}) - V(\underline{\mu})$. Then, there is a continuous linear $v : \Delta(C) \to I\!R$ such that $V(\nu) = v(\nu^1)$ for all $\nu \in \Delta$.*

Proof: Fix $\mu^2 \in \Delta(Z)$ and define $v : \Delta(C) \to I\!R$ by

$$v(\mu^1) := V(\mu^1 \times \mu^2)$$

Take any $\nu \in \Delta$ and let μ be the product measure $\nu^1 \times \mu^2$. By continuity, there exists $\alpha > 0$ small enough so that

$$U(\alpha\nu + (1-\alpha)\underline{\mu}) + V(\alpha\nu + (1-\alpha)\underline{\mu}) < U(\bar{\mu}) + V(\bar{\mu})$$
$$U(\alpha\mu + (1-\alpha)\underline{\mu}) + V(\alpha\mu + (1-\alpha)\underline{\mu}) < U(\bar{\mu}) + V(\bar{\mu})$$
$$V(\alpha\nu + (1-\alpha)\underline{\mu}) > V(\bar{\mu})$$
$$V(\alpha\mu + (1-\alpha)\underline{\mu}) > V(\bar{\mu})$$

Axiom 8 implies that $W(\{\alpha\nu + (1-\alpha)\underline{\mu}, \bar{\mu}\}) = W(\{\alpha\mu + (1-\alpha)\underline{\mu}, \bar{\mu}\})$. From the representation ((2)) it now follows that $V(\alpha\nu + (1-\alpha)\underline{\mu}) = V(\alpha\mu + (1-\alpha)\underline{\mu})$. Since V is linear, we have $V(\nu) = V(\mu) = v(\nu^1)$ as desired. □

To complete to proof, we show that the conclusion of Lemma 10 holds in all cases. By non-degeneracy U is not constant. If V is constant the conclusion of Lemma 10 holds trivially. So, we assume that neither U, nor V is constant.

Suppose $V = \alpha U + \beta$ for some $\alpha, \beta \in I\!R$. Since V is not constant, by non-degeneracy, this implies $0 \neq \alpha >_1$. If $\alpha > 0$, replace V with $V' = 0$ and U with $U' = U + V$. Then, $W(z) := \max_{\mu \in z}\{U'(\mu) + V'(\mu)\} - \max_{\nu \in z} V'(\nu)$. Hence, the conclusion of Lemma 10 holds.

Finally, if $\alpha \in (-1, 0)$ or if V is not an affine transformation of U then V is not a positive affine transformation of $U + V$. Hence, the preferences represented by V and $U + V$ are different and non-trivial (i.e. neither V nor $U + V$ is constant). Therefore, there exists $\bar{\nu}$, $\underline{\nu}$ such that either $U(\bar{\nu}) + V(\bar{\nu}) \geq U(\underline{\nu}) + V(\underline{\nu})$ and $V(\bar{\nu}) < U(\underline{\nu})$ or $U(\bar{\nu}) + V(\bar{\nu}) > U(\underline{\nu}) + V(\underline{\nu})$ and $U(\bar{\nu}) \leq U(\underline{\nu})$. In either case, since neither U nor V is constant, we can use the linearity of U and V to find $\bar{\mu}$ close to $\bar{\nu}$ and $\underline{\mu}$ close to $\underline{\nu}$ for which all of the above inequalities are strict and apply Lemma 10. □

7.3 Proof of Theorem 4

A regular preference $\succeq$ represented by (u, v, δ) has the property that $U := u + \delta W$ is not constant and v is not constant. Consequently, there exists no $\alpha, \beta \in I\!R$ such that

$v = \alpha U + \beta$. Then, we may apply the proof of Theorem 4 in Gul and Pesendorfer (2000) to conclude that if (u, v, δ) and (u', v', δ') both represent $\succeq$ then $u' + \delta W' = \alpha(u + \delta W) + \beta_u$, $v' = \alpha v + \beta_v$ and therefore $W' = \alpha W + \beta_u + \beta_v$ for some $\alpha > 0, \beta_u, \beta_v \in I\!R$. By regularity this implies that $\delta' = \delta$ and $u' = \alpha u + (1 - \delta)\beta_u - \delta\beta_u$. The proof of the converse is straightforward and therefore omitted. □

7.4 Proof of Theorem 5

We note that any binary $\succeq$ on Z that satisfies Axioms $1 - 8$ induces a the following binary relation $\succeq^*$ on $\mathcal{A} = \mathcal{K}(\Delta(C))$: $A \succeq^* B$ iff $x \succeq y$ for some $x, y \in Z_{II}$ such that $A := \{\mu^1 \mid \text{ for some } \mu \in x\}, B := \{\mu^1 \mid \text{ for some } \mu \in y\}$ and $\mu^2 = \hat{\mu}^2$ for all $\mu \in x, \hat{\mu} \in y$. The preference $\succeq^*$ is well-defined since it can be represented as in Theorem 2. Moreover, $\succeq^*$ satisfies Axioms $1 - 4$ and (u, v) represents $\succeq^*$ if and only if (u, v, δ) represents $\succeq$, for some $\delta \in (0, 1)$. Hence, $\succeq_1$ has more preference for commitment than $\succeq_2$ iff $\succeq_2^*$ has a preference for commitment at $A \in \mathcal{A}$ implies $\succeq_2^*$ has a preference for commitment at $A \in \mathcal{A}$. Since $\succeq$ is instantaneously regular, $\succeq^*$ satisfies the regularity as defined in Gul and Pesendorfer (2000). Therefore we may apply Theorem 8 in Gul and Pesendorfer (2000) to yield the desired result. □

7.5 Proof of Theorem 6

Define $\succeq^*$ as in the proof of Theorem 5. Again, $\succeq^*$ satisfies Axioms $1 - 4$ and (u, v) represents $\succeq^*$ if and only if (u, v, δ) represents $\succeq$, for some $\delta \in (0, 1)$. Hence, $\succeq_1$ has more self-control than $\succeq_2$ iff $\succeq_2^*$ has self-control at $A \in \mathcal{A}$ implies $\succeq_2^*$ has self-control at $A \in \mathcal{A}$. Therefore we may apply Theorem 9 in Gul and Pesendorfer (2000) to yield the desired result. □

7.6 Proof of Corollary

The "if" parts of both statements are straightforward and omitted.

Theorem 5, the fact that $\succeq_1$ has more preference for commitment than $\succeq_2$ and the fact that each $\succeq_i$ is instantenously regular imply that there exists representation $(u_i, v_i\delta)$ of $\succeq_i$ for $i = 1, 2$ and a non-singular, non-negative matrix $\mathbf{\Theta}$ and $\beta \in I\!R^2$ such that

$$\begin{pmatrix} u_2(\mu_1) \\ v_2(\mu_1) \end{pmatrix} = \mathbf{\Theta} \cdot \begin{pmatrix} u_1(\mu_1) \\ v_1(\mu_1) \end{pmatrix} + \beta \qquad (7)$$

for all μ_1.

Similarly, by Theorem 6, $\succeq_1$ has more self-control than $\succeq_2$ implies that there exists a non-singular, non-negative matrix $\mathbf{\Theta}'$ and $\beta' \in I\!R^2$ such that

$$\begin{pmatrix} u_2(\mu_1) + v_2(\mu_1) \\ v_2(\mu_1) \end{pmatrix} = \mathbf{\Theta}' \cdot \begin{pmatrix} u_1(\mu_1) + v_2(\mu_1) \\ v_1(\mu_1) \end{pmatrix} + \beta' \tag{8}$$

for all μ_1.

Suppose $\succeq_1$ and $\succeq_2$ have the same preference for commitment and $\succeq_1$ has more self-control than $\succeq_2$. Without loss of generality we can choose (u_2, v_2) such that $\beta = (0,0)$ in (7). That is, for some non-singular, non-negative matrix $\mathbf{\Theta}$

$$\begin{pmatrix} u_2(\mu_1) \\ v_2(\mu_1) \end{pmatrix} = \mathbf{\Theta} \cdot \begin{pmatrix} u_1(\mu_1) \\ v_1(\mu_1) \end{pmatrix} \tag{9}$$

for all $\mu_1 \in M_1$. Similarly, reversing the roles of (u_1, v_1) and (u_2, v_2) in (7) yields a non-singular, non-negative $\hat{\mathbf{\Theta}}$ and $\hat{\beta} \in I\!R^2$ such that

$$\begin{pmatrix} u_1(\mu_1) \\ v_1(\mu_1) \end{pmatrix} = \hat{\mathbf{\Theta}} \cdot \begin{pmatrix} u_2(\mu_1) \\ v_2(\mu_1) \end{pmatrix} + \hat{\beta} \tag{10}$$

for all $\mu_1 \in M_1$. Equation (9) implies that $\hat{\beta} = 0$ and $\hat{\mathbf{\Theta}} = \mathbf{\Theta}^{-1}$. But since both $\hat{\mathbf{\Theta}}$ and $\mathbf{\Theta}$ are non-negative, this implies

$$\hat{\mathbf{\Theta}} = \begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\gamma} \end{pmatrix} \tag{11}$$

for some $\alpha > 0, \gamma > 0$. Again, without loss of generality, we assume $\alpha = 1$. To conclude the proof, we show that $\gamma \geq 1$. Since, $\succeq_1$ has more self-control than $\succeq_2$, the instantenous regularity of $\succeq_2$, equations $(7), (9) - (11)$ and the fact that $\hat{\beta} = 0$ imply that for some non-negative, non-singular $\tilde{\mathbf{\Theta}}$ and $\tilde{\beta}$,

$$\begin{pmatrix} u_2(\mu_1) + v_2(\mu_1) \\ v_2(\mu_1) \end{pmatrix} = \tilde{\mathbf{\Theta}} \cdot \begin{pmatrix} u_1(\mu_1) + v_1(\mu_1) \\ v_1(\mu_1) \end{pmatrix} + \tilde{\beta} = \tilde{\mathbf{\Theta}} \cdot \begin{pmatrix} u_2(\mu_1) + \frac{1}{\gamma} v_2(\mu_1) \\ \frac{1}{\gamma} v_2(\mu_1) \end{pmatrix} + \tilde{\beta}$$

for all $\mu_1 \in M_1$. Since, $\succeq_2$ is instantenously regular and $\tilde{\mathbf{\Theta}}$ is non-negative, we conclude $1 + a = \gamma$ for some $a \geq 0$. Hence, $\gamma \geq 1$ as desired.

Suppose $\succeq_2$ has more preference for commitment than $\succeq_1$, and $\succeq_1$ and $\succeq_2$ have the same self-control. Following the line of argument above, we obtain (u_2, v_2, δ_2), a representation of $\succeq_2$ such that $u_2 + v_2 = u_1 + v_1$ and $v_2 = \gamma v_1$ for some $\gamma > 0$. Then, since $\succeq_2$ has more preference for commitment than $\succeq_1$, (7) implies that there is a non-negative, non-singular $\mathbf{\Theta}$ and β such that

$$\begin{pmatrix} u_2(\mu_1) \\ v_2(\mu_1) \end{pmatrix} = \mathbf{\Theta} \cdot \begin{pmatrix} u_1(\mu_1) \\ v_1(\mu_1) \end{pmatrix} + \beta = \mathbf{\Theta} \cdot \begin{pmatrix} u_2(\mu_1) + (1-\gamma) v_2(\mu_1) \\ \gamma v_2(\mu_1) \end{pmatrix} + \beta$$

for all $\mu_1 \in M_1$. It follows from the instanenous regularity of $\succeq_1, \succeq_2$ that $1 - \gamma + a\gamma = 0$ for some $a \geq 0$. Since $\gamma > 0$, $\gamma \geq 1$ as desired. □

7.7 Proof of Proposition 1

Consider the corresponding economy truncated in period τ. We first establish existence for that truncated economy.

Let $v_i^* : I\!R_{++} \times I\!R_+ \to I\!R_+$ be defined as $v_i^*(\alpha, m) = v_i(\frac{m}{\alpha})$ and note that $v^*(p, \cdot)$ is convex since v_i is convex. The consumer' maximization problem can be written as

$$\max_{\{c_t\}} \sum_{t=1}^{\tau} \delta^{t-1} \left(w_i(c_t) - v_i^* \left(p_t, \sum_{t'=1}^{\tau} p_{t'} \omega_{it'} - \sum_{t'=1}^{t-1} p_{t'} c_{t'} \right) \right)$$

subject to

$$c_t \leq \sum_{t'=1}^{t'} p_{t'} \omega_{it'} - \sum_{t'=1}^{t-1} p_{t'} c_{t'}$$

Note that the objective function is strictly concave, strictly increasing, and the feasible set of consumption choices is compact. Therefore, we may apply a standard argument (for example, Proposition 17.C.1 in Mas-Colell et.al.) to establish the existence of an equilibrium in the truncated economy.

Re-normalize equilibrium prices in the truncated economy so that $p_{11} = 1$. There is an $h < \infty$ such that $p_{tl} \leq \delta^{t-1} h$ for all (t, l) and every truncation τ. This follows from a standard argument since v is increasing, u is continuous, strictly increasing and since aggregate endowment is bounded away from zero.

Let $(p(\tau), \mathbf{c}_i(\tau))$ denote the price, consumption pair where $\mathbf{c}_{it}(\tau) = \omega_{it}$ and $p_t(\tau) = 0$ for $t > \tau$ and $(c_t(\tau), p_t(\tau))$ is equal to the equilibrium price consumption pair of the τ period truncation for $t \leq \tau$. Consider a sequence of equilibria $(p(\tau), \mathbf{c}_i(\tau))$ and note that $(p_t(\tau), c_t(\tau))$ has a convergent subsequence for every t. Hence, there is a $(p, \mathbf{c}_i)$ and a subsequence τ_k such that $(p(\tau_k), \mathbf{c}_i(\tau_k))$ converges to $(p, \mathbf{c}_i)$. We claim that $(p, \mathbf{c}_i)$ is an equilibrium. Since market clearing holds for every τ it must hold also in the limit. It suffices therefore to show that $\mathbf{c}_i$ solves the optimization problem for individual i. Observe that $p(\tau)\omega_i \leq h \sum_{t=1}^{\infty} \delta^{t-1} \sum_l \omega_{itl} < \infty$. Hence, by the dominated convergence theorem, $\lim p(\tau)\omega_i \to \sum_t \lim p_t(\tau)\omega_{it} = p\omega_i$.

Define $m_t(p, \mathbf{c}_i) := p\omega_i - \sum_{t'=1}^{t-1} p_{t'} \mathbf{c}_{it'}$. Suppose $p\mathbf{c}'_i \leq p\mathbf{c}_i$ and

$$\sum_{t=1}^{\infty} \delta^{t-1}[u_i(\mathbf{c}'_{it}) + v_i(\mathbf{c}'_{it}) - v_i^*(p, m_t(\mathbf{c}'_i, p))] \geq \sum_{t=1}^{\infty} \delta^{t-1}[u_i(\mathbf{c}_{it}) + v_i(\mathbf{c}_{it}) - v_i^*(p_t, m_t(p, \mathbf{c}_i))] + \beta$$

for some $\beta > 0$. Let T be such that $\delta^T \max_{x \in K} u(x)/(1-\delta) < \beta/4$. It follows that

$$\sum_{t=1}^{T} \delta^{t-1}[u_i(\mathbf{c}'_{it}) + v_i(\mathbf{c}'_{it}) - v_i^*(p_t, m_t(p, \mathbf{c}'_i))]$$
$$\geq \sum_{t=1}^{T} \delta^{t-1}[u_i(\mathbf{c}_{it}) + v_i(\mathbf{c}_{it}) - v_i^*(p_t, m_t(p, \mathbf{c}_i))] + 3\beta/4$$

Since $p\mathbf{c}'_i - p\omega_i \leq 0$ and $p(\tau_k)\omega_i - p\omega_i \to 0$, and since v_i^* and u_i are continuous, it follows that for large τ_k we can find a $\mathbf{c}''_i$ with $p(\tau_k)\mathbf{c}''_i - p(\tau_k)\omega = 0$ and

$$\sum_{t=1}^{\tau} \delta^{t-1}[u_i(c''_{it}) + v_i(c''_{it}) - v_i^*(p_t(\tau_k), m_t(p(\tau_k), \mathbf{c}''_i))] >$$
$$\sum_{t=1}^{\tau_k} \delta^{t-1}[u_i(c_{it}(\tau_k)) + v_i(c_{it}(\tau_k)) - v_i^*(p(\tau_k), m_t(p(\tau_k), \mathbf{c}_i(\tau_k)))] - \beta/4$$

contradicting the optimality of $\mathbf{c}_i(\tau_k)$. □

7.8 Proof of Proposition 2

In a steady state equilibrium $c_{it} = c_{i0}$ for all t and for some c_{i0} and $p_t/t_{t+1} = 1 + r$ for all t. Furthermore, $m_i^* = (1+r)c_{i0}/r$ must solve the following maximization problem.

$$\max_{\{m_{it}\}} \sum_{t=1}^{\infty} \delta^{t-1} \left[w_i \left(m_{it-1} - \frac{m_{it}}{1+r} \right) - v_i(m_{it}) \right]$$
$$\text{subject to: } m_{i1} = m_i^*; m_{it} \geq 0 \text{ for all } t;$$
$$m_{it-1} - \frac{m_{it}}{1+r} \geq 0 \text{ for all } t$$

Hence, a necessary condition for a steady state is

$$w_i'(c_{i0}) = \delta(1+r)w_i'(c_{i0}) - \delta\lambda_i(1+r)v'\left(\frac{(1+r)c_{i0}}{r}\right)$$

or

$$\frac{\delta(1+r)-1}{\delta(1+r)\lambda_i}w_i'(c_{i0}) = v'\left(\frac{(1+r)c_{i0}}{r}\right) \tag{12}$$

Together with market clearing, $\sum c_{i0} = \omega$, this condition is also sufficient.

Choose i_0 such that $\lambda_{i_0} = \min \lambda_i$. We define $\bar{r}$ by the equation

$$\frac{\delta(1+\bar{r})-1}{\delta(1+\bar{r})\lambda_{i_0}}w_{i_0}'(\bar{\omega}) = v'\left(\frac{(1+\bar{r})\bar{\omega}}{\bar{r}}\right)$$

Note that $\bar{r}$ is well-defined since the l.h.s. of the above equation is increasing in r and the r.h.s. is decreasing. Moreover, as $r \to \infty$, the l.h.s. converges to $w_i'(\bar{\omega})/\lambda_{i_0} + v'(\alpha\bar{\omega}) > v'(\bar{\omega})$ the r.h.s. converges to $v'(\alpha)$. Hence, there is a unique $\bar{r}$ that satisfies the above equation. Furthermore, $\bar{r} > \frac{1-\delta}{\delta}$.

Let $r \in (\frac{1-\delta}{\delta}, \bar{r}]$. For every r in that range there is a unique $c_{i0}(r) \leq \bar{\omega}$ that satisfies (12). To see this observe that since $w_i^{'}(c_t) \to \infty$ as $c_t \to 0$, the l.h.s. goes to infinity as $c_{i0} \to 0$. The r.h.s. stays bounded. For $c_{i0} = \bar{\omega}$ and $r \leq \bar{r}$, we have

$$\frac{\delta(1+r)-1}{\delta\lambda_i(1+r)}w_i'(\bar{\omega}) \leq \frac{\delta(1+r)-1}{\delta\lambda_{i_0}(1+r)}w_{i_0}'(\bar{\omega}) = v'\left(\frac{(1+\bar{r})\bar{\omega}}{\bar{r}}\right) \leq v'\left(\frac{(1+r)\bar{\omega}}{r}\right)$$

Since w_i' is decreasing and v' is increasing, there is a unique solution, $c_{i0}(r)$.

Now, observe that $c_{i0}(r)$ is strictly increasing in r. To see this, take a total derivative of equation (12) to find that

$$\left(\frac{c_{i0}v''}{r^2} + \frac{w_i'}{\delta\lambda_i\,(1+r)^2}\right)dr = \left(\frac{(1+r)v''}{r} - \frac{[\delta(1+r)-1]w_i''}{\delta\lambda_i(1+r)}\right)dx_i$$

Since all terms are positive the assertion follows.

Thus, $\sum_i c_{i0}(r)$ is increasing in r with

$$\lim_{r\to\frac{1-\delta}{\delta}} \sum_i c_{i0}(r) = 0 \text{ and } \sum_i c_{i0}(\bar{r}) \geq \bar{\omega}$$

Continuity implies that there is a unique r^d such that $\sum_i c_{i0}(r^d) = \bar{\omega}$.

Finally we need to show that $c_{i0} > c_{j0}$ if $\lambda_i < \lambda_j$. Examine

$$\frac{\delta(1+r)-1}{\delta\lambda_i(1+r)} w_i'(c_{i0}) = v'\left(\frac{(1+r)c_{i0}}{r}\right)$$

to see that if $\lambda_i \leq \lambda_j$ and $c_{i0} \leq c_{j0}$ then the necessary condition for an optimum must be violated for either i or j. $\square$

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