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# ON DELAY IN BARGAINING WITH ONE-SIDED UNCERTAINTY

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Recently, attention has been given to a model of two-person bargaining in which the parties alternate making offers and there is uncertainty about the valuation of one party. The purpose of the analysis has been to identify delay to agreement with a screening process, where agents with relatively lower valuations distinguish themselves by waiting longer to settle. We point out a fundamental difficulty with this program by demonstrating that the assumptions used in the literature allow for delay only in so far as the time between offers is significant.

KEYWORDS: Non-cooperative bargaining, delay to agreement, one-sided uncertainty about valuations, alternating offers, sequential equilibrium.

Key words too long

## 1. INTRODUCTI

THE FOLLOWING FORMULATION of an alternating offer bargaining process is now standard. There are two agents: a buyer and a seller. They have common rates of time preference and their valuations of the object to be exchanged are common knowledge. The seller makes the first offer, and this can be accepted or declined by the buyer. If the buyer declines the offer, then he can counteroffer and the counteroffer can be accepted or declined by the seller. If the seller declines the counteroffer, then he can make a second offer, etc. The time between offers is a parameter of the model.

Rubinstein (1982) demonstrated that there is a *unique* subgame perfect equilibrium of the above bargaining game. In other words, given the institution of alternating offer bargaining, a unique division of the gains from trade is determined by the time preference of the agents, the time between offers and the specification of who moves first. Furthermore Rubinstein showed that as the time between offers approaches zero, the advantage from moving first disappears and equilibrium converges to the equal division of the gains from trade. These results have received a great deal of attention and deservedly so. Economists have sometimes regarded the pure bilateral monopoly problem as indeterminate and the result that a unique division of surplus is predicted by such a simple and compelling framework came as something of a surprise and invited further analysis.

Given uniqueness, it is clear that the Rubinstein equilibrium results in the buyer accepting the first offer made by the seller. For suppose that equilibrium

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It is a pleasure to acknowledge our debt to Robert Wilson for his role in the development of our ideas. Both the substance of the result and the technique of proof are related to work done with Bob on bargaining with one-sided uncertainty and one-sided offers. In addition to encouraging our work on the alternating offers model, Bob shared with us his construction of a sequence of equilibria of the variety considered in Section 4.

prescribed the exchange to take place after the $n$th offer ($n > 1$). At time $n - 1$ the agent who is to offer could make a proposal, which if accepted, would make both agents better off than the unique equilibrium that follows if the offer is declined. The fact that trade always occurs without delay has been troublesome since one of the primary reasons for analyzing the bargaining problem has been to develop insights concerning the possibility of bargaining impasses. When one has as a goal a theory that uniquely determines a division of surplus, in order to create the possibility of delay to agreement it was thought compelling to introduce incomplete information. (For concreteness, suppose that the seller's valuation and the distribution of the buyer's type is common knowledge, but that the realization of the buyer's type is known only to the buyer.) With incomplete information during the bargaining process agents might be expected to signal their valuation with their offers, and this takes time. One hopes for a theory in which agents communicate their private information by revealing their willingness to delay agreement and that in a significant number of realizations agreement will occur only after some delay.

In this paper we point out a fundamental difficulty with the above program. Consider the formulation in which there is incomplete information regarding the buyer's type. We prove that if one confines attention to (pure strategy) sequential equilibria in which offers and acceptances of the buyer depend in a simple way on the seller's last offer and his beliefs when he makes that offer, then with the addition of some mild monotonicity and a condition that rules out "free screening," the expected time to agreement goes to zero as the time between offers goes to zero. When the time between offers is small, the fact that one expects there to be many offers before agreement is reached does not guarantee that bargaining will take a long time. Our restrictions allow for many offers to be made in equilibrium; however, in all of the equilibria the market will close quickly when the time between offers is sufficiently small, and this implies that all serious offers will be at approximately the same terms.[2]

Before beginning our analyses a few words regarding solution concepts are in order. For the bargaining game with complete information, if subgame perfection is not required, then it is easy to see that almost any division can be supported as a Nash equilibrium. For example, consider the division that gives the seller three quarters of the gains from trade that are left at the time $t^*$. Such an equilibrium is supported by the strategies that have both the seller and the buyer demand all

[2] The result is related to the Coase conjecture (see Stokey (1982)); the same conclusion for markets with one-sided offers is obtained by Gul, Sonnenschein, and Wilson (1985).

With significant time between offers there can be significant delay to agreement (see Grossman and Perry (1986)). The time between offers might be explained by the cost of making offers or the possibility of committing to silence for some period of time. The latter approach is studied by Admati and Perry (1986).

When the time between offers is small, significant delay is exhibited if there is two-sided uncertainty and overlapping supports. Vincent (1986) examines models with one-sided uncertainty but correlated valuations. These models have a unique sequential equilibrium and exhibit delay even as the time between offers goes to zero.

of the surplus at each point in the game other than $t^*$, and at $t^*$ demand at least their share of the indicated division. The punch of Rubinstein's theorem is that it shows that, for the case of complete information, the restriction to subgame perfect Nash equilibrium (Selten (1975)) leaves a unique solution.

The leading solution concept for games with incomplete information is sequential equilibrium; however, for the bargaining game with incomplete information, the restriction to sequential equilibrium (Kreps and Wilson (1982)) does very little to refine the possible divisions that can result from bargaining. To see why, suppose that whenever the buyer makes an off-the-equilibrium path offer, the seller concludes that he is dealing with the highest valuation buyer. This makes it unattractive for the buyer to deviate from equilibrium path behavior; as a consequence, with these beliefs almost any pair of strategies which are sequentially rational *along the equilibrium path* can be supported as a sequential equilibrium. In the bargaining problem with incomplete information the lack of sufficient restrictions on beliefs leads to a continuum of equilibria.

Attempts to extend Rubinstein's analysis of complete information bargaining to the case of incomplete information have focused on finding adequate refinements of sequential equilibrium and centered on equilibrium in pure strategies.[3] With this background we once again state the result: For all (pure strategy) equilibria that satisfy our stationarity, monotonicity, and no free screening conditions, the likelihood that bargaining will take place for more than any preprescribed length of time approaches zero as the time between offers approaches zero. In short, under the above hypotheses, delay to agreement can only be explained by the time between offers. One can conclude further that under our hypotheses, almost all types of buyers must be trading at approximately the same price. This follows from the fact that if the time between two serious offers is small, then the difference between these offers must be small.

We are not *per se* interested in refining the notion of sequential equilibrium, either in general or for bargaining games. Our results are relevant for a variety of sequential equilibria. In particular, our hypotheses admit the equilibrium considered by Grossman and Perry (1986) when it exists. Also, although our hypotheses are not formulated so as to apply to the Rubinstein's version of bargaining with uncertainty (Rubinstein (1985)), since his equilibrium converges to immediate acceptance as the time between offers goes to zero, his results square nicely with ours. Our conditions impose restrictions on the strategies of the buyer but do not identify a unique sequential equilibrium. Grossman and Perry (1986) construct equilibria that satisfy our conditions and in these equilibria the lowest price offered tends toward the Rubinstein price when a buyer with valuation $l$ faces a seller with valuation zero. In Section 4 (for the case $l < h/2$) we present an alternate sequential equilibrium in which the lowest price offered tends toward the lowest of the buyer's valuations.

Before embarking upon the formal analysis, let us consider the intuition underlying the theorem. In an equilibrium in which the time between offers is

[3] See Grossman and Perry (1986) and Rubinstein (1985).

very small, one would expect that agents are screened very finely. By this we mean that buyers with significantly different valuations pay different prices and buy at different times. However, when the time between offers is small, the amount of screening that can be achieved in a small amount of time may become large, since in a given amount of time the seller can make many different offers. In this case one expects that the incremental value of screening is small. Equilibrium for the seller requires that he cannot increase his expected utility by dropping some offers early on (this means that he screens somewhat less) in order to reach the lower valuation buyers a bit earlier. Our stationarity assumption buys for us the fact that provided screening has taken place to the same level of buyer valuation, the remaining buyers will not behave differently if they are approached with the same offer, but sooner. This suggests a tendency for bargaining to take place quickly when the time between offers is small. In such a regime there can be almost perfect screening in a short period of time, although all serious offers will have approximately the same value.

## 2. THE MODEL

There is a single buyer and a single seller. They alternate making offers and the time between offers is $\Delta$. The buyer is defined by his valuation $b$, which is a positive number and is private information of the buyer. The distribution of buyer valuations is common knowledge and is given by a continuous probability distribution function $F$ with support $B = [l, h]$. An outcome of the game is a pair $(P, n)$, with the interpretation that the seller sells the object to the buyer in period $n$ at price $P$. Agents are equally impatient, in fact we assume a common discount rate, $r$, and this leads to the von Neumann–Morgenstern utility functions over outcomes of $U_s(P, n) = Pe^{-\Delta nr}$ (for the seller) and $U_b(P, n) = e^{-\Delta nr}(b - P)$ (for the buyer with valuation $b$).

The seller moves in even periods (by convention, the initial period is zero), and the buyer moves in odd periods. An agent can accept the offer that has just been made (this action is denoted by the symbol $Y$) or choose a nonnegative counteroffer $P \in R_+$. Thus, the strategy of the seller is formally a sequence of functions $\sigma_s^n \colon H^{n-1} \to \{Y\} \cup R_+$, $n = 0, 2, \ldots$, where $H^{n-1}$ denotes the history of the game at time $n$ if no agreement has been reached. Similarly, the strategy of the buyer with valuation $b$ is a sequence of functions $\sigma_b^n \colon H^{n-1} \to \{Y\} \cup R_+$, $n = 1, 3, \ldots$. A strategy profile is denoted by $\sigma = (\sigma_s, \sigma_B)$, where $\sigma_B$ specifies for each $b \in B$ a buyer's strategy $\{\sigma_b^n\}$, $n = 1, 3, \ldots$. We will assume that buyers' strategies are measurable in types.

Let $W$ denote the set of probability distributions on $B$ and $Z$ denote the set of conditional distributions of $F$ on intervals $[a, c]$, where $l \leqslant a \leqslant c \leqslant h$; elements of $Z$ are denoted by $(a, c)$. Beliefs of the seller are specified for each history of the game and are formally defined by a $g \colon \bigcup_{n=-1}^{\infty} H^n \to W$. This definition does not impose any rationality (this will come later); however, we require that the seller's beliefs not change after his own move. In the obvious notation $g(h^n, P) = g(h^n)$ for all $h^n$, $P$, and $n$ odd.

The strategy profile-belief pair $(\sigma, g)$ is a sequential equilibrium if (a) $g$ is derived from $F$ by Bayesian updating (whenever possible), and (b) given the beliefs defined by $g$, the strategies $\sigma_s$ and $\sigma_b$ (for all $b \in B$) are optimal after every history that occurs when all other agents play according to $\sigma$.

We confine attention to (pure strategy) sequential equilibria. This is not to say that we believe that equilibria in mixed strategies are unlikely. With minor adjustments our results will cover equilibria where there is mixing, as long as it is not on the equilibrium path. So far the literature on bargaining with uncertainty has largely ignored the analysis of mixed strategy equilibria, and we would be disappointed if mixing along the equilibrium path was necessary for delay. At the very least this would mean that in order to get delay one would have to give up the determinacy of equilibrium prices. Finally one might question the descriptive relevance of solutions to the bargaining problem that are characterized by mixing. In any case our analysis presents a result about models in which mixing on the equilibrium path has not been considered.

The sequential equilibria $(\sigma, g)$ that we consider are required to satisfy three conditions. Before stating these conditions, we present a Lemma which shows that after any history $h^n \in H^n$ and for each price $P$ that the seller might charge, there is a unique number $b \in B$ so that a buyer will accept the offer if his valuation exceeds $b$ and not accept it if his valuation is less than $b$. Call this number the marginal valuation at $h^n$ prescribed by $\sigma$. (The proof is standard.)

LEMMA 0: *For any sequential equilibrium and any odd number $n$, there exists a function $\beta$: $H^n \times R_+ \to B$ such that for all $P \in R_+$ and for all $h^n \in H^n$, $\sigma_c(h^n, P) = Y$ whenever $c > \beta(h^n, P)$ and $\sigma_c(h^n, P) = Y$ implies $c \geqslant \beta(h^n, P)$.*

In general, the response of a buyer $b$ to an offer can depend on the entire history of offers and counteroffers. The first condition we consider requires that two histories $(h^n, P)$ and $(h^m, Q)$ prescribe the same response for all buyer types still remaining if the marginal buyer to purchase at price $P$ after history $h^n$ is the same as the marginal buyer to buy at price $Q$ after history $h^m$; that is $\beta(h^n, P) = \beta(h^m, Q)$. The condition precludes the consideration of equilibria in which the actions of buyers depend in a complicated way upon the history of the game: From the point of view of a given buyer's response to an arbitrary history, all that can matter is the marginal valuation prescribed by the equilibrium.

CONDITION 1 (Stationarity of Buyer's Strategies): *For each $b \in B$, $P$, $Q \in R_+$, $n$ and $m$ odd, $h^n \in H^n$ and $h^m \in H^m$, if $\beta(h^n, P) = \beta(h^m, Q)$, $b < \beta(h^n, P)$, then $\sigma_b^{n+2}(h^n, P) = \sigma_b^{m+2}(h^m, Q)$.*

The next condition requires that the marginal valuation function display some natural monotonicity. In particular, the possibility of additional high valuation buyers does not lead a low valuation buyer to lower his acceptance price.

CONDITION 2 (Monotonicity)[4]: *For all* $P \in R_+$, $h^n \in H^n$, $h^m \in H^m$, *and n and m odd, if* $g(h^n) = z = (l, c) \in Z$, $g(h^m) = w = (l, d) \in Z$, *and* $d \geq c$, *then there exists* $Q \geq P$ *such that* $\beta(h^n, P) = \beta(h^m, Q)$.

Finally, we restrict attention to equilibria $(\sigma, g)$ in which the offer of a buyer can influence the beliefs of the seller only if $\sigma$ specifies that the offer be accepted. This permits buyers to signal their valuation by making an offer that will not be accepted; however, it does not permit them to differentiate themselves from other buyers who plan to keep the bargaining open.

CONDITION 3 (No Free Screening): *For all even n,* $P, Q \in R_+$ *and for all histories* $h^n \in H^n$, *if* $g(h^n, P) \neq g(h^n, Q)$ *and* $\{b \mid \sigma_b(h^n) = \alpha\} \neq \varnothing$ *for* $\alpha = P, Q$, *then either* $\sigma_s(h^n, P) = Y$ *or* $\sigma_s(h^n, Q) = Y$.

Condition 3, together with the restriction to pure strategies, forces all unaccepted buyer offers, in a given period, to be the same. We should emphasize once again that our restrictions do not in general lead to a unique equilibrium. In Section 4 we sketch the construction of two types of sequential equilibria that satisfy Conditions 1–3. These equilibria are associated with very different offers along their equilibrium paths.

## 3. THE THEOREM

Let $\Delta$ denote the length of time between offers, and let $\Sigma(\Delta)$ denote the set of sequential equilibria of the alternating offer bargaining game that satisfy Conditions 1–3. The following theorem establishes that, with the above hypotheses, delay to agreement can be significant only to the extent that the time between offers is significant.

THEOREM: *For any* $\varepsilon > 0$, *there exists* $\bar{\Delta} > 0$ *such that for all positive* $\Delta < \bar{\Delta}$ *and for any sequential equilibrium* $(\sigma, g) \in \Sigma(\Delta)$, *the probability that the game will be terminated within* $\varepsilon$ *time is at least* $1 - \varepsilon$.

The proof of the theorem is preceded by three lemmas.

LEMMA 1: *In any equilibrium* $(\sigma, g) \in \Sigma(\Delta)$ *and after any history* $h^n \in H^n$ *at which no trade has taken place, the expected return to the seller at* $n + 1$ *is at least* $le^{-r\Delta}/(1 + e^{-r\Delta})$.

Rubinstein (1982) shows that if the buyer were known to have valuation $l$, he would choose to accept any price below $l/(1 + e^{-r\Delta})$ since the seller would never charge a lower price. The lemma merely states that the possibility of the buyer

[4] This is a hybrid assumption that implies the time stationarity of the buyer's acceptance strategies as well as certain regularities regarding how the acceptance strategies can change as a function of the seller's belief. The term monotonicity describes an important part of this regularity.

having a higher valuation does not alter this result. The proof follows along similar lines to the one presented by Rubinstein.

LEMMA 2: *Call an offer P at n serious relative to an equilibrium* $(\sigma, g) \in \Sigma(\Delta)$ *if the probability that P will be made at n and accepted is positive. Then, no two distinct offers at n can be serious.*

This is clear, since if there were two serious prices the buyer would never charge the higher one.

Lemma 3 is of great importance and states that sequential equilibria satisfying our hypotheses have a simple structure along the equilibrium path. First, the beliefs of the seller regarding the distribution of buyer valuations is at every point in time a truncation of the original distribution. Second, after each offer by the seller, buyers divide into (at most) three intervals: the highest valuation buyers accept, the next highest valuation buyers make an acceptable counteroffer, and the remaining buyers make an unacceptable counteroffer.

LEMMA 3: *For all* $\sigma \in \Sigma(\Delta)$ *and any integer i, there exists (A) if i is even, a unique* $h^i_\sigma \in H^i$ *such that the probability of* $h^i_\sigma$ *given* $\sigma$ *is positive, and (B) a unique nonincreasing sequence* $a_0, a_1, a_2, \ldots$ *called the cutoffs generated by* $\sigma$, *such that*

(1) $a_0 = h,$

(2) *if i is even, then* $g(h^{i-1}_\sigma) = (l, a_i) \in Z$, *or* $\sigma^i_s(h^{i-1}_\sigma) = Y$

(3) *if i is odd, then*

$$\begin{aligned} \sigma^i_b(h^{i-1}_\sigma) &= Y \quad \text{if} \quad b \in (a_{i+1}, a_i) \\ &= P_i \quad \text{if} \quad b \in (a_{i+2}, a_{i+1}) \text{ and } P_i \text{ is serious at } i, \\ &= \overline{P}_i \quad \text{if} \quad b \in [l, a_{i+2}), \end{aligned}$$

*where* $\overline{P}_i$ *is the unique equilibrium nonserious offer. (See the remark following Condition 3.)*

PROOF: See the Appendix.

PROOF OF THE THEOREM: The proof is by contradiction and we begin with a sketch of the argument. Suppose that in an equilibrium $\sigma$ the time between offers is very small, and yet the probability that the market will remain open for more than $\varepsilon$ time is greater than $\varepsilon$. This means that there is a $b^\varepsilon$ such that no buyer with valuation below $b^\varepsilon$ will accept until more than $\varepsilon$ time has elapsed and also that the probability of an agent having valuation in $[l, b^\varepsilon]$ is at least $\varepsilon$. We define $\eta$ so that the expected loss from sacrificing $\eta$ on every realization of buyer type above $b^\varepsilon$ is less than the expected gain from reaching buyers with valuation no more than $b^\varepsilon$ in one half the time. When the time between offers is sufficiently small it is possible to contradict the optimality of the seller's strategy $\sigma$ by constructing an alternative strategy that sacrifices no more than $\eta$ on each of the buyers with valuation at least $b^{\varepsilon}$ and takes no more than $\varepsilon/2$ of time to reach $b^{\varepsilon}$. This uses the monotonicity and stationarity hypotheses, which buy us the fact that, provided screening has taken place to the same level of buyer valuation, the remaining buyers will not behave differently if they are approached with the same offer, but sooner.

We now present the formal argument. If the result is false, then there exists $\varepsilon > 0$ such that for all $\bar{\Delta} > 0$, there exists $\Delta < \bar{\Delta}$ and $(\sigma, g) \in \Sigma(\Delta)$ such that the probability of no trade by time $t = \varepsilon$ exceeds $\varepsilon$. Suppose there is such an $\varepsilon$. Define $\eta = l\varepsilon e^{-r}(e^{-r\varepsilon/2} - e^{-r\varepsilon})/(1 + e^{-r})$ and $\bar{\Delta} = \min\{(\varepsilon/4N), 1\}$, where $N$ is the least integer that exceeds $h/\eta$. Now let $\Delta < \bar{\Delta}$ and $(\sigma_s, \sigma_B, g) \in \Sigma(\Delta)$ have the property that the probability of no trade by $t = \varepsilon$ exceeds $\varepsilon$. A contradiction will be obtained by showing that $\sigma_s$ is not a best response to $\sigma_B$.

Let $b^{\varepsilon}$ be the highest type of buyer that will not buy by $t = \varepsilon$ according to $\sigma$. By the definition of $b^{\varepsilon}$, all buyers in $(b^{\varepsilon}, h]$ will have bought by $\varepsilon$. Also, by the optimality of buyers' actions, no buyer in $[l, b^{\varepsilon})$ will have bought by $\varepsilon$. Thus, by assumption $F(b^{\varepsilon}) > \varepsilon$.

Next we will construct an alternative strategy $\bar{\sigma}_s$ for the seller that improves on $\sigma_s$; it has the following properties:

A. For any realization of $b$ in $(b^{\varepsilon}, h]$, trade with $b$ will take place:
   1. by $\varepsilon/2$;
   2. no later than it would have taken place according to $\sigma$;
   3. at a price no less than $P - \eta$, where $P$ is the price at which trade would have occurred according to $\sigma$.

B. For any realization of $b$ in $[l, b^{\varepsilon}]$, trade will take place:
   1. at a price no less than the price that would have occurred under $\sigma$;
   2. by time $s - \varepsilon/2$, where $s$ is the time at which $b$ trades in $\sigma$.

This is accomplished as follows. From each interval of the form $(P^{\varepsilon} + n(h - P^{\varepsilon})/N,\ P^{\varepsilon} + (n+1)(h - P^{\varepsilon})/N]$ for $0 \leqslant n \leqslant N - 1$, choose the smallest serious offer (when one exists), where $P^{\varepsilon}$ is the lowest serious price at which a buyer $b \in (b^{\varepsilon}, h]$ buys. If this offer is made in period $k$ in $\sigma$, then denote it by $P_k$ in part to keep track of whether it is a buyer's or a seller's offer. Call these offers *good offers*. Now start with the highest good offer: call it $P_k$. If $k$ is even; that is, if $\sigma$ specifies that the seller make the offer $P_k$ in period $k$, then let $\theta_0$ be the reservation price of the highest valuation buyer that accepts $P_k$ in period $k$ according to $\sigma$ and have the seller offer $\theta_0$. With the sequence $a_0, a_1, a_2, \ldots$ defined as in Lemma 3, $a_{k+1} = \beta(\theta_0, h^{-1})$, where $h^{-1}$ denotes the null history. If $k$ is odd, that is if $\sigma$ specifies (for some realization of $b$) the buyer to make the offer $P_k$ in period $k$, then let $\theta_0$ specify the acceptance price of the highest type buyer that would buy the good by period $k - 1$ according to $\sigma$ and have the seller offer $\theta_0$; thus $a_k = \beta(\theta_0, h^{-1})$. By definition, nobody with valuation less than $a_k$ will accept $\theta_0$. If the next good offer $P_n$ is $\sigma_b^1(\theta_0)$ for some $b$, then the seller should accept it if it is made. If the offer $P_n$ is not made, then he should reject the offer and consider the good offer after $P_n$, call it $P_m$. If $m$ is even, define $\theta_1$ implicitly by $a_{m+1} = \beta(\bar{h}^1, \theta_1)$; where $\bar{h}^1 = (\theta_0, \sigma_b^1(\theta_0))$. If $m$ is odd, define $\theta_1$ by $a_m = \beta(\bar{h}^1, \theta_1)$. The seller is prescribed to offer $\theta_1$. If $\sigma_b^1(\theta_0)$ is not $P_n$ for any $b$, then the seller should accept it if it is serious at $n+1$ and if it is not serious he should consider $P_n$. If $n$ is even (respectively odd), then proceed as above to generate $\theta_1$; that is, define $\theta_1$ by $a_{n+1} = \beta(\bar{h}^1, \theta_1)$ (respectively $a_n = \beta(\bar{h}^1, \theta_1)$), etc. Finally, after the lowest good offer is made, revert to $\sigma$; that is, make the offers that would generate the same cutoffs as $\sigma$.

We verify A1 by noting that if $b \in (b^\varepsilon, h]$ then since there are at most $N$ good offers, there are at most $2N$ offers before the game ends when $\bar{\sigma}_s$ is employed; hence, at most $\Delta 2N \leqslant \varepsilon/2$ amount of time elapses before the game ends. Property A2 follows from the fact that the sequence of cutoffs generated by $\bar{\sigma} = (\bar{\sigma}_s, \sigma_B)$, call it $\{\bar{a}_i\}$, is a subsequence of the cutoffs generated by $\sigma$.

Next we verify A3. Assume that $b$ purchases (in $\sigma$) in period $i$; that is, $b \in [a_{i+i}, a_i)$. If $i$ is odd, and $P_i$ is a good offer, then by construction $a_i = \beta(h_{\bar{\sigma}}^{j-2}, \theta_{j-1})$, where $b$ buys in period $j$ according to $\bar{\sigma} = (\bar{\sigma}_s, \sigma_B)$. But then by stationarity $\sigma_b^j(h_{\sigma}^{j-1}) = \sigma_b^i(h_{\sigma}^{i-1})$, which gives the result for this case.

If $i$ is even and if $P_i$ is a good offer, then by the observation that the cutoffs of $\bar{\sigma}$ are a subsequence of the cutoffs of $\sigma$, we have that $a_i \geqslant \bar{a}_j$. But by the definition of cutoffs $\bar{a}_{j+1} = \beta(h_{\sigma}^{i-1}, \sigma_s(h_{\sigma}^{i-1}))$, $g(h_{\bar{\sigma}}^{j-1}) = (l, \bar{a}_j)$, and $g(h_{\sigma}^{i-1}) = (l, a_i)$, and so by monotonicity (C2) we have $\theta_j \geqslant \sigma_s^i(h_{\sigma}^{i-1})$.

If $P_i$ is not a good offer, then let $P_k$ be the next good offer. Regardless of whether $k$ is even or odd, based on the arguments in the preceding two paragraphs there exist $\bar{P}_j \geqslant P_k$ such that $b$ buys (according to $\bar{\sigma}$) no later than time $j$, at a price no less than $\bar{P}_j$. Furthermore, by the definition of a good offer, the next good offer $P_k$, is at least $P_i - \eta$. Thus $\bar{P}_j \geqslant P_i - \eta$.

Next B1 follows from the definition of $\bar{\sigma}_s$ and B2 follows from A1.

Finally, from A1–A3, if $b \in (b^\varepsilon, h]$, then the use of $\bar{\sigma}_s$ (as opposed to $\sigma_s$) against $\sigma_B$ results in a loss $L$ not exceeding $\eta$. By B1, if $b \in [l, b^\varepsilon)$, then $\bar{\sigma} = (\bar{\sigma}_s, \sigma_B)$ will have trade take place at least $\varepsilon/2$ sooner than it would have taken place in $\sigma$. Hence, for these cases, by Lemma 1, $\bar{\sigma}_s$ results in a gain $G$ no less than

$$(e^{-r\varepsilon/2} - e^{-r\varepsilon})\left(\frac{e^{-\Delta r}}{1+e^{-\Delta r}}\right) l \geqslant (e^{-r\varepsilon/2} - e^{-r\varepsilon})\left(\frac{e^{-r}}{1+e^{-r}}\right) l.$$

As a consequence, the net change in expected utility from the employment of $\bar{\sigma}_s$ is at least

$$G\,\mathrm{Prob}\{b \in [l, t^\varepsilon)] - L\,\mathrm{Prob}(b \in (b^\varepsilon, h]\}$$
$$\geqslant (e^{-r\varepsilon/2} - e^{-r\varepsilon})\left(\frac{e^{-r}}{1+e^{-r}}\right) l\varepsilon - \eta(1-\varepsilon) = \eta\varepsilon > 0,$$

which contradicts the assumption that $\sigma$ is an equilibrium.

## 4. AN EXAMPLE

We will sketch the construction of a sequential equilibrium that leads to a very different division of surplus than is the case for the equilibrium presented by Grossman and Perry (1986). Our purpose is not to advocate this alternative equilibrium, but rather to demonstrate that the axioms under consideration admit a variety of equilibria. We assume familiarity with Fudenberg, Levine, and Tirole (1983) and Gul, Sonnenschein, and Wilson (1986).

EXAMPLE: The distribution of buyers' valuations is uniform. Assume $l < h/2$ and consider the equilibrium examined by Fudenberg, Levine, and Tirole (1983) in which only the seller makes serious offers. There, equilibrium is in pure strategies, however, it is not sequential since any buyer who makes an offer (other than zero) is taken to have valuation $K > h$.

Since $h/2 > l$, by Theorem 3, in Gul, Sonnenschein, and Wilson, for $\delta = e^{-\Delta r}$ sufficiently close to one, no buyer will accept an offer above $h/2$. This means that the Fudenberg, Levine, and Tirole equilibrium can be made sequential by specifying that any buyer who makes an offer other than zero is believed to have valuation $h$. It is easy to verify that our Conditions 1–3 are met (using Theorem 1 of Gul, Sonnenschein, and Wilson (1986)). Furthermore, using Proposition 1′ of Fudenberg, Levine, and Tirole (1983) and their observation that the uniform distribution case satisfies Axiom S, we can establish that the equilibrium above is what they call strong Markov; that is, that it involves only pure strategies on or off the equilibrium path.

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## APPENDIX

PROOF OF LEMMA 3: To prove (A) assume that there exist $i$, and $h_1^i, h_2^i \in H^i$ such that $i$ is even, $h_1^i \neq h_2^i$ and the probability of both $h_1^i$ and $h_2^i$ given $\sigma$ is positive. Let $h_1^j$ and $h_2^j$ be the shortest subhistories of $h_1^i$ and $h_2^i$ such that $h_1^j \neq h_2^j$. Obviously $j$ is odd; that is, $j$ is the move of the buyer. Hence, there exist disjoint sets $B_1, B_2 \subset B$ such that $h_1^j = (h^{j-1}, \sigma_{b_1}^j(h^{j-1}))$ for all $b_1 \in B_1$ and $h_2^j = (h^{j-1}, \sigma_{b_2}^j(h^{j-1}))$ for all $b_2 \in B_2$, where $h^{j-1}$ is the common subhistory of $h_1^j$ and $h_2^j$. Since $(\sigma, g)$ is a sequential equilibrium $g(h_1^j) \neq g(h_2^j)$. However, since $j \leqslant i$, and since $i$ is even and $j$ is odd, $j+1 \leqslant i$. Also, since by definition there exists no "$Y$" in $h_1^i$ and $h_2^i$, $\sigma_s^{j+1}(h_1^j) \neq Y$ and $\sigma_s^{j+1}(h_2^j) \neq Y$, which contradicts Condition 3.

To prove (B) define $a_0 = h$, $a_i = \inf\{b \mid \sigma_b^i(h_\sigma^{i-1}) = Y\}$ if $i$ is odd and $a_i = \inf\{b \mid \sigma_s^i(h^{i-2}, \sigma_b^{i-1}(h^{i-2})) = Y\}$ if $i$ is even. (If the game ends with probability 1 in period $k$, then $a_i = l$ for all $i \geqslant K+1$.) Obviously $\{a_i\}$ satisfies 1. Property 2 follows from C3 and Lemma 0. To verify 3, note that for odd $i$ the fact that $\sigma_b^i(h_\sigma^{i-1}) = Y$ for all $b \in (a_{i+1}, a_i)$ follows from Lemma 0. Similarly for all $b \in (a_{i+2}, a_{i+1})$, $\sigma_b^i(h_\sigma^{i-1}) = P_i$ for some $P_i$ that is serious at $i$ follows from Lemma 0 and Lemma 2. Finally, the fact that all buyers making nonserious offers make the same offer; that is $\sigma_b^i(h_\sigma^{i-1}) = \overline{P}_i$ for all $b \in [l, a_{i+2})$ follows from C3.

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