Unobservable Investment and the Hold-Up Problem†

Faruk Gul
Princeton University

January 2000

Abstract

We study a two person bargaining problem in which the buyer may invest and increase his valuation of the object before bargaining. We show that if all offers are made by the seller and the time between offers is small, then the buyer invests efficiently and the seller extracts all of the surplus. Hence, bargaining with frequently repeated offers remedies the hold-up problem even when the agent who makes the relation-specific investment has no bargaining power and contracting is not possible. We consider alternative formulations with uncertain gains from trade or two-sided investment.

† I am grateful to Avinash Dixit, Gene Grossman, Alessandro Lizzeri and Ennio Stacchetti and Andrea Wilson for their help and comments. Financial support from the National Science Foundation is gratefully acknowledged.

1. Introduction

Consider the following simple model of specific investment. One agent, the buyer, takes an observable action that determines his own utility of later consumption. Afterwards, the buyer and the other agent, the seller, bargain. Such a situation is studied in a number of recent papers on the hold-up problem, contract theory and the theory of the firm. If the bargaining process is such that the seller can extract all surplus and there is no contractual commitment then, in equilibrium, the buyer will not invest at all. This is due to the fact that the investment of the buyer is sunk-cost at the bargaining stage and will not be compensated for by the seller. This result is a very extreme form of the hold-up problem and holds regardless of whether the seller makes a single take-it-or-leave-it offer or is able to make repeated offers.

Next, consider the same problem, only this time assume that the buyer's investment decision cannot be observed by the seller. In Proposition 1, we show that if the bargaining consists of a single take-it-or-leave-it-offer then in equilibrium, the buyer and the seller still obtain the same payoffs as in the case of observable investment. Thus, as suggested by Gibbons (1992), in a one-shot interaction, the hold-up problem continues to be extremely costly even with unobservable investment. However, Proposition 5 establishes that if the investment decision is unobservable *and* the seller makes repeated offers then as the time between offers becomes arbitrarily small, the equilibrium investment decision of the buyer converges to the efficient (i.e., surplus maximizing) level. Moreover, the expected delay converges to zero. Hence, all inefficiency disappears. Regardless of the time between offers, the buyer's equilibrium payoff is zero. Therefore, the seller extracts full surplus. Neither unobservable investment nor frequently repeated offers alleviates the hold-up problem; yet, the two together completely resolve it.

The purpose of this paper is twofold. First, we wish to note the tendency toward efficient investment created by this interaction between unobservable investment and dynamic bargaining. When investment is unobservable, self-selection constraints imply that the price confronted by the buyer is independent of his type. Then, any setting where efficient trade (i.e., immediate agreement) is guaranteed, unobservable investment implies that the buyer is the residual claimant on his investment and hence, leads to the first best outcome. In the one-sided offer bargaining setting efficient trade is guaranteed by the Coasian effect. Thus, we would expect our results to extend to other settings provided the underlying bargaining model yields immediate agreement.

The second objective of this paper is to emphasize the role of allocation of information as a tool in dealing with the hold-up problem. Audits, disclosure rules or privacy rights could be used to optimize the allocation of rents and guarantee the desired level of investment. Controlling the flow of information may prove to be a worthy alternative to controlling bargaining power in designing optimal organizations. That private information rents might substitute for bargaining power and ameliorate the hold-up problem has been noted by Rogerson (1992) and others. Our purpose is to demonstrate the importance of this effect by noting that in a Coasian setting, incomplete information may completely remedy the hold-up problem even when the investing agent has no bargaining power.

There are two central assumptions in this paper. First, we assume that commitment is not possible. Undoubtedly, there are many applications were some commitment either implicit or through long term contracts, is feasible. Nevertheless, this extreme form of incomplete contracting may be useful both as a benchmark and for understanding the disagreement payoffs associated with any contracting game.

The second main assumption is unobservable investment. Again, we recognize that there are situations where investment decisions will be observable. Nevertheless, in many settings, it will be possible for one agent to have less than perfect information about her opponent's earlier decision.[1] For the argument of Proposition 5 to apply, a small amount of asymmetric information between the buyer and the seller regarding the buyer's investment level may be sufficient.[2] Consequently, we would expect our analysis to be relevant in many situations, even when the seller cannot be kept totally uninformed about the decision of the buyer.

With one major difference, the setting of Proposition 5 is closely related to the work on one-sided offer, one-sided incomplete information bargaining (see for example Fudenberg, Levine and Tirole (1985)) and the Coase conjecture (see Gul, Sonnenschein and Wilson

[1] This is likely to be the case if for example, the hidden investment is investment in intangible assets or some other type of unobservable effort.

[2] See the example with a random buyer valuation, in Section 6.

(1986), henceforth GSW, and Ausubel and Deneckere (1989)). In the earlier papers, the distribution of the buyer's valuations is exogenously given whereas in the current paper the buyer's valuation is determined by a strategic choice and uncertainty arises from the unobservability of that choice and the buyer's use of a mixed strategy. Nevertheless, understanding the Coase conjecture is important for understanding Proposition 5.

Assume, as is done in the literature on the Coase conjecture, that the distribution of the buyer's valuation is fixed. Then, if the lowest valuation in the support of this distribution is greater than the cost of production, the equilibrium strategy of the buyer will be described by an acceptance function that determines the willingness to pay of each buyer type independent of history. The seller's problem is to find the optimal level of price discrimination given her own impatience and the "demand" function defined by the buyer's strategy. As the time between offers becomes arbitrarily small, the seller will be able to move down this demand function in an arbitrarily small amount of real time. But with a fixed distribution of valuations the fact that expected time until trade is converging to zero implies that the buyer with valuation arbitrarily close to the lowest in the support of the distribution is buying almost immediately, which means that the price charged by the seller must fall to the lowest possible valuation almost immediately. This is the Coase conjecture.

The conclusion that expected delay converges to zero as the time between offers converges to zero carries over to the current setting. However, since the distribution of the buyer's valuation is no longer fixed, this does not mean that the buyer with valuation arbitrarily close to the lower end of possible valuations is buying almost immediately. As the time between offers becomes small, the probability that the buyer invests at the efficient level approaches 1. Hence, only a buyer with a high valuation is buying early, in spite of the fact that investing zero is always in the support of the buyer's strategy. In equilibrium, the expected delay is small but there is positive probability that delay will occur. At each moment in time, the seller believes that there is a high probability of making a sale at a high price in the near future. This keeps the seller from decreasing prices too quickly. The fact that the distribution of the buyer's valuation is endogenously determined enables one part of the Coase conjecture to survive without the other; no delay is expected in equilibrium but the market price does not collapse immediately.

This analysis explains why the Coase conjecture is compatible with Proposition 5, that is, why Proposition 5 might be true. To see why it must be true, observe that if v is the lowest buyer valuation in the support of the distribution, then a price below v will never be charged by the seller. Hence, investing zero must be in the support of the buyer's strategy and his equilibrium payoff must be zero. (If the lowest level investment in the support of buyer's strategy were strictly above zero, the buyer's equilibrium payoff would be negative.) The Coasian effect guarantees no delay in expectation. But if the buyer expects to trade immediately, it is optimal for him to invest at the efficient level. Hence, the outcome is efficient and the seller extracts all the surplus.

We consider two extensions of our model. First, we assume that the constant marginal cost of production is random and not known at the time the buyer undertakes his investment but is commonly known before the bargaining begins. Hence, gains from trade are no longer certain. This new setting brings our framework closer to many models studied in the incomplete contracting literature. We assume that the expected gains from trade are strictly quasi-concave in the buyer's level of investment and that the probability of positive gains from trade given zero investment is not zero. Then, we prove that if offers can be made frequently, sequential equilibrium outcomes with stationary buyer strategies are efficient. Again, the seller extracts all surplus. The restriction to stationary strategies was used by GSW for proving the Coase conjecture for the "no-gap" case (i.e., when a strictly positive lower bound on the gains from trade is not assumed). We impose the same restriction in order to extend the "no expected delay" result to the case where gains from trade are uncertain.

In our second extension, we investigate the consequences of the seller having an investment decision as well. We assume that the seller makes observable, cost reducing investment prior to the bargaining stage. We continue to assume that the buyer's investment is unobservable. We show that the earlier efficiency result holds if the buyer is able to observe the seller's investment prior to making his own. However, if the two agents make their investment decisions simultaneously, then there will be a tendency for the *seller* to underinvest but the buyer will still invest efficiently.

As noted above, a number of recent papers on incomplete contracting also deal with the hold-up problem and the its impact on relation specific investment. Grout (1984) studies the relationship between shareholders and workers. In his setting, the shareholders make the relation specific investment. He compares the case where binding contracts can be made prior to investment with the case where no binding contracts can be made and the *ex post* distribution of surplus is determined exogenously, according to a generalized Nash bargaining solution. He notes that without binding contracts, unless the bargaining solution gives all of the power to the shareholders, there will be underinvestment.

Grossman and Hart (1986) also model the *ex post* bargaining stage in closed form (i.e., as a cooperative game), where the *ex ante* specified distribution of control determines the disagreement payoffs. They observe that greater control, like greater bargaining power in Grout's model, tends to create greater incentives to invest. The optimal organizational form is determined by the relative benefits of giving control to one agent versus the other.

Aghion, Dewatripont and Rey (1994) have investigated the possibility of designing the negotiation process optimally so as to overcome the hold-up problem. Thus, unlike the two papers above, they do not take the renegotiation or bargaining stage as exogenous. They show that first best outcomes can be obtained by the optimal renegotiation procedure (i.e., noncooperative bargaining game), in a wide range of settings.

The papers discussed above and much of the contracting literature assume observable investment. Observable investment in a relation-specific asset may cause a hold-up problem which creates underinvestment. The focus of the literature is to investigate possible contractual resolutions and other remedies to the hold-up problem and to interpret existing institutions as expressions of these remedies. Our main result establishes that unobservability of investment may be an alternative remedy to the hold-up problem. The important role played by unobservability in the context of the extensive form games studied in this paper suggests that the hold-up problem may not be so severe in many contexts. Therefore, in certain applications, the relevant benchmarks for the analysis of contractual resolutions may need to be reconsidered.

A common feature of many of the models studied in the literature on moral hazard and renegotiation (see for example, Che and Chung (1995), Che and Hausch (1996), Fudenberg and Tirole (1990), Hermalin and Katz (1991), Ma (1991), (1994), and Matthews (1995)) is that a pure strategy by the agent generates too severe a response by the principal. Therefore, in equilibrium, the agent randomizes, generating asymmetric information. This phenomenon also plays an important role in our analysis.

2. The Simple Models

In the simple bargaining game with perfect information, the buyer chooses a level of investment x, then this choice is observed by the seller who chooses a price p. The buyer either accepts this price ($\sigma^b(x,p) = 1$) or he rejects it ($\sigma^b(x,p) = 0$). In either case, the game ends. If the buyer invests x and gets the good at price p, then the seller's utility is p and the buyer's utility is $v(x) - x - p$. If the buyer does not end-up with the good, that is; if he rejects the seller's offer, then the payoffs for the seller and buyer are zero and $-x$ respectively. The formal description below of the simple bargaining game with perfect information mentions only the set of pure strategies. However, we permit the use of mixed strategies throughout this paper. Hence, statements regarding the uniqueness of equilibria never entail a restriction to pure strategies.

Let $[0, M]$, $M > 0$, be the set of possible investment choices for the buyer. The non-decreasing function $v : [0, M] \to \mathbf{R}$ defines the valuation of the buyer as a function of the level of investment he has undertaken. We will assume that $v(M) \leq M, v(0) > 0$.

Definition: $B^0 = (\Sigma^b, \Sigma^s)$ *is called a simple bargaining game with perfect information, where* $\Sigma^b := [0, M] \times \{\sigma^b : [0, M] \times \mathbf{R}_+ \to \{0, 1\}\}$ *and* $\Sigma^s := \{\sigma^s : [0, M] \to \mathbf{R}_+\}$.

It is well-known and easily verified that in the simple bargaining game the hold-up problem rules out any possibility of investment. At the bargaining stage, the buyer's investment is sunk-cost and is not taken into account by the seller. Since the seller makes a take-it-or-leave-it offer, he extracts all surplus. Knowing this, the buyer cannot afford to undertake any investment. We record this observation as Proposition 0 below.

Proposition 0: *There is a unique subgame perfect Nash equilibrium of the game* B^0. *In this equilibrium the buyer invests zero, the seller charges* $p = v(0)$ *and the buyer accepts. In equilibrium the seller and buyer obtain utility* $v(0)$ *and* 0 *respectively.*

It follows from Proposition 0 that the hold-up problem causes inefficiency whenever 0 is not an efficient investment level. Next, we will show that if the investment decision is not observed by the seller then the nature of equilibrium is changed but the equilibrium payoffs remain the same. Hence, the unobservability of the investment decision by itself does not remedy the hold-up problem.

Definition: *$B = (\Sigma^b, \Sigma^s)$ is called a simple bargaining game, where $\Sigma^b := [0, M] \times \{\sigma^b : [0, M] \times \mathbf{R} \to \{0, 1\}\}, \Sigma^s := \mathbf{R}_+$.*

Note that the only difference between B^0 and B is that in B the price charged by the seller does not depend on the investment decision of the buyer. This difference accommodates the fact that the investment choice is no longer observable to the seller.

In this paper, a sequential equilibrium is a behavioral strategy profile σ and an assessment μ that satisfy the following: (i) σ is sequentially rational given μ, (ii) if information set h' can be reached by σ given that h is reached, then, the assessment at h' is obtained from the assessment at h by Bayes' Law. This is the solution concept used throughout the bargaining literature.

Proposition 1: *In any sequential equilibrium of B the seller and buyer achieve utility $v(0)$ and 0 respectively. The buyer accepts any offer $p < v(x)$ and rejects any $p > v(x)$. If v is strictly concave and continuous then:*

Equilibrium exists and is unique. In equilibrium, the seller randomizes according to the cumulative F, where $F(p) = 0$ if $p < v(0)$, $F(p) = \frac{1}{v'^{+}(v^{-1}(p))}$ for $p \in [v(0), v(x^*))$, $F(p) = 1$ for $p \geq v(x^*)$, v'^{+} denotes the right-derivative of v and x^* is the unique maximizer of $v(x) - x$. In his investment decision, the buyer randomizes according to the cumulative G, where $G(x) = 0$ for $x < 0$, $G(x) = 1 - \frac{v(0)}{v(x)}$ for $x \in [0, x^*)$ and $G(x) = 1$ for $x \geq x^*$.

Proof: See Appendix.

Proposition 1 establishes that in equilibrium, the payoffs to the players are the same as the equilibrium payoffs when the investment choice is observable. Problem 2.23 of Gibbons (1992) makes the same point in a game with only two investment levels.

For the case in which v is strictly concave and continuous, Proposition 1 also yields existence and uniqueness of the equilibrium and describes the equilibrium strategies.[3] While the unobservability of the investment decision alters the nature of equilibrium behavior, it does not change the equilibrium payoffs (i.e., the extent of inefficiency). The source of the inefficiency changes (underinvestment is reduced but the possibility of disagreement

[3] In fact, existence does not require any assumptions other than continuity of v. Note that if $v'(0) < \infty$, F has a discontinuity at $v(0)$. G has a discontinuity at x^*.

is added) but the amount of inefficiency is not decreased by the ability of the buyer to conceal his investment decision.

Next, we consider the following modifications of the games B^0 and B: Instead of a single take-it-or-leave-it offer, we permit the seller to make a new offer each time her offer is rejected. Hence, after the investment stage, the seller and buyer are engaged in an infinite horizon bargaining game with one-sided offers. The payoffs of the players are computed in the standard way: If an agreement is reached at price p, in period $k = 0, 1, 2...$ of the bargaining stage, then the payoff of the buyer is $[v(x) - p]e^{-k\Delta} - x$ while the payoff of the seller is $pe^{-k\Delta}$, where Δ is the time interval between successive offers. Hence, we normalize the time units so that the interest rate describing the players' impatience is 1. We will denote the infinite horizon versions of B^0 and B, $B^0(\Delta)$ and $B(\Delta)$ respectively. If no agreement is reached, the seller's and buyer's payoffs are 0 and $-x$, respectively.

The problem of bargaining with one-sided uncertainty and one-sided offers has been studied extensively. The only difference between the game $B(\Delta)$ and the game studied by Fudenberg, Levine and Tirole (1985) and GSW is the initial investment stage. In the earlier models asymmetric information is assumed while in $B(\Delta)$ it arises endogenously, due to the unobservable investment decision of the buyer. In the game $B^0(\Delta)$ the investment decision is observable. Therefore, there is complete information at the bargaining stage. It is easy to show and well-known in the bargaining literature that if one side makes all the offers in a complete information setting, then she gets all the surplus. To see this, suppose that p, the infimum of all prices ever offered in any equilibrium is strictly below v, the buyer's valuation. Then, any price strictly less than $(1 - e^{-\Delta})v + pe^{-\Delta}$ would be immediately accepted by the buyer. Which means no price strictly less than $(1 - e^{-\Delta})v + pe^{-\Delta}$ will be offered. So, $p \geq (1 - e^{-\Delta})v + pe^{-\Delta}$ which implies $p \geq v$, a contradiction. Since no price below v is ever charged, such a price would always be accepted if it were offered. This establishes that the equilibria of B^0 and $B^0(\Delta)$ are essentially identical. Hence, we have:

Proposition 2: *There is a unique subgame perfect equilibrium of $B^0(\Delta)$. This equilibrium yields the same outcome as the unique equilibrium of B^0; the buyer invests 0, the seller offers $v(0)$ in the initial period and the buyer accepts.*

Thus, neither the unobservability of investment nor the possibility of repeated offers alleviates the hold-up problem. In either case the equilibrium payoffs for the buyer and seller remain 0 and $v(0)$ respectively. However, in the next section we will show that the unobservability of the investment decision *together* with the possibility of repeated offers does remedy the hold-up problem.

3. A Resolution to the Hold-Up Problem

In this section we will study the game $B(\Delta)$ and provide the following result: Unobservable investment together with repeated bargaining yields efficiency when offers can be made arbitrarily frequently. Hence, the ultimatum game and the one-sided repeated offers game yield identical payoffs with complete information but the corresponding games lead to very different payoffs when the investment decision is unobservable. Propositions 4 and 5 rely on the analysis of the problem of one-sided bargaining with one-sided uncertainty due to GSW. The relevant results from that paper are summarized in Proposition 3 below. Let H be any distribution over valuations such that $H(0) = 0$, $H(v) = 1$ for some $v < \infty$ and let B_H denote the corresponding one-sided bargaining game with one-sided uncertainty.

GSW characterize all stationary equilibria of the game B_H. They show that stationary equilibria can be described by two functions, a function which determines the behavior of the buyer on and off the equilibrium path and a function that determines the behavior of the seller along the equilibrium path and after some off-equilibrium path histories. GSW also provide conditions under which all equilibria are stationary. Ausubel and Deneckere (1989) use a similar construction to study stationary equilibria. The definition below is closely related to their approach. The function q describes the buyer behavior, the function r describes the seller behavior after certain histories (in particular, along the equilibrium path) and the function Π describes the seller's expected payoff after those histories.

Definition: *The non-increasing, left-continuous functions* $q : \mathbf{R}_+ \to [0,1]$, $r : q(\mathbf{R}_+) \to \mathbf{R}_+$ *and the non-increasing, continuous function* $\Pi : q(\mathbf{R}_+) \cup \{0\} \to \mathbf{R}_+$ *are called a consistent collection if*

(*i*) $\Pi(q^0) := \sum_{k=0}^{\infty} [q^{k+1} - q^k] p^k e^{-k\Delta}$, *for all* $q^0 \in q(\mathbf{R}_+) \cup \{0\}$, *where* $p^k := r(q^k)$,

$q^{k+1} := q(p^k)$ *for* $k \geq 0$.

(ii) $p = r(q^0)$ *maximizes* $[q(p) - q^0]p + e^{-\Delta}\Pi(q(p))$ *for all* $q^0 \in q(\mathbf{R}_+)$.

(iii) $v - p = e^{-\Delta}[v - r(q(p))]$ *whenever* p *maximizes* $[q(p) - q^0]p + e^{-\Delta}\Pi(q(p))$ *for some* $q^0 \in q(\mathbf{R}_+) \cup \{0\}$ *and* $v = \sup\{v' \mid H(v') \leq 1 - q(p)\}$.

Definition: *A sequential equilibrium of* B_H *is stationary if there exists a consistent collection* q, r, Π *such that*

(i) If p^0 *is in the support of prices charged in the initial period then* $p = p^0$ *maximizes* $q(p)p + e^{-\Delta}\Pi(q(p))$.

(ii) If p^l *is the lowest price charged in some (possibly off-equilibrium path)* $k-1$ *period history and* $p^l = r(q^0)$ *for some* $q^0 \in [0,1]$, *then in period* k, *the price* $r(q(p^l))$ *is charged and the probability of agreement is* $\frac{q(r(q(p^l)))-q(p^l)}{1-q(p^l)}$.

It follows from the two definitions above that in a stationary sequential equilibrium, along the equilibrium path the seller does not randomize after the initial period.[4] The function r describes the seller's behavior along the equilibrium path and the function q describes the buyer's behavior on and off the equilibrium path. To see how q defines the behavoir of each type of the buyer, consider any p such that H is strictly incresing and continuous at v such that $H(v) = 1 - q(p)$. If the seller charges the price p, all buyer types $v' \geq v$ accept the current offer. All other types reject p.[5] The seller's payoff conditional on q^0 is $\frac{\Pi(q^0)}{1-q^0}$.

GSW show that given a consistent collection q, r, Π any pricing strategy that satisfies (i) in the definition of a stationary equilibrium and subsequently chooses prices according to r is a best response to the buyer strategy implied by q. Conversely, given a consistent collection, the buyer strategy implied by q is a best response to any seller strategy that satisfies (i) in the definition of a stationary sequential equilibrium and subsequently chooses prices according to r. Finally, GSW establish that given any consistent collection q, r, Π

[4] Ausubel and Deneckere (1989) call stationary equilibria, weak-Markov equilibria. If the function q is strictly increasing then they say that the equilibrium is a strong-Markov equilibrium. Both GSW and Ausubel and Deneckere (1989) use a slightly different description of stationary sequential equilibrium. The translation from the latter authors' definition to the one presented above is straightforward.

[5] Deriving the buyer's strategy from q is slightly more complicated (see the proof of Proposition 4) if H is not invertable in a neighborhood of $1 - q(p)$.

and an initial period pricing rule F satisfying (i), a stationary sequential equilibrium can be constructed by specifying off-equilibrium behavior appropriately. Hence, we sometimes refer to such F, q, r, Π as a stationary sequential equilibrium of B_H.

Proposition 3 (GSW): *Let $v^l > 0$ denote the lower boundary of the support of H (i.e., $v^l = sup\,\{v \mid H(v) = 0\}$) in the game B_H.*

(0) In any sequential equilibrium a price below v^l is never charged and the probability that agreement will never be reached is 0. After any history, if a buyer with valuation v accepts price p then all types with higher valuation also accept p.

If for every sequence $v_t > v^l$, v_t converges to v^l and $\lim \frac{H(v_t)}{v_t - v^l} = \alpha$ implies $\alpha > 0$, then:

(1) For some $K < \infty$, the probability that the game ends by period K is 1.

(2) There is a unique consistent collection q, r, Π.

(3) A sequential equilibrium exists and every sequential equilibrium is stationary.

If there exists a sequence $v_t > v^l$, converging to v^l such that $\lim \frac{H(v_t)}{v_t - v^l} = 0$, then:

(4) A price $p \leq v^l$ will not be charged after any history. Hence, for any $K < \infty$ the probability that the game ends after period K is strictly positive.

The claims (0)-(3) above are established in the proof of Theorem 1 in GSW. The proof of (4) is in the Appendix.

We would like to use Proposition 3 in our subsequent analysis. In particular, we would like to conclude that a sequential equilibrium of $B(\Delta)$ induces a sequential equilibrium of the game B_H where $H = G \circ v^{-1}$ and G is the equilibrium investment strategy of the buyer. This is not an immediate consequence of the definition of a sequential equilibrium. A deviation by the buyer, during the bargaining stage of $B(\Delta)$ may cause the seller to assign positive probability to a set of valuations not in the support of H. Such a situation cannot arise in a sequential equilibrium of B_H. Nevertheless, possible deviations by the buyer play no significant role in the analysis of either game and the buyer is the only player that has a strategic choice prior to the bargaining stage. Hence, it is fairly easy to show that Proposition 3 applies to the bargaining stage of the game $B(\Delta)$.

We refer to the collection G, F, q, r, Π as a strategy profile for $B(\Delta)$, where G is a probability distribution over investment levels, F is a probability distribution over initial period prices and q, r, Π is a consistent collection. Obviously, these are not the only strategy profiles one could have for the game $B(\Delta)$. However, in the proof of Proposition 5 it is shown that in any sequential equilibrium of $B(\Delta)$, $v_t > v^l$, v_t converges to v^l and $\lim H(v_t)/(v_t - v^l) = \alpha$ implies $\alpha > 0$, where H is the distribution of valuations at the bargaining stage and $v^l = sup\,\{v \mid H(v) = 0\}$. Hence, by Proposition 3, every sequential equilibrium of $B(\Delta)$ will indeed be of the form G, F, q, r, Π. Proposition 4 below establishes the existence of a sequential equilibrium for the game $B(\Delta)$.

Throughout the remainder of this section and in Section 4, we assume that v is increasing, strictly concave and continuously differentiable on $(0, M)$. Define $v'(0) := \lim_{x \to 0^+} v'(x)$. We also require that $v'(0) < \infty$.

Proposition 4: *The set of sequential equilibria of the game $B(\Delta)$ is non-empty. In any sequential equilibrium, the seller's payoff is least $v(0)$, the buyer's payoff is 0 and 0 is in the support of the buyer's investment strategy.*

Proof: The proof of existence of a sequential equilibrium is in the Appendix. Here we prove that the seller's equilibrium payoff is at least $v(0)$, the buyer's equilibrium payoff is 0 and 0 is in the support of the buyer's investment strategy.

Let $x^l := \sup\{x \mid G(x) = 0\}$. Clearly, $x^l \geq 0$. By an analogous argument to the one used in establishing Proposition 2, no price below $v(x^l)$ is charged in equilibrium and hence any such price would be accepted with probability 1 if it were offered. This establishes that the seller's payoff is at least $v(x^l) \geq v(0)$. Since x^l is in the support of G, it is an optimal choice of investment (see (2) in the proof of Proposition 1). Then, since no price below $v(x^l)$ is ever charged the equilibrium payoff to the buyer is at most $-x^l$. But 0 is an attainable payoff. Hence, $x^l = 0$. □

The proof of existence of a sequential equilibrium for the game $B(\Delta)$ is constructive. When v is twice continuously differentiable and $v'' < 0$, the equilibrium constructed has the following features: The buyer randomizes in his investment decision according to the continuous distribution G, where G has support $[0, x^*]$, is strictly increasing throughout its support and is piecewise differentiable. The efficient investment level x^* is the only point of discontinuity of G. The seller randomizes according to the distribution F, where F is continuous and strictly increasing throughout its support. Given the distribution of valuations H induced by the investment strategy G, the subsequent behavior of the agents constitute a stationary equilibrium of the resulting bargaining game B_H. The main task in the proof is to choose G and F so that the stationary equilibrium yields 0 utility for every investment decision in the support of G. When v is not twice continuously differentiable (or $v''(x) = 0$ for some $x \in (0, M)$), we construct a sequence of v_n that satisfies these properties that converge to v. Then, we show that the sequential equilibria for v_n converge to a sequential equilibrium for v.

Proposition 5 below states that as the time between offers becomes arbitrarily small, the equilibrium outcome converges to efficient investment and immediate agreement at a price that compensates the buyer fully for his investment ($p = v(x^*) - x^*$). To understand the result better, consider the following incorrect argument: As we know from Proposition 4, 0 must be in the support of the buyer's investment decision. Then, the Coase conjecture for the gap case (i.e., $v^l = v(0) > 0$) states that as the time between offers becomes arbitrarily small the first price charged in equilibrium must fall to $v(0)$. But this implies that the buyer should maximize $v(x) - v(0) - x$, that is; choose $x = x^*$ with probability 1. Hence, 0 is not in the support of the buyer's investment decision, a contradiction. Where is the flaw in this argument? We have already proven the first assertion, that 0 must be in the support of the buyers investment decision (Proposition 4). The final step is also correct; if the price were to fall to $v(0)$ instantaneously, then it would indeed be uniquely optimal for the buyer to invest x^*. What is incorrect is the appeal to the Coase conjecture which establishes that for a *given* distribution of buyers valuations, the price must fall to the lowest valuation in the support of the distribution, as the time between offers becomes arbitrarily small. But in the current setting, the distribution of buyer's valuations is not given exogenously but is determined *in equilibrium.* The Coase conjecture is not uniformly true over all distributions with the same lowest valuation in their support.

Proposition 5: *For every $\epsilon > 0$ there exists $\Delta^* > 0$ such that $\Delta < \Delta^*$ implies that in any sequential equilibrium of $B(\Delta)$ the probability of agreement by time ϵ (i.e., period ϵ/Δ) is at least $1 - \epsilon$ and the probability that the buyer's investment is in $[x^* - \epsilon, x^*]$ is* *at least $1-\epsilon$. That is, as the time between offers becomes arbitrarily small the sequential equilibrium outcomes become efficient and the seller extracts all of the surplus.*

Proof: Let G_n be the cumulative distribution describing the buyer's investment behavior in some sequential equilibrium σ_n of $B(\Delta_n)$. Thus, after the investment stage, the seller and buyer are engaged in a one-sided offer bargaining game with one-sided uncertainty as described in Proposition 3, where $H_n = G_n \circ v^{-1}$. By Proposition 4, $v(0) > 0$, is the lowest value in the support of H_n.

First, we show that for all $\Delta > 0$, in any sequential equilibrium of $B(\Delta)$, $v_t > v(0)$ converges to $v(0)$ and $\lim \frac{H(v_t)}{v_t - v(0)} = \alpha$ implies $\alpha > 0$. Suppose this condition is not satisfied. Let σ a sequential equilibrium of $B(\Delta)$. By (0) and (4) of Proposition 3, there exist a sequence k_j converging to infinity such that $v(x_{k_j}) > v(0)$, $v(x_{k_j})$ converges to $v(0)$, each $v(x_{k_j})$ is in the support of H (i.e., x_{k_j} is in the support of G) and the buyer with valuation $v(x_{k_j})$ buys in period k_j. By Proposition 4, the buyer's equilibrium payoff is 0 and since a price below $v(0)$ is never charged (Proposition 3), we have, $[v(x_{k_j}) - v(0)]e^{-k_j\Delta} - x_{k_j} \geq 0$ for all k_j. Re-arranging terms in the above inequality yields $\lim_{j\to\infty} \frac{v(x_{k_j})-v(0)}{x_{k_j}} e^{-k_j\Delta} \geq 1$. But the first term on the left-hand side goes to $v'(0) < \infty$ and the second term goes to zero; hence the product goes to zero, a contradiction.

It follows from the above argument that $(0) - (3)$ of Proposition 3 applies. Suppose that the first assertion of the Proposition is false. Then, there exists $\epsilon > 0$, a sequence Δ_n converging to 0 and a corresponding sequence of equilibria σ_n of $B(\Delta_n)$, such that in all of these equilibria, the probability of agreement by time ϵ is no greater than $1-\epsilon$. Let q_n, r_n, Π_n denote the consistent collection associated with σ_n.

Re-define q_n at its discontinuity points so that it is right-continuous. Then, each $1-q_n$ is a probability distribution and these distributions have uniformly bounded supports (each support is contained in $[0, M]$), by Helly's Selection Theorem (see Billingsley (1986) Theorem 25.9 and Theorem 25.10), there exists a function q and a subsequence n_j such that the subsequence $q_{\Delta_{n_j}}$ converges to q weakly (i.e., $1-q_{n_j}$ converges to $1-q$ in distribution). Again, without loss of generality, we take this subsequence to be the sequence itself. We now show that the maximized value of Π^{q_n} converges to $\int_{[v(0),\infty)} p d(1-q)(p)$. To see this, first assume that $q_{\Delta_n} = q$ for all n. Then, as the time between offers becomes arbitrarily small, the seller can price-discriminate arbitrarily finely in an arbitrarily small interval of real time. Hence, she is able to, in the limit, extract all of the area under the "demand". When q_n is not equal to q but converges to it, the argument is still the same: For any ϵ' the seller can find a finite sequence of k prices such that if these prices were to be charged in the first k periods and the buyer behaves according to q the expected revenue of the seller (ignoring discounting) would be within ϵ' of $\int\limits_{[v(0),\infty)} pd(1-q)(p)$. Then, by lowering each price by ϵ', for n sufficiently large, the probability of sale in the first k periods can be kept at least as high as against q_n as it was against q with the original sequence of prices. In the limit, the effect of discounting goes away and expected revenue is at least $\int\limits_{[v(0),\infty)} pd(1-q)(p) - \epsilon'$. Since this is true for all ϵ', the result follows.

But under our hypothesis that the probability of agreement by time ϵ is at most $1-\epsilon$, the limiting equilibrium payoff of the seller is at most $\int\limits_{[v(0),\infty)} pd(1-q)(p) - \epsilon v(0)(1-e^{-\epsilon})$. Thus, the equilibrium strategy that results in an ϵ probability of disagreement until time ϵ yields a payoff bounded above by $\int\limits_{[v(0),\infty)} pd(1-q)(p) - \epsilon v(0)(1-e^{-\epsilon})$ when an alternative strategy that can extract all surplus and achieve a payoff arbitrarily close to $\int\limits_{[v(0),\infty)} pd(1-q)(p)$ exists, a contradiction. So the first assertion is established.

A buyer who plans to purchase at time t will choose his investment level to maximize $E[(v(x)-p(t))e^{-t}]-x$, where $p(t)$ is the price charged at time t and the expectation is over the possible randomization of the seller in period 0. As t approaches 0, the maximizing level of investment x approaches x^*. It follows from the first part of the proof that in the limit, the equilibrium investment behavior of the buyer must converge to the optimal investment level conditional on buying at time 0 (i.e., x^*). By Proposition 4, the buyer's payoff is 0. Hence, the final assertion of the Proposition follows. □

To understand why Proposition 5 holds, consider the simpler case in which the buyer has only two options: He can either invest 0 or $x^* > 0$. As noted in Proposition 4, in equilibrium, the buyer must choose 0 with positive probability and his utility must be 0. As the time between offers goes to 0, the Coasian effect precludes delay and would force the initial price to $v(0)$ if the probability of investing 0 did not go to zero. But, this would mean that investing x^* with probability 1 yields strictly positive utility to the buyer which we argued cannot be. Consequently, the probability of investing 0 must go to 0 as the time between offers goes to 0. The key observation is that when the time between offers goes to zero, it may take a positive amount of time to ensure that trade takes place with probability 1, but the Coasian effect still guarantees that the *expected* time until trade approaches zero.

It is of some interest to figure out what the behavior of the buyer (i.e., q) is like and in particular, how this behavior prevents the seller from succumbing to the temptation of running down the buyer's demand curve in the "blink of an eye". As the time between offers becomes arbitrarily small, the probability of the buyer purchasing the good at the price to be charged at time t becomes arbitrarily large compared to the probability of purchasing at the price to be charged at time $t + \epsilon$. This is true in spite of the fact that prior probability of purchase at or after time t is going to zero for all t greater than 0. Thus, conditional on not reaching agreement prior to time t, the probability of the game ending almost immediately after t is arbitrarily close to 1. This makes it not worthwhile for the seller to try to speed up the process.

Proposition 5 has a peculiar observational implication: In situations satisfying the assumptions of the Proposition, in equilibrium, nearly always the buyer will invest nearly efficiently and nearly always trade will take place almost immediately at a price that covers the (sunk) cost of investment. An observer might be inclined to conclude that the compensation of the buyer's investment by the seller is due to some extra-strategic notion of fairness or a possible repeated game effect supporting such a norm. However, in our model, a purely strategic one-shot relationship is able to support this apparently paradoxical outcome.

4. Uncertain Gains From Trade

A crucial assumption in the preceding analysis was $v(0) > c$ where c, the constant marginal cost of production was normalized to 0. Hence, we had assumed strictly positive gains from trade, even with 0 investment. Relaxing this assumption is important not only because the case of uncertain gains from trade is of some interest but also to facilitate comparisons between the current work and the incomplete contracts literature since much of this literature focuses on this case of uncertain gains from trade.

To allow for uncertain gains from trade, we modify the model of Section 3 by assuming that the cost of production C, is random. We assume that the random variable C is non-negative and has finite support. We also assume that the cost of production is incurred by the seller at the time of agreement. When the realization of C is above $v(x)$ there are no gains from trade. The realization of C is observed by both agents prior to the bargaining stage. Let $B^C(\Delta)$ denote the game with uncertain gains from trade.

Let S denote the expected gains from trade as a function of the buyer's level of investment. That is, $S(x) = \sum_{c<v(x)}(v(x) - c)Prob\{C = c\} - x$. Note that since v is continuously differentiable, S is continuously differentiable at all x such that $Prob\{C = v(x)\} = 0$. It is easy to verify that the left-derivative of S at x is $v'(x)Prob\{C < v(x)\} - 1$. Let $X_C := \{x \mid Prob\{C = v(x)\} > 0\}$ and $X^* := \{x \mid v'(x)Prob\{C < v(x)\} = 1\}$. We make the following assumption:

Assumption A: *(i) $X_C \cap (X^* \cup \{0\}) = \emptyset$. (ii) S is strictly quasi-concave.*

Part (i) of Assumption A is a genericity requirement. For any S that fails (i) there exist $\epsilon > 0$ such that replacing C with $C - \zeta$ for any $\zeta \in (0, \epsilon)$ ensures that (i) is satisfied. Moreover, if C satisfies (ii) then ϵ can be chosen sufficiently small so that $C - \zeta$ satisfies both (i) and (ii). By (ii), there is a unique maximizer x^*, of S. Since $S'(x) < 0$ for all $x < \min X_C$ and $0 \notin X_C$, (ii) of Assumption A implies that either $x^* = 0$ or $Prob\{C < v(0)\} > 0$. The other important consequence of Assumption A is that it ensures that the left-derivative of S is strictly positive on $(0, x^*)$

We prove below, a result analogous to Proposition 5 and show that as the the time between offers becomes arbitrarily small, the buyer's investment strategy converges to

x^* and the probability of agreement by time ϵ converges to $Prob\{C < v(x^*)\}$. That is, investment and trade become efficient as the time between offers converges to 0.

In the current setting, it could be that at the bargaining stage, the probability that the buyer's valuation is in the interval $(c, c+\epsilon)$ is strictly positive for all positive ϵ. Hence, we are in what is called the "no-gap" case of the bargaining problem.[6] It is well-known that the Coase conjecture is not valid in the no-gap case without further assumptions. For the gap case, Proposition 3 guarantees that sequential equilibria are stationary. To prove the Coase conjecture in the no-gap case, GSW restrict attention to stationary sequential equilibria of B_H. The existence of stationary sequential equilibria of B_H, for the no-gap case is established by Ausubel and Deneckere (1989). We take the same approach as GSW and restrict attention to sequential equilibria of $B^C(\Delta)$ that specify a stationary equilibrium for the bargaining stage.

A sequential equilibrium σ of $B^C(\Delta)$ is stationary if for each c in the support of C, σ specifies a strategy profile that constitutes a stationary sequential equilibrium of the bargaining stage given cost c. In the no-gap case, the consistent collection q^c, r^c, Π^c associated with a given investment decision and cost c need not be unique.

Proposition 6: *Suppose the game $B^C(\Delta)$ satisfies Assumption A. Then, for every $\epsilon > 0$, there exists $\Delta^* > 0$ such that $\Delta < \Delta^*$ implies that in any stationary sequential equilibrium of $B^C(\Delta)$, the probability that the buyer's investment is in $[x^* - \epsilon, x^*]$ is at least $1 - \epsilon$. Moreover, conditional on $C < v(x^*)$, the probability of agreement by time ϵ is at least $1 - \epsilon$. That is, as the time between offers becomes arbitrarily small the stationary sequential equilibrium outcomes become efficient and the seller extracts all of the surplus.*

Proof: See Appendix.

To see why the restriction to stationary sequential equilibria is necessary, note that the later the buyer expects to trade, the lower will be his optimal level of investment. But when there is no-gap, it is known that many equilibria can be sustained, including ones in which there is substantial delay.[7]

---

[6] See GSW and Ausubel and Deneckere (1989).

[7] Ausubel and Deneckere (1989) prove that in the no-gap case, virtually any level of inefficiency and any division of surplus can be sustained with non-stationary sequential equilibria.

To see why assumption A is necessary, suppose that C has two elements in its support $c_1 > c_0$. Let x^* be the unique maximizer of S and x^{**} maximize $Prob\{C = c_0\}(v(x) - c_0) - x$. Hence, x^* is the efficient investment level while x^{**} is the efficient level of investment in the game $B^{\hat{C}}(\Delta)$ where $Prob\{\hat{C} = c_0\} = Prob\{C = c_0\}$ and $Prob\{\hat{C} = M\} = Prob\{C = c_1\}$. The game $B^{\hat{C}}(\Delta)$ is just like the game $B^C(\Delta)$ except that instead of c_1, $\hat{C}$ takes on a value that precludes gains from trade. In general, it is possible for x^{**} to be strictly less than x^*. Moreover, if assumption A is not satisfied then it is possible that $v(x^{**}) \leq c_1$. Assume that the buyer invests according to some stationary sequential equilibrium $\hat{\sigma}$ of $B^{\hat{C}}(\Delta)$. Suppose that if $C = c_1$ the seller charges M in every period and the buyer never accepts and if $C = c_0$ then both the seller and the buyer play as they would in $\hat{\sigma}$ after observing c_0. It is easy to see that this strategy profile is a stationary sequential equilibrium of $B^C(\Delta)$. As Δ goes to zero, investment in this type of equilibrium converges to x^{**} which is an inefficient local maximizer of S. Assumption A rules out this situation by ensuring that $v(x^{**}) > c_1$.

To prove Proposition 6, we consider a convergent sequence of equilibrium outcomes as Δ approaches zero. The finiteness of the support of C ensures the existence such a sequence. To see how the proof of Proposition 6 works, let G be the limiting distribution of the buyer's investment decision. If x is in the support of G, the buyer who invests x should not be able to increase his payoff with a small change in his investment without changing his buying strategy. In a stationary sequential equilibrium, the analysis of Proposition 5 suffices to show that the expected time until agreement is reached conditional on strictly positive gains from trade converges to 0. With unobservable investment the buyer invests efficiently given the probability and timing of trade. Hence, if $S'(x)$ is well-defined and x in the support of G then $S'(x) = 0$. If $S'(x)$ is not well-defined and x is in the support of G then the left-derivative of S at x must be 0. Given Assumption A, the only G consistent with this requirement is the distribution with unit mass at x^*. Finally, the argument used in Proposition 4 establishes that in any stationary sequential equilibrium of $B^C(\Delta)$ the buyer's payoff is zero and hence the seller extracts all surplus.

Proposition 6 enables us to replace the requirement $v(0) > 0$ used in Proposition 5 with the weaker requirement that there should be *some* chance that C is less than $v(0)$. Proposition 7 below shows that this condition is essential.

Proposition 7: *If* $Prob\{C \geq v(0)\} = 1$ *then in any sequential equilibrium of* $B^C(\Delta)$ *the buyer invests 0 and both players receive 0 utility.*

Proof: See Appendix.

5. Two-Sided Investment

In this section we consider a game in which both the buyer and the seller invest prior to the bargaining stage. For simplicity, we assume that the buyer invests either 0 or $x^* > 0$, while the seller invests either 0 or $y^* > 0$. As in the previous sections, the investment of the buyer x, determines his valuation v. Now, the investment of the seller y, determines her constant marginal cost of production c. In order to avoid trivial cases, we assume that 0 investment is inefficient for both agents. Hence $v^* - v^0 - x^* > 0$ and $c^0 - c^* - y^* > 0$ where v^0, c^0 denote the valuation and cost associated with 0 investment and v^*, c^* denote the corresponding values for positive investment. Also, we assume $v^0 > c^0$. Hence, it is common knowledge that there are strictly positive gains from trade.

We consider two different extensive form games. In the first game, the buyer makes his unobservable investment *after* learning the investment decision of the seller. Then, the bargaining stage begins. The bargaining stage is the same repeated offer, infinite horizon game studied in Sections 3 and 4. In the second game, the seller and buyer invest simultaneously, prior to the bargaining stage. The investment of the seller is observable. Hence, only the buyer has private information during the bargaining stage. The first game enables the seller to commit to a particular level of investment while the second does not.

Let $B^{sb}(\Delta)$ denote the bargaining game with two-sided investment where the seller invests first and let $B^2(\Delta)$ denote the game in which investment decisions are made simultaneously. Proposition 8 below, establishes that the reasoning of Proposition 5 carries over to the case of two-sided investment if the seller can commit to her investment level. By Proposition 5, no matter what the seller invests, the buyer's investment will be efficient and the seller will extract all surplus as Δ approaches 1. This implies that the unique optimal action of the seller is to invest y^*. Formally, Proposition 5 considers only the case in which the set of investment decisions for the buyer is an interval. However, neither the proof of Proposition 5 nor Proposition 3 from GSW require a continuum of types. Given Proposition 5, the proof of Proposition 8 below is straightforward and omitted.

Proposition 8: *For every $\epsilon > 0$ there exists $\Delta^* > 0$ such that $\Delta < \Delta^*$ implies that in any sequential equilibrium of $B^{sb}(\Delta)$ the probability of agreement by time ϵ is at least $1 - \epsilon$, the probability that the seller invests y^* is 1 and the probability that the buyer invests x^* is at least $1 - \epsilon$. That is, as the time between offers becomes arbitrarily small, the sequential equilibrium outcomes are efficient and the seller extracts all the surplus.*

In contrast to Proposition 8, there is a potential source of inefficiency when investment decisions are made simultaneously. To see this, note that a higher constant marginal cost of production renders the seller less impatient to run down the buyer's demand curve (i.e., q^c). Thus, for a fixed investment strategy of the buyer, a higher c results in a higher initial price and hence, higher expected revenue in equilibrium. But this means that some of the cost of the inefficient choice of y is passed on to the buyer. Consequently, the seller has an incentive to underinvest. Therefore, when the efficiency gain from the seller's investment, $c^0 - c^* - y^*$ is sufficiently small, she will not invest at all.

Proposition 9: *Fix v^0, v^*, x^*, c^0, c^*. Then, there exists $\delta > 0$ such that for all $y^* > c^0 - c^* - \delta$ and $\epsilon > 0$, there exists $\Delta^* > 0$ such that $\Delta < \Delta^*$ implies that in any sequential equilibrium of $B^2(\Delta)$, the probability of agreement by time ϵ is at least $1-\epsilon$, the probability that the seller invests 0 is 1 and the probability that the buyer invests x^* is at least $1 - \epsilon$.*

Proof: See Appendix.

In Proposition 9, the need for the efficiency gain to be small is an artifact of the discrete investment choice. Presumably, if v were a differentiable function of y, the equilibrium investment level would be bounded away from the efficient investment y^*. However, the proof of Proposition 9 entails constructing the consistent collection associated with any possible investment strategy of the buyer. This task is not feasible when the investment choice is a continuous variable.

6. Conclusion

The result that unobservable investment and the information rents it creates may provide sufficient incentives for optimal investment, even when the agent investing has no bargaining power, appears to be robust to a number of extensions or modifications of our basic model.[8] We conclude by discussing a few other possible extensions and speculate on the implications of our results for the problem of organization design under incomplete contracting.

If the buyer's valuation given his investment is random, then it could be possible for the buyer to enjoy strictly positive surplus in equilibrium. For example, suppose that the cost $c = 0$ is known. Assume that there are two possible investment levels, 0 and the efficient level $x^* > 0$. Let $v^0 > 0$ be the deterministic valuation that results from 0 investment. However, assume that x^* leads to the random valuation V^* with support $[a, b]$. Suppose that only the buyer observes the realization of V^* (prior to the bargaining stage) and $v^0 < a < x^*$. Then, as the time between offers goes to 0, every sequential equilibrium outcome will be efficient and yield the buyer expected utility equal to the entire surplus minus a (i.e., $E[V^*] - x^* - a$). Hence, the salient feature of Proposition 5 is the tendency towards immediate agreement and efficient investment, not the fact that the buyer achieves 0 surplus.

Next, consider the possibility of the buyer making offers. The literature on bargaining provides much fewer general results when privately informed agents make repeated offers. It is often assumed that many outcomes can be sustained in such situations. Nevertheless if, as is sometimes done in the bargaining literature, a refinement criterion that guarantees immediate agreement is imposed, then the conclusions of Proposition 5 would carry over to this setting.[9] More generally, whenever the framework precludes "a folk theorem" in the spirit of Ausubel and Deneckere (1989), unobservable investment will tend to guarantee some rents to the investor. The first objective of this paper has been to demonstrate how such rents can be sustained in equilibrium, and even lead to efficient investment.

The literature on incomplete contracting considers the allocation of *ex post* bargaining power, either through the allocation of property rights or through the detailed specification

[8] Note that in Proposition 9, it is the player with the observable decision that is investing inefficiently.
[9] See for example, Gul and Sonnenschein (1988).

of the non-cooperative renegotiation game, as the proper tool for controlling the *ex ante* incentives to invest. Our second objective has been to suggest the possible use of information regarding investment as an alternative tool for organizational design and remedy for the hold-up problem. This approach may be particularly useful when contracting and enforcement is difficult and re-allocating bargaining power through organizational design is costly. In general, when more than one agent has to undertake specific investment, a given institutional arrangement may give some agents too little incentive to invest while it gives others too much. Changing the flow of information by increasing monitoring and hence forcing one agent to disclose, perhaps partially, what she knows or granting her the right to withhold such information, will influence the rents enjoyed by all agents and hence their incentives to invest. Comparing the example with a random buyer valuation above, with the model of Section 3 suggests that by altering the flow of information both efficiency and a wide range of distributions of surplus may be achieved.

7. Appendix

Proof of Proposition 1: Let $p^l = \sup\{p \,|\, F(p) = 0\}$ and $p^h = \inf\{p \,|\, F(p) = 1\}$. Define x^l and x^h for G in an analogous fashion. Let z be called a point of increase of a cumulative distribution function if either z is point of discontinuity or if for every $\epsilon > 0, H(z+\epsilon) > H(z)$. First, we will make a number of simple observations:

(1) In equilibrium, the buyer will always accept any offer below his valuation and reject any offer above his valuation. The seller will not charge a price below $v(x^l)$ or above $v(x^h)$.

(2) If z is a point of increase of F (G) then z is an optimal strategy for the seller (buyer).

(3) $x^l = 0$ and $p^l = v(0)$.

Moreover, if v is strictly concave and continuous, then,

(4) $x^h = x^*$ and if $x \in [0, x^*)$ then x is a point of increase of G.

The proofs of (1)-(4) are straightforward and are omitted. Since x^l and p^l are points of increase, the first sentence of the Proposition follows from (2) and (3). (1) establishes the second sentence.

To conclude the proof, note that since the equilibrium payoff of the buyer is 0, (2) and (4) imply $\int\limits_{v(0)}^{v(x)} (v(x) - p)dF(p) - x = 0$ for all $x \in [0, x^*)$ establishing that F is the desired function. But this implies, again by (2), that every $p \in [v(0), v(x^*))$ is optimal for the seller. Hence $(1 - G(x))v(x) = v(0)$, yielding the desired G. □

Proof of Part (4) of Proposition 3: Let $\nu(p) = [p - v(0)e^{-\Delta}]/[1 - e^{-\Delta}]$ and note that since a price below $v(0)$ is never charged, in any equilibrium, a buyer with valuation $v > \nu(p)$ will accept price p immediately. Let $\bar{H}$ be the distribution of valuations after some history h. By part (0) of Proposition 3, a buyer type $v' < v$ will never purchase before a buyer type v. Therefore, after history h, there is a valuation $\bar{v}$ and α, such that $H^-(\bar{v}) \leq \alpha \leq H(\bar{v})$, $\bar{H}(v) = 1$ for all $v \geq \bar{v}$ and $\bar{H}(v) = H(v)/\alpha$ for all $v < \bar{v}$, where $H^-(\bar{v})$ is the left-limit of H at $\bar{v}$. We will refer to h as a history with state $\bar{v}$. Suppose that the price $v(0)$ has not been charged in the past. Hence, $H^-(\bar{v}) > 0$. The payoff to the seller, after a history with state $\bar{v}$ is at least

$$\pi_l(\bar{v}, p) = [1 - \bar{H}(\nu(p))]p$$

Straightforward calculations yield,

$$\frac{\pi_l(\bar{v}, p) - v(0)}{p - v(0)} = 1 - \frac{p\bar{H}(\nu(p))}{p - v(0)} \geq 1 - \frac{pH(\nu(p))}{H^-(\bar{v})[p - v(0)]}$$

Note that since $\nu' = 1/(1 - e^{-\Delta})$ and by assumption, $\lim H(v_t)/[v_t - v(0)] = 0$ for some sequence v_t, we can find $p > v(0)$ such that $1 - \frac{pH(\nu(p))}{H^-(\bar{v})[p - v(0)]} > 0$. Then, $\pi_l(v, p) - v(0) > 0$. So, after any history in which $v(0)$ has never been charged, there exists some $p > 0$ that yields a higher payoff to the seller than $v(0)$. □

Proof of Proposition 4 (Existence of a Sequential Equilibrium):

Since v is concave, if $v'(0) \leq 1$ then the buyer investing 0 with probability 1 and accepting any offer less than or equal to $v(0)$ and the seller asking $p = v(0)$ after every history is clearly an equilibrium strategy profile. Henceforth, we assume $v'(0) > 1$.

A stationary equilibrium of the game B_Δ consists of G a distribution of investment decisions which yields a distribution of valuations H, an initial period pricing strategy F and a consistent collection (given H) q, r, Π. To ensure that G, F, q, r, Π is a stationary equilibrium of B_Δ we need to verify that q, r, Π is a consistent collection given the distribution of valuations induced by G, the sellers first period pricing strategy F is optimal given q, r, Π and G is optimal given F, q, r, Π. In our proof, we assume that G is piecewise differentiable and strictly increasing. Then, we construct F, q, r, Π with the desired properties which determines an equation defining the equilibrium G. Finally, we provide a solution to this equation.

Let $\delta := e^{-\Delta}$. For any v satisfying the hypothesis of the Proposition, let $T = \min\{k \geq 1 \,|\, \delta^k \leq \frac{1}{v'(0)}\}$. Since $v'(0) > 1$, $T \geq 1$ is well-defined. In the equilibrium we construct, bargaining will continue for at most $T + 1$ periods. In constructing the equilibrium we consider two cases: First, we assume that v is twice continuously differentiable. This makes it possible to characterize a particular equilibrium investment strategy by T differential equations. Then, we use the fact that twice continuously differentiable functions are dense within the set of v's covered by the Theorem to construct an equilibrium for all such v.

In Theorem 3 and the related definition of a consistent collection, q^0, the probability of acceptance until the current period, is the state variable. In constructing an equilibrium for the entire game (i.e. including the investment stage) it is more convenient to use as the state variable the investment decision x rather than q^0. Hence, instead of H the distribution of valuations, we utilize G, the distribution of investment decision. Similarly, instead of r which determines the price charged in the current period given state q^0, we use ρ which determines state in the next period as a function of the current state x. Finally, instead of F we use $\hat{F}$ and instead of q we use P to complete the translation from the state space of q^0's to x's.

Define $\rho, \eta : [0, M] \to [0, M]$ as follows: $\rho(x) = 0$ if $\frac{v'(x)}{\delta} \geq v'(0)$ and $\rho(x) = v'^{-1}(\frac{v'(x)}{\delta})$ otherwise. Let $\bar{x} := \max\{z \,|\, \rho(z) = 0\}$, $\eta(0) = \bar{x}$, $\eta(x) = \rho^{-1}(x)$ for all $x \in (0, \rho(x^*))$ and $\eta(x) := x^*$ for all $x \in [\rho(x^*), M]$. Since ρ is continuous everywhere and strictly increasing on $[\bar{x}, M]$, η is well-defined. Moreover, η is continuous everywhere and is strictly increasing on $[0, \rho(x^*)]$. At any state less than $\bar{x}$, the seller makes an offer that is accepted with probability 1, (i.e. $p = v(0)$).

Let $\rho^0(x) := x$ for all $x \in [0, M]$ and for $t \geq 1$, let $\rho^t(x) := \rho(\rho^{t-1}(x))$. Define η^t in a similar fashion. The function $P : [0, M] \to \mathbb{R}$ describes the highest price the buyer accepts if he has invested x. This function is defined as follows: $P(x) = (1-\delta)\sum_{t=0}^{\infty} v(\rho^t(x))\delta^t$. The function P is continuous and strictly increasing everywhere. Next, we describe the the seller's pricing strategy in period 1. The distribution $\hat{F}$ has support $[\rho(x^*), x^*]$ and $1-\hat{F}(x)$ is the probability that x will be the state at the end of period 1. $\hat{F}$ is defined as follows: $\hat{F}(x) = 0$ if $x < \rho(x^*)$, $\hat{F}(x) = \frac{1}{(1-\delta)v'(x)} - \frac{\delta}{1-\delta}$ if $x \in [\rho(x^*), x^*]$ and $\hat{F}(x) = 1$ if $x > x^*$.

For any G that is continuous and strictly increasing in the interval $[0, x^*)$ such that $G(0) = 0$, $G(x^*) = 1$, define $\hat{\Pi}(x)$, the expected present value of profit conditional on x as follows: $\hat{\Pi}(x) := \frac{1}{G^-(x)}\sum_{t=0}^{\infty}[G^-(\rho^t(x)) - G(\rho^{t+1}(x))]P(\rho^{t+1}(x))\delta^t$ for $x \in (0, x^*]$ and $\hat{\Pi}(0) = v(0)$ where G^- denotes the left-limit of the function G. Note that $G^-(x) = G(x)$ whenever $x \neq x^*$. The expected present value of profit at the start of the bargaining stage is denoted $\hat{\Pi}(M) := [1 - G^-(x^*)]P(x^*) + \delta\Pi(x^*)$.

Next, we describe how F, q, r, Π can be derived from $\hat{F}, P, \rho, \hat{\Pi}$: For $p \leq P(\rho(x^*))$, $F(p) := 0$; for $p \in (P(\rho(x^*)), P(x^*))$, $F(p) := \hat{F}(P^{-1}(p))$ and for $p \geq P(x^*)$, $F(p) := 1$. For $p > P(x^*)$, $q(p) := 0$; $q(P(x^*)) = 1 - G^-(x^*)$; for $p \in (v(0), P(x^*))$, $q(p) := 1 - G(P^{-1}(p))$ and for $p \leq v(0)$, $q(p) := 1$. Let $r(1 - G^-(x^*)) := P(\rho(x^*))$; for $q^0 \in (1 - G^-(x^*), 1)$, $r(q^0) := P(\rho(G^{-1}(1 - q^0)))$ and $r(1) := v(0)$. Define $\Pi(0) := \hat{\Pi}(M)$; $\Pi(1 - G^-(x^*)) = \hat{\Pi}(x^*)$; for $q^0 \in (1 - G^-(x^*), 1)$, $\Pi(q^0) := \hat{\Pi}(G^{-1}(1 - q^0))$ and $\Pi(1) := v(0)$. We seek a G that ensures that the resulting G, F, q, r, Π is a sequential equilibrium of $B(\Delta)$. In Step 1 below, we identify the conditions that define the equilibrium G. In Step 2, we prove the existence of G satisfying these conditions when v is twice continuously differentiable and $v'' < 0$. In Steps $3 - 5$ we take care of the case in which v is merely strictly concave and continuously differentiable.

Step 1: Suppose,

A) G is strictly increasing and continuous on the interval $[0, x^*)$, $G(0) = 0, G(x^*) = 1$.

B) For all $x \in (0, x^*)$, $\hat{\Pi}(x) \geq \frac{G(x)-G(y)}{G(x)}P(y) + \delta\frac{G(y)}{G(x)}\hat{\Pi}(y)$ for all $y \in [0, x]$.

C) $[1 - G(x)]P(x) + \delta G(x)\hat{\Pi}(x)$ is constant for all $x \in [\rho(x^*), x^*)$.

Then, G, F, q, r, Π is a sequential equilibrium of $B(\Delta)$.

Proof: Note that (B) above is necessary for the optimality of the seller behavior after the initial period while (C) is necessary for the optimality of the initial period randomization.

Clearly, if (A) is satisfied q, r, Π are well-defined, q, r are left-continuous, Π is continuous except at 0 and q is non-increasing. Since ρ is non-decreasing, r is non-increasing. For $y \geq x$,

$$\begin{aligned}\hat{\Pi}(x) &= \left[\frac{G(x) - G(\rho(x))}{G(x)} P(\rho(x)) + \delta \frac{G(\rho(x))}{G(x)} \hat{\Pi}(\rho(x))\right] \\ &\leq \left[\frac{G(y) - G(\rho(x))}{G(y)} P(\rho(x)) + \delta \frac{G(\rho(x))}{G(y)} \hat{\Pi}(\rho(x))\right] \\ &\leq \hat{\Pi}(y)\end{aligned}$$

The first line above holds by definition, the second line follows from the fact that P is increasing so $P(z) \geq \hat{\Pi}(z)$ for all z and the last inequality follows from (B). Hence, $\hat{\Pi}$ is non-decreasing, so Π is non-increasing. By construction (i) and (iii) in the definition of a consistent collection are satisfied.

To conclude Step 1, it remains to show that the buyer's investment strategy is optimal given the specified behavior afterward. Note that given the specified behavior of the seller, $V(x, y)$, the expected utility of investing x and accepting any offer at or below $P(y)$ is

$$V(x, y) := \int_{\rho^t(z) \leq y} [v(x) - P(\rho^{t-1}(z))]\delta^{t-1} d\hat{F}(z) + \int_{\rho^t(z) > y} [v(x) - P(\rho^t(z))]\delta^t d\hat{F}(z) - x$$

whenever $y \in (\rho^{t+1}(x^*), \rho^t(x^*)]$. Let $W(x)$ be the equilibrium payoff of a buyer who invests x. That is, $W(x) := V(x, x)$. Both V and W are continuous. The definitions of P and ρ ensure that for any realization of x^0 in the initial period, along the price sequence $P(\rho^t(x^0))$, we have

$$v(\rho^t(x^0)) - P(\rho^t(x^0)) = \delta[v((\rho^t(x^0)) - P(\rho^{t+1}(x^0))]$$

Hence conditional on any investment level $x \in [0, x^*]$ it is optimal to buy in period $t - 1$ if $x \in [\rho^t(x^0), x^*]$ and in period t if $x \in [\rho^t(x^*), \rho^{t-1}(x^0))$. That is;

$$V(x, x) - V(x, y) \geq 0 \tag{1}$$

Take any $y \in (\rho^{t+1}(x^*), \rho^t(x^*))$ for $t < T$. Let $x \in [0, M]$ and $\rho^t(y^1) = y$ for some $y^1 \in (\rho(x^*), x^*)$. Simple manipulations of the above expression for V yield

$$V(y, y) - V(x, y) = [v(y) - v(x)]\delta^t[\hat{F}(y^1) + (1 - \hat{F}(y^1))\delta] - y + x$$

Since $y = \rho^t(y^1)$, $v'(y) = v'(\rho^t(y^1)) = \frac{v'(y^1)}{\delta^t}$ and v is concave with $v' \geq 1$, we have $\frac{v(y)-v(x)}{y-x} \geq 1$. Hence, the above expression implies that $V(y,y) - V(x,y) \geq 0$ for all $y \in (0, x^*]$, $x \in [0, M]$ and $\lim_{x \to y} \frac{V(y,x)-V(x,x)}{y-x} = \lim_{x \to y} \frac{V(y,y)-V(x,y)}{y-x} = 0$ whenever $x \in (0, x^*]$. Then, by (1)

$$V(y,x) - V(x,x) \leq W(y) - W(x) \leq V(y,y) - V(x,y)$$

Dividing all terms by $y - x$ and taking a limit as x goes to y establishes that $W'(y) = 0$. Since W is continuous and has zero derivative outside of a finite set, W is constant on $[0, x^*]$. Note that conditional on investing more than x^*, the optimal behavior in the bargaining stage is to buy in the initial period at whatever price the seller offers. Recall that given G, the bargaining behavior is optimal. Hence, the optimality of G follows from $W(x) = W(x^*) \geq V(y, x^*)$ for all $x \in [0, x^*]$, $y \in (x^*, M]$ and the fact that $[0, x^*]$ is the support of G.

Given Step 1, the task of establishing existence is reduced to finding a probability distribution G satisfying $(A), (B)$ and (C) above. For the case of a twice continuously differentiable v, we will ensure that (B) and (C) are satisfied by finding G that solves the corresponding differential equations (B^*) and (C^*) below.

Call v neoclassical if v is twice continuously differentiable, $v'' < 0$ on $(0, M)$ and $\lim_{x \to 0^+} v''(x) < 0$. Let $Z^0 := \{z \in [0, M] \,|\, v'(z) = \delta^t v'(0) \text{ for some } t \geq 0\}$, $Z^* := \{z \in [0, x^*] \,|\, \delta^t v'(z) = v'(x^*) \text{ for some } t \geq 0\}$ and $X^0 := (0, M) \backslash [Z^0 \cup Z^*]$. Clearly, $Z^0 \cup Z^*$ is a finite set and $x \in X^0$ implies $\eta(x) \in X^0$ and either $\rho(x) = 0$ or $\rho(x) \in X^0$. Note that when v is neoclassical, ρ and P are differentiable at every $x \in (0, M) \backslash Z^0 \supset X^0$ and $\rho'(x) > 0$, $P'(x) > 0$ for all $x \in (0, M) \backslash Z^0$. Moreover, P' can be continuously extended to any interval $[\eta^t(0), \eta^{t+1}(0)]$ for any $t = 0, \ldots, T-1$.

Step 2: If v is neoclassical then G satisfying the hypothesis of Step 1 exists.

Proof: Consider the following differential equations on $[0, x^*]$:

$$[G(x) - G(\rho(x))]P'(\rho(x)) - g(\rho(x))[P(\rho(x)) - \delta P(\rho^2(x))] = 0 \qquad (B^*)$$

$$[1 - G(x)]P'(x) - g(x)[P(x) - \delta P(\rho(x))] = 0 \qquad (C^*)$$

where $g := \frac{dG}{dx}$. To prove Step 2, we first show that if G satisfies (A) of Step 1, (B^*) at all $x \in X^0$ and (C^*) at all $x \in (\rho(x^*), x^*) \cap X^0$ then G satisfies $(B), (C)$ of Step 1. Then, we conclude the proof by constructing such a G.

Let G be a distribution function that satisfies (A) of Step 1, (B^*) at all $x \in X^0$ and (C^*) at all $x \in (\rho(x^*), x^*) \cap X^0$. For all $x \in (0, x^*)$ and $y \in [0, x]$, define $\pi(x, y) := \frac{G(x)-G(y)}{G(x)} P(y) + \delta \frac{G(y)}{G(x)} \hat{\Pi}(y)$. Since G and $\hat{\Pi}$ are continuous on $[0, x^*)$ so is π. Moreover, since (B^*) is satisfied and v is neoclassical $\pi(x, \cdot)$ is differentiable at every $y \in X^0$. If (B^*) is satisfied at every $x \in X^0$ then at every such x we have

$$\frac{dG(x)\hat{\Pi}(x)}{dx} = g(x)P(\rho(x)) + \frac{\delta dG(\rho(x))\hat{\Pi}(\rho(x))}{dx} - \delta g(\rho(x))P(\rho^2(x))\rho'(x)$$

Hence, an inductive argument establishes that $\frac{dG(x)\hat{\Pi}(x)}{dx} = g(x)P(\rho(x))$ for all $x \in X^0$. Since $[G(\eta(y)) - G(y)]P'(y) - g(y)[P(y) - \delta P(\rho(y))] = 0$ for all $y \in X^0$,

$$\frac{d\pi(x, y)}{dy} = \frac{G(x) - G(\eta(y))}{G(x)} P'(y)$$

whenever $y \in X^0$. The term on the right-hand side above is strictly greater than 0 whenever $x > \eta(y)$ or equivalently, $\rho(x) > y$. Similarly, this term is less than 0 whenever $\rho(x) < y$. Hence, for all $x, y \in X^0$, $\pi(x, \rho(x)) \geq \pi(x, y)$. By continuity $\pi(x, \rho(x)) \geq \pi(x, y)$ for all $x \in (0, x^*), y \in [0, x]$ and therefore (B) of Step 1 holds.

Since $\frac{dG(x)\hat{\Pi}(x)}{dx} = g(x)P(\rho(x))$, the equation (C^*) implies $\frac{d\left[[1-G(x)]P(x)+\delta G(x)\hat{\Pi}(x)\right]}{dx} = 0$ for all $x \in (\rho(x^*), x^*)\backslash Z^0$. Then, (C) of Step 1 follows.

Let $A(x) := \frac{P'(x)}{P(x)-\delta P(\rho(x))}$ and $B(x) := \int_0^x A(\xi)d\xi$. Recall that A is well-defined at $x \in (0, M)\backslash Z^0$ and that for any interval $[\eta^t(0), \eta^{t+1}(0)]$, there is a unique continuous function A_t on this interval that agrees with A at every $x \in (\eta^t(0), \eta^{t+1}(0))$. Clearly, $A_t > 0$ for all $t = 0, \ldots, T-1$. Let $\mathcal{F}$ denote the space of all continuous functions on $[0, x^*]$ endowed with the sup norm. Consider the class of first order linear differential equations of the form

$$A(x)[f(x) - L(x)] = L'(x)$$

Note that (B^*) has this form. We say that L is a solution to this differential equation if L is a continuous function on $[0, x^*]$ and satisfies the equation at every $x \in (0, M)\backslash Z^0$.

Clearly, for any initial condition $L(y) = a$ and $f \in \mathcal{F}$, there is a unique solution to this equation and this solution has a right-derivative at every $x \in [0, M)$. Let $\mathcal{L}_y(f, a)$ denote this unique solution. Since P has a right-derivative at every $x < M$ so does $\mathcal{L}_y(f, a)$.

Claim 1: $f(x) > \hat{f}(x)$ for all $x \in [0, y]$, $L := \mathcal{L}_y(f, a)$ and $\hat{L} := \mathcal{L}_y(\hat{f}, a)$ implies $L(x) < \hat{L}(x)$ for all $x \in [0, y)$.

Proof: Clearly, $L(x) < \hat{L}(x)$ for all $x \in [y - \epsilon, y)$ for sufficiently small ϵ. If the claim is false, $x := \max\{x' \,|\, L(x') \geq \hat{L}(x')\}$ is well-defined and $L(x) = \hat{L}(x)$. If $x \notin Z^0$, we have $L(x) = \hat{L}(x)$, $L'(x) > \hat{L}'(x)$ and $L(z) < \hat{L}(z)$ for all $z \in (x, y)$ a contradiction. If $x \in Z^0$ replace the derivative of $L, \hat{L}$ with the corresponding right-derivatives to get a similar contradiction.

Claim 2: For any $x \in [0, y)$, $\mathcal{L}_y(\cdot, \cdot)(x)$ is a continuous function from $\mathcal{F} \times I\!\!R$ to $I\!\!R$ and $\mathcal{L}_y(\cdot, \cdot)$ is a continuous function from $\mathcal{F} \times I\!\!R$ to $\mathcal{F}$.

Proof: Assume $|f - \hat{f}| < \epsilon$ and $|a - \hat{a}| < \epsilon$ and let $L := \mathcal{L}_y(f, a)$ and $\hat{L} := \mathcal{L}_y(\hat{f}, \hat{a})$. Let $\phi(x) = \epsilon \cdot (1 - e^{B(y) - B(x)})$. Then, $\phi = \mathcal{L}_y(I_\epsilon, 0)$ and $-\phi = \mathcal{L}_y(-I_\epsilon, 0)$ where I_ϵ denotes the constant function ϵ. Therefore, by Claim 1, $\epsilon e^{B(M)} > |\mathcal{L}_y(f - \hat{f}, 0)(x)| = |\mathcal{L}_y(f, 0)(x) - \mathcal{L}_y(\hat{f}, 0)(x)|$. Also, note that $\epsilon e^{B(M)} > |\mathcal{L}_y(I_0, a - \hat{a})(x)| = |\mathcal{L}_y(\hat{f}, a)(x) - \mathcal{L}_y(\hat{f}, \hat{a})(x)|$. Hence, $|\mathcal{L}_y(f, a)(x) - \mathcal{L}_y(\hat{f}, \hat{a})(x)| \leq |\mathcal{L}_y(f, a)(x) - \mathcal{L}_y(\hat{f}, a)(x)| + |\mathcal{L}_y(\hat{f}, a)(x) - \mathcal{L}_y(\hat{f}, \hat{a})(x)| < 2\epsilon e^{B(M)}$ establishing both the continuity of $\mathcal{L}_y(\cdot, \cdot)(x)$ and $\mathcal{L}_y(\cdot, \cdot)$.

Let $y^t := \rho^t(x^*)$, $G_a^0 := \mathcal{L}_{y^0}(I_1, a)$ and for $t = 1, \ldots, T$ define G_a^t inductively as $G_a^t := \mathcal{L}_{y^t}(G_a^{t-1} \circ \eta, G_a^{t-1}(y^t))$. Note that $G_a^0(x) = 1 - [1 - a]e^{B(x^*) - B(x)}$ and hence $G_1^t(x) = 1$ for all $t \geq 0$ and $x \in [0, 1]$. Since $A(x) > 0$ for all $x \in X^0$, an inductive argument establishes that G_a^t is stricly increasing on $[y^t, y^{t-1}]$ whenever $a \in (0, 1)$. Define the distribution G_a as follows: for $x \geq x^*$, $G_a(x) := 1$, for $x \in [\rho^t(x^*), \rho^{t-1}(x^*))$ and $t = 1, \ldots, T$, $G_a(x) := G_a^{t-1}(x)$, for $x \in [0, \rho^{T-1}(x^*))$, and for $x \leq 0$, $G_a(x) = G_a^{T-1}(0)$. The construction above ensures that G_a solves (B^*). To conclude the proof we need to find $a < 1$ such that $G_a(0) = 0$.

By Claim 2, $G_a^{T-1}(0) = G_a(0)$ is a continuous function of a. Since each G_a^{t-1} is strictly increasing on $[y^t, y^{t-1}]$, G_a is strictly increasing on $[0, x^*]$. Hence $G_0(0) < 0$ and $G_1(0) = 1$. Therefore, there exists $a^* < 1$ such that $G_{a^*}(0) = 0$. The function $G := G_{a^*}$ satisfies all the desired properties.

Let X, Y, Z be arbitrary compact intervals in $I\!R$.

Step 3:

A) If $f_n : X \to Y$ and $g_n : Y \to Z$ converge uniformly to f and g respectively and f_n, g_n are continuous, then $g_n \circ f_n$ converges uniformly to $g \circ f$.

B) If each $f_n : X \to Y$ is a continuous bijection and f_n converges uniformly to the bijection f then f_n^{-1} converges uniformly to f^{-1}.

C) If $f_n : X \to Y$ is non-decreasing and f_n converges to the continuous function f at every point then f_n converges to f uniformly.

Proof of A: Since uniform limits of continuous functions on compact sets are continuous, f, g are continuous and hence uniformly continuous. Pick $\epsilon > 0$. By assumption, there exists $\epsilon_1 > 0$ such that $|g(y) - g(y')| < \frac{\epsilon}{2}$ whenever $|y - y'| < \epsilon_1$. Also, there exists N such that $|f - f_n| < \epsilon_1$ and $|g - g_n| < \frac{\epsilon}{2}$ whenever $n \geq N$. Hence for $n \geq N$,

$$|g(f(x)) - g_n(f_n(x))| \leq |g(f(x)) - g(f_n(x))| + |g(f_n(x)) - g_n(f_n(x))| < \epsilon$$

Proof of B: Note that f is continuous therefore f^{-1} is continuous and hence uniformly continuous. So, for all $\epsilon > 0$ there exists $\epsilon_1 > 0$ such that $|f^{-1}(y) - f^{-1}(y')| < \epsilon$ whenever $|y - y'| < \epsilon_1$. By assumption, there exists N such that $n \geq N$ implies $|f - f_n| < \epsilon_1$. For any $n \geq N$ and $y \in Y$, set $x_n = f_n^{-1}(y)$ and $y_n = f(x_n)$. By construction $|y - y_n| < \epsilon_1$. So, $|f^{-1}(y) - f_n^{-1}(y)| = |f^{-1}(y) - f^{-1}(y_n)| < \epsilon$.

Proof of C: Suppose f_n is non-decreasing for all n (hence, f is non-decreasing). For any $\epsilon > 0$, pick a finite set $Z \subset X$ such that for every $x \in X$ there exists $z^1, z^2 \in Z$ satisfying $z^1 \leq x \leq z^2$ and $f(z^2) - f(z^1) < \frac{\epsilon}{2}$. Since f is non-decreasing and X is bounded this can be done. Since Z is finite we can pick N large enough so that $|f(z) - f_n(z)| < \frac{\epsilon}{2}$ for all $n \geq N$ and $z \in Z$. Then, for any $x \in X$, there are $z^1, z^2 \in Z$ such that $z^1 \leq x \leq z^2$ and $0 \leq f(z^2) - f(z^1) < \frac{\epsilon}{2}$. Hence, $f(x) - f_n(x) \leq f(x) - f(z^1) + f(z^1) - f_n(z^1) < \epsilon$. Similarly, $f(x) - f_n(x) \geq f(x) - f(z^2) + f(z^2) - f_n(z^2) > -\epsilon$. Hence, $|f(x) - f_n(x)| < \epsilon$ as desired. The proof for the case of non-increasing f_n is similiar and omitted.

For any v that is not neoclassical, define $\rho, \eta, P, \hat{F}$ as above. We construct a sequence v_n such that each v_n is neoclassical and v_n, v_n' converge uniformly to v, v' respectively. Let $v_n'(x) := v'(x)$ for all $x \in [0, M]$ such that $v'(x) = v'(0) + \frac{k}{n}[v'(M) - v'(0)]$ for some integer

$k \leq n$. This defines v'_n as a strictly decreasing function at $n+1$ different points. This function can be extended to the entire intverval $[0, M]$ in a manner that ensures that the resulting function v'_n is strictly decreasing, continuously differentiable with a derivative that is bounded below by some $\alpha > 0$. Clearly, v_n defined by $v(x) = \int_0^x v'_n(y)dy$ is neoclassical and the sequence v_n converges uniformly to v. Define $\rho_n, \eta_n, P_n, \hat{F}_n$ as above, after replacing v with v_n, ρ with ρ_n etc. in the corresponding definitions. Let G_n be the equilibrium distribution of investment decisions guaranteed by Step 2. By Helley's Selection Theorem, there exists a subsequence along which G_n converges in distribution to some G. Without loss of generality we assume that this subsequence is the sequence itself. Define $\hat{\Pi}_n$ as above by replacing ρ with ρ_n and G with G_n.

Step 4: The functions ρ_n, η_n, P_n converge uniformly to ρ, η, P respectively.

Proof: Parts (A) and (B) of Step 3 suffice to show that ρ_n converges to ρ uniformly. Define $\hat{\rho}$ on $[-1, M]$ as follows: $\hat{\rho}(x) := \frac{x}{1+\bar{x}} - \frac{\bar{x}}{1+\bar{x}}$ for all $x \in [-1, \bar{x})$ and $\hat{\rho}(x) := \rho(x)$ for all $x \in [\bar{x}, M]$. Define $\hat{\rho}_n$ by replacing ρ with ρ_n and $\bar{x}$ with $\bar{x}_n$. Note that $\bar{x}_n$ converges to $\bar{x}$ and ρ_n converges uniformly to ρ. Therefore, $\hat{\rho}_n$ converges uniformly to $\hat{\rho}$. By (B) of Step 3, $\hat{\rho}_n^{-1}$ converges uniformly to $\hat{\rho}^{-1}$. Hence, η_n converges uniformly to η.

Pick an integer K such that $\delta^K < \frac{v'(M)}{v'(0)}$. Then, for all $x \in [0, M]$, $\rho^K(x) = 0$ and $\rho_n^K(x) = 0$ for all n. Hence, $P_n(x) = (1-\delta)\sum_{t=0}^{K-1} v_n(\rho_n^t(x))\delta^t + \delta^K v_n(0)$ and $P(x) = (1-\delta)\sum_{t=0}^{K-1} v(\rho^t(x))\delta^t + \delta^K v(0)$. Since ρ_n, v_n converge uniformly to ρ, v, part (A) of Step 3 implies $v_n(\rho_n^t(x))$ converges uniformly to $v(\rho^t(x))$ and hence P_n converges uniformly to P.

Step 5: G satisfies (A) of Step 1.

Proof of Step 5: First, we prove that G is continuous at every $\bar{y} < x^*$. Suppose not and let $\bar{y} > 0$ be a point of discontinuity and J be the size of the jump at $\bar{y}$. Pick x, y continuity points G sufficiently close so that $0 < |P(y) - P(x)| < \frac{J[P(y)-\delta P(\rho(y))]}{3+2[J+P(y)-\delta P(\rho(y))]}$ and $\rho(y) < x < \bar{y} < y$. Then, choose $\epsilon > 0$ such that $\epsilon < \frac{J[P(y)-\delta P(\rho(y))]}{3+2[J+P(y)-\delta P(\rho(y))]}$ and $\epsilon < \frac{P(y)-\delta P(\rho(y))}{2}$. Also, choose N such that $n \geq N$ implies $|P_n - P| < \epsilon$, $|P_n \circ \rho_n - P \circ \rho| < \epsilon$, (Steps 3 and 4 ensure that this can be done), $|G_n(x) - G(x)| < \epsilon$ and $|G_n(y) - G(y)| < \epsilon$. Since G_n satisfies (B) of Step 1, for $w_n := \eta_n(y)$, we have

$$\begin{aligned}&\frac{G_n(w_n) - G_n(y)}{G_n(w_n)} P_n(y) + \delta \frac{G_n(y) - G_n(\rho_n(y))}{G_n(w_n)} P_n(\rho_n(y)) + \delta^2 \frac{G_n(\rho_n(y))}{G_n(w_n)} \hat{\Pi}_n(\rho_n(y)) \\ \geq\ &\frac{G_n(w_n) - G_n(x)}{G_n(w_n)} P_n(x) + \delta \frac{G_n(x) - G_n(\rho_n(y))}{G_n(w_n)} P_n(\rho_n(y)) + \delta^2 \frac{G_n(\rho_n(y))}{G_n(w_n)} \hat{\Pi}_n(\rho_n(y))\end{aligned}$$

Hence,

$$\begin{aligned}0 &\leq G_n(w_n)[P_n(y) - P(y) + P(y) - P(x) + P(x) - P_n(x)] \\ &\qquad - G_n(y)[P_n(y) - \delta P_n(\rho_n(y))] + G_n(x)[P_n(x) - \delta P_n(\rho_n(y))] \\ &< 3\epsilon - G_n(y)[P_n(y) - \delta P_n(\rho_n(y))] + G_n(x)[P_n(y) - \delta P_n(\rho_n(y))] \\ &< 3\epsilon - [G_n(y) - G(y) + G(y) - G(x) + G(x) - G_n(x)][P(y) - \delta P(\rho(y)) - 2\epsilon] \\ &< 3\epsilon - [J - 2\epsilon][P(y) - \delta P(\rho(y)) - 2\epsilon] < 0\end{aligned}$$

a contradiction.

Since $G_n(x) = 0$ for all $x < 0$ and G is right-continuous, to conclude the proof of continuity we need to show that $G(0) = 0$. Assume not and pick $\epsilon \in (0, \frac{G(0)}{2})$, $\epsilon < \frac{G(0)v(0)}{6}$ and $y \in (0, \bar{x})$ such that $v(y) - v(0) < \epsilon$. Again, let $w_n := \eta_n(y)$ and note that since G_n satisfies (B) of Step 1, we have $\hat{\Pi}_n(w_n) \geq \frac{G_n(w_n) - G_n(x)}{G_n(w_n)} P_n(x) + \delta \frac{G_n(x)}{G_n(w_n)} \hat{\Pi}_n(x)$ for all $x \in [0, y]$. In particular, the above inequality holds for $x = 0$. Recall that $\rho(\bar{x}) = 0$. Then, since $y < \bar{x}$, for n sufficiently large $\hat{\Pi}_n(w_n) = \frac{G_n(w_n) - G_n(y)}{G_n(w_n)} P_n(y) + \delta \frac{G_n(y)}{G_n(w_n)} v_n(0)$. Hence, we have

$$\frac{G_n(w_n) - G_n(y)}{G_n(w_n)} P_n(y) + \delta \frac{G_n(y)}{G_n(w_n)} v_n(0) \geq v_n(0)$$

Since $P_n(y) = (1 - \delta)v_n(y) + \delta v_n(0)$ the above expression yields

$$G_n(w_n)[v_n(y) - v_n(0)] \geq G_n(y) v_n(y)$$

Since G is continuous at y, for n large enough $|G(y) - G_n(y)| < \epsilon$. Similarly, for n large enough we have $|v_n(y) - v(y)| < \epsilon$, $|v(0) - v_n(0)| < \epsilon$. So, for n large enough, $3\epsilon > [v_n(y) - v(y) + v(y) - v(0) + v(0) - v_n(0)] \geq G_n(w_n)[v_n(y) - v_n(0)] \geq G_n(y)v_n(y) \geq [G(y) - \epsilon]v(0) \geq [G(0) - \epsilon]v(0) > \frac{G(0)v(0)}{2}$, a contradiction.

By construction, $G_n(x) = 1$ for all $x > x_n^*$ and n, x_n^* converges to x^*. Moreover, G is right-continuous. Hence, $G(x^*) = 1$.

Since G is continuous at every $x < x^*$, so is $\hat{\Pi}$. Therefore, by part (C) of Step 3, G_n and $\hat{\Pi}_n$ converge uniformly to G and $\hat{\Pi}$ respectively on $[0, \alpha]$ for any $\alpha < x^*$. Suppose G is not strictly increasing. Then, we can find $0 < x < y < x^*$ such that $\rho^t(x^*) \notin [x, y]$ for all t and $G(x) = G(y)$. Hence, $y < \eta(x)$. Suppose $x < \rho(x^*)$. Since G_n satisfies (B) of Step 1 and $\hat{\Pi}_n$, ρ_n are non-decreasing, we have

$$\begin{aligned}\hat{\Pi}_n(\eta_n(x)) =& \frac{G_n(\eta_n(x)) - G_n(x)}{G_n(\eta_n(x))} P_n(x) + \delta \frac{G_n(x)}{G_n(\eta_n(x))} \hat{\Pi}_n(x) \\ \geq& \frac{G_n(\eta_n(x)) - G_n(y)}{G_n(\eta_n(x))} P_n(y) + \delta \frac{G_n(y)}{G_n(\eta_n(x))} \hat{\Pi}_n(y) \\ \geq& \frac{G_n(\eta_n(x)) - G_n(y)}{G_n(\eta_n(x))} P_n(y) + \delta \frac{G_n(x)}{G_n(\eta_n(x))} \hat{\Pi}_n(x)\end{aligned} \tag{2}$$

After some manipulation and taking limits using Step 4, the above expression yields $G(x) = G(\eta(x))$. Hence $G(z) = G(x)$ for all $z \in [x, \eta(y))$. If $\eta(x) < \rho(x^*)$, repeating the above argument with $\eta(x)$ in place of x and $z \in (\eta(x), \eta(y))$ in place of y yields $G(\eta^2(x)) = G(\eta(x)) = G(x)$. Continuing in this fashion, we will eventually obtain $\rho(x^*) < x < y < x^*$ such that $G(x) = G(y)$. For $\rho(x^*) < x < y < x^*$, by (C) of Step 1, we can replace $G_n(\eta_n(x))$ in (2) above, with 1, repeat the previous argument and get $G_n(x) = G_n(\eta_n(x)) = 1$. Letting n go to infinity yields $G(x) = 1$. But then (C) of Step 1 and Step 4 yield

$$\begin{aligned}\hat{\Pi}(x) =& \lim \left[\frac{G_n(x) - G_n(\rho_n(x))}{G_n(x)} P_n(\rho_n(x)) + \delta \frac{G_n(\rho_n(x))}{G_n(x)} \hat{\Pi}_n(\rho_n(x)) \right] \\ \leq& \lim \left[[1 - G_n(\rho_n(x))] P_n(\rho_n(x)) + \delta G_n(\rho_n(x)) \hat{\Pi}_n(\rho_n(x)) \right] \leq \lim \hat{\Pi}_n(M) \\ =& \lim \left[[1 - G_n(x)] P_n(x) + \delta G_n(x) \hat{\Pi}_n(x) \right] = \delta \hat{\Pi}(x)\end{aligned}$$

So, $\hat{\Pi}(x) \geq v(0) > 0$ and $\hat{\Pi}(x) = \delta \hat{\Pi}(x)$, a contradiction.

To conclude the proof of existence we will show that G satisfies $(B), (C)$ of Step 1 as well. Recall that G_n and $\hat{\Pi}_n$ converge uniformly to G and $\hat{\Pi}$ respectively on $[0, \alpha]$ for any $\alpha < x^*$. By Step 4 and (A) of Step 3, for every $\epsilon > 0$ and $x < x^*$ there exists N such that $n \geq N$ implies $\frac{G(x) - G(\rho(x))}{G(x)} P(\rho(x)) + \delta \frac{G(\rho(x))}{G(x)} \hat{\Pi}(\rho(x)) > \frac{G_n(x) - G_n(\rho_n(x))}{G_n(x)} P_n(\rho_n(x)) +$

$\delta\frac{G_n(\rho_n(x))}{G_n(x)}\hat{\Pi}_n(\rho_n(x)) - \epsilon$ for all $0 \leq y \leq x$. But since G_n satisfies (B) of Step 1, we have $\frac{G_n(x)-G_n(\rho_n(x))}{G_n(x)}P_n(\rho_n(x)) + \delta\frac{G_n(\rho_n(x))}{G_n(x)}\hat{\Pi}_n(\rho_n(x)) \geq \frac{G_n(x)-G_n(y)}{G_n(x)}P_n(y) + \delta\frac{G_n(y)}{G_n(x)}\hat{\Pi}_n(y)$, for all $0 \leq y \leq x$. Again, we can choose n large enough so that the last term above is greater than $\frac{G(x)-G(y)}{G(x)}P(y)+\delta\frac{G(y)}{G(x)}\hat{\Pi}(y)-\epsilon$. Hence, $\hat{\Pi}(x) > \frac{G(x)-G(y)}{G(x)}P(y)+\delta\frac{G(y)}{G(x)}\hat{\Pi}(y)-2\epsilon$. Since ϵ is arbitrary, it follows that G satisfies (B). Verifying (C) requires a similar argument. $\square$

Proof of Proposition 6: Since C has a finite support, if the Proposition is false, we can find an $\epsilon > 0$, a sequence $\Delta_n > 0$ converging to 0, and a sequence of stationary sequential equilibria σ_n, of the games $B^C(\Delta_n)$ such that either $G_n(x^*)-G_n(x^*-\epsilon) < 1-\epsilon$ for all n or for some $c < v(x^*)$, the probability of agreement conditional on $C = c < v(x^*)$ by time ϵ is less than $1-\epsilon$ for all n. By Helly's Selection Theorem, we can assume, without loss of generality that G_n converges in distribution to some distribution G. Call x an active point of G if $G(x + \alpha) - G(x - \alpha) > 0$ for all $\alpha > 0$. Let x^0 denote the minimum of the support of G.

Step 1: For all $\epsilon > 0$ conditional on $C = c < v(x^0)$, the probability of agreement by time ϵ converges to 1 as Δ_n converges to 0.

Proof of Step 1: If the statement above is false, then there exists $\epsilon' > 0, c < v(x^0)$, a subsequence of games $B^C(\Delta_{n_j})$ and equilibria σ_{n_j} of $B^C(\Delta_{n_j})$ satisfying the following properties: In the equilibrium σ_{n_j}, conditional on observing c and the price $p^0_{n_j}$ in the support of the seller's initial period pricing strategy $F^c_{n_j}$, the buyer buys at time $t \geq \epsilon'$ with probability greater than ϵ'. Recall that in a stationary sequential equilibrium, the seller can only randomize in the initial period. Hence, the buyer knows the entire sequence after observing the first price $p^0_{n_j}$. Therefore, his decision is conditioned on both the realization of c and the entire price sequence $p^k_{n_j}$. Without loss of generality we will assume that the subsequence of stationary sequential equilibria is the sequence itself and that ϵ' is small enough so that $v(x^0) > c + \epsilon'$.

Let $Q^c_n(t)$ denote the probability of sale by period $k(t) := \min\{l \,|\, l\Delta_n > t\}$, in the equilibrium σ_n, conditional on c and the price sequence p^k_n. Define $P^c_n(t) := p^{k(t)}_n$. Since σ_n is a stationary sequential equilibrium, for each n, p^k_n is a non-increasing sequence. Hence, $Q^c_n(t)$ and $P^c_n(t)$ are right-continuous and monotone functions. Assume that Q^c_n and P^c_n converge weakly; to some monotone Q^c and P^c respectively. Observe that this assumption is without loss of generality since we can use Helly's Selection Theorem to find some convergent subsequence with the desired properties and apply the analysis below to that subsequence. By assumption $Q_n^c(\epsilon') \leq 1-\epsilon'$ for all n and hence, $Q^c(\epsilon'/2) \leq 1-\epsilon'/2$. Let $\epsilon = \epsilon'/2$. Hence, $Q^c(\epsilon) \leq 1-\epsilon$.

We note that $P^c(\epsilon) > c$. To see this, recall that the buyer's behavior is optimal, given the sequence of prices chosen by the seller. A price below c is never charged in equilibrium and by assumption, the buyer waits until time $\epsilon' = 2\epsilon$, with probability greater than ϵ'. Since $G(v^{-1}(c+\epsilon')) = 0$, with probability approaching 1, the buyer's valuation is above $c+\epsilon$. For such a buyer, it would not be optimal to wait an extra ϵ if the price were approaching c near time ϵ.

We consider two cases: First, we assume $Q^c(\epsilon/3) - Q^c(\epsilon/4) > 0$. In this case, we show that the seller can do better by moving down the demand curve (i.e., q_n^c) faster, prior time $t = \epsilon/5$. This will save on the expected delay and still get arbitrarily close to the equilibrium expected profit on buyer types that would have purchases prior to $\epsilon/5$. On the other hand, if $Q^c(\epsilon/3) - Q^c(\epsilon/4) = 0$, then we show that the seller can do better by moving faster through the prices charged during the time interval $(\epsilon/4, \epsilon/3)$ and reaching the buyer types that purchase after time $\epsilon/3$ sooner. Since the function P^c and Q^c are continuous almost everywhere, there are points arbitrarily close to $\epsilon/3, \epsilon/4$ and $\epsilon/5$ at which both of these functions are continuous. Therefore, without loss of generality, we assume that both of these functions are continuous at all three of these points.

Case 1: $Q^c(\epsilon/3) - Q^c(\epsilon/4) > 0$

Proof of Case 1:

We establish a contradiction by showing that an alternative sequence of prices r_n^k would yield a higher expected profit for the seller then the sequence p_n^k which is charged with positive probability.

Let $p_n = \max\{p_n^k \leq P^c(\epsilon/5)\}$ and construct a new sequence of prices r_n^k from p_n^k as follows: Set $r_n^0 = p_n^0$. Then, divide the interval $[p_n, p_n^0)$ into $(p_n^0 - p_n)/\Delta_n^{1/2}$ equal sized, left-closed, right-open intervals (an obvious adjustment needs to be made when $(p_n^0 - p_n)/\Delta_n^{1/2}$ is not an integer). Remove all but the lowest price in each interval. Since the new price sequence r_n^k has at most one price in any interval of size $\Delta_n^{1/2}$, it will take at most $(p_n^0 - p_n)/\Delta_n^{1/2}$ periods before the p_n is charged and r_n^k reverts to the equilibrium price sequence. If the seller were to switch to the sequence r_n^k instead of the equilibrium sequence p_n^k, this would have two effects. In equilibrium, if the buyer were a type that buys prior to time $\epsilon/5$, then the alternative price path r_n^k may achieve less price discrimination than the equilibrium sequence. On the other hand, if the buyer were planning to purchase after time $\epsilon/5$, the new strategy would decrease the amount of delay until such a buyer purchases.

Note that for sufficiently large n, the equilibrium strategy takes close to $\epsilon/5$ units of time until it reaches a price lower than p_n, while the alternative strategy takes no more than $(p_n^0 - p_n)\Delta_n^{1/2}$. Moreover, the alternative strategy sells to each type of the buyer at a price at most $\Delta_n^{1/2}$ lower than the equilibrium purchase price. Hence, the potential loss with the alternative strategy (compared to the equilibrium strategy) on these buyer types is at most $\Delta_n^{1/2}$. The gain associated with the alternative strategy is at least $A := [Q^c(\epsilon/3) - Q^c(\epsilon/4)][e^{-(\epsilon/3-\epsilon/5)} - e^{-\epsilon/3}]e^{-\epsilon/2}[P^c(\epsilon/2) - c]$ as n goes to ∞. To see why A is a lower bound on the gain and is strictly greater than 0, note that the time saved on types that would have purchased between $\epsilon/4$ and $\epsilon/3$, approaches $\epsilon/5$. These sales are made no later than time $\epsilon/2$. Also, the probability that agreement is reached between time $\epsilon/4$ and $\epsilon/3$, is no less than $Q_c(\epsilon/3) - Q_c(\epsilon/4) > 0$, for n sufficiently large. Since P^c is non-increasing, these sales are made at a price no less than $P^c(\epsilon/2) \geq P^c(\epsilon) > c$. Hence, the net effect is no less than A. On the other hand, the loss associated with the alternative sequence r_n^k converges to 0, and hence the net effect is $A > 0$, contradicting the optimality of the equilibrium strategy.

Case 2: $Q^c(\epsilon/3) - Q^c(\epsilon/4) = 0$

Proof of Case 2:

Again, we define an alternative sequence of prices r_n^k and show that for large enough n, the expected profit associated with this sequence is larger than the profit associated with the equilibrium price sequence p_n^k. Let $p_n(\alpha) := \min\{p_n^k > P^c(\epsilon/3)\}$, $p_n(\beta) := \max\{p_n^k \leq P^c(\epsilon/4)\}$.

This time, r_n^k is constructed from p_n^k as follows: Divide the interval $[p_n(\alpha), p_n(\beta))$ into $1/B\Delta_n$ equal sized, left-closed, right-open intervals (choose B such that $1/B\Delta_n$ is an integer). Remove all but the lowest price in each interval. The new price sequence r_n^k contains at most $1/B\Delta_n$ prices between $p_n(\alpha)$ and $p_n(\beta)$. After these prices are charged the sequence reverts to the equilibrium price sequence.

As in case 1, there are two effects associated with the seller switching to the sequence r_n^k instead of the equilibrium sequence p_n^k. If in equilibrium, the buyer were planning to purchase between $\epsilon/4$ and $\epsilon/3$, then the alternative price path r_n^k may achieve less price discrimination than the equilibrium sequence. On the other hand, if the buyer were planning to purchase after time $\epsilon/3$, the new strategy may decrease the amount of delay until such a buyer purchases.

Let $\Pi(\alpha\beta)$ denote the expected profit associated with the equilibrium strategy, conditional on the buyer being a type that would have purchased between $\epsilon/4$ and $\epsilon/3$ in the original equilibrium. Let $\Pi(\beta)$ be the equilibrium expected profit conditional on the buyer purchasing after time β. Compared to the equilibrium price sequence, the price sequence r_n^k loses at most $MB\Delta_n$ on each type of buyer that purchases between α and β. (Recall that M is the largest feasible value of v.) Moreover, with r_n^k, it takes at most $1/B\Delta_n$ many periods and hence $1/B$ units of time to sell to these types of the buyer. Hence, the expected the expected gain with the new strategy conditional on the buyer being a type that would not have purchased by time $\epsilon/3$ with the equilibrium strategy approaches $(e^{-[\epsilon/4+1/B]} - e^{-\epsilon/3})\Pi(\beta)$ as n gets large. This is the saving in delay multiplied by the expected profit conditional on the delay. Therefore, a lower bound on the net expected change in profit by switching to the alternative strategy approaches

$$[1 - Q^c(\epsilon/3)](e^{-[\epsilon/4+1/B]} - e^{-\epsilon/3})\Pi(\beta) - [Q^c(\epsilon/3) - Q^c(\epsilon/4)]MB\Delta_n$$

as n goes to infinity. A price below c will never be charged. Consequently, a buyer with valuation at least $v(x^0)$ will purchase immediately at any price less than $(1 - e^{-\Delta_n})v(x) + e^{-\Delta_n}c$. Hence, $\Pi(\beta) \geq [1 - e^{-\Delta_n}][v(x^0) - c] > 0$ for n sufficiently large. Recall that $Q^c(\epsilon/3) - Q^c(\epsilon/4) = 0$. Therefore, the net effect of switching to the alternative strategy divided by Δ_n is bounded below by a term converging to $[1 - Q^c(\epsilon/3)][e^{-[\epsilon/4+1/B]} - e^{-\epsilon/3}][v(x^0) - c]$. Since $Q^c(\epsilon/3) \leq Q^c(\epsilon) < 1$ and $v(x^0) - c > 0$, for $B > 12/\epsilon$, this term is strictly positive, contradicting the optimality of the equilibrium strategy.

Step 2: $x^0 = x^*$ and $G(x^*) = 1$.

Proof of Step 2: Assume $x^0 < x^*$.

To get a contradiction first assume that $Prob\{C = v(x^0)\} = 0$. Since S is strictly quasi-concave, $S(x) > S(x^0)$ for any $x \in (x^0, x^*]$. Let c' be the maximum of the set of c's such that $c < v(x^0)$ and $Prob\{C = c\} > 0$. By Assumption A, c' is well defined. Also, let c^* be the lowest element in the set of c's such that $c > c'$ and $Prob\{C = c\} > 0$. (If this set is empty, let $c^* = \infty$). Note that for any x such that $x^0 < x < x^*$ and $v(x) < c^*$, we have $S(x) - S(x^0) = [v(x) - v(x^0)]Prob\{C \leq c'\} - x + x^0 > 0$. Pick any such x and choose α small enough so that $S(x) - S(x') > 0$ whenever x' is in the set $[x^0 - \alpha, x^0 + \alpha]$. Then, pick $\epsilon < \alpha$ such that $S(x)e^{-\epsilon} - (1 - e^{-\epsilon})x - S(x') > 0$ for all $x' \in [x^0 - \alpha, x^0 + \alpha]$. This is possible since S is continuous.

Pick x_n in the support of G_n such that $|x_n - x^0| < \alpha$. This is possible since x^0 is in the support of G. Next, assume that instead of investing x_n the buyer invests x and follows exactly the same buying strategy as he would have had he invested x_n in the game $B^C(\Delta_n)$. The expected payoff associated with investing x_n is $\sum_{c \leq c'} Prob\{C = c\}E\{[v(x_n) - p_n(c)]e^{-t_n(c)}\} - x_n$, where $t_n(c)$ and $p_n(c)$ are respectively the time and price at which at which the buyer reaches agreement conditional on x, c and the price sequence. The expectation is over the possible randomization of the seller in the initial period, given c, which in a stationary equilibrium determine the entire price sequence. Similarly, the expected payoff associated with the alternative strategy is $\sum_{c \leq c'} Prob\{C = c\}E\{[v(x) - p_n(c)]e^{-t_n(c)}\} - x$. By Step 1, for n sufficiently large, $E[e^{-t_n(c)}] > e^{-\epsilon}$ for all $c \leq c'$. Hence the difference in utility between choosing x and x_n is $\sum_{c \leq c'} Prob\{C = c\}[v(x) - v(x_n)]E[e^{-t_n(c)}] - x + x_n \geq S(x)e^{-\epsilon} - (1 - e^{-\epsilon})x - S(x_n) > 0$. But this contradicts the fact that x_n is in the support of G_n and hence, is optimal.

If $Prob\{C = v(x^0)\} > 0$, let c' be the largest c such that $c < v(x^0)$ and $Prob\{C = c\} > 0$. Let $g(z) := \sum_{c < v(x^0)}[v(z) - c]Prob\{C = c\} - z$. Note that the function g is continuously differentiable on $(0, M)$ and $g(z) = S(z)$ for all z such that $v(z) \in [c', v(x^0)]$. By Assumption A, the left-derivative of S is strictly positive at $x^0 > 0$ and therefore the derivative of g at x^0 must also be positive. (Here, we are also using the assumption $Prob\{C = v(0)\} = 0$; otherwise we cannot rule out the possibility that $x^0 = 0$ and hence, the left-derivative of S is not defined at x^0). Pick any $x > x^0$ such that $g(x) - g(x^0) > 0$. Choose α small enough so that the interval $[v(x^0 - \alpha), v(x^0 + \alpha)]$ contains no c such that $Prob\{C = c\} > 0$ other than $c = v(x^0)$. Then, choose $\epsilon < \alpha$ small enough so that $g(x)e^{-\epsilon} - (1 - e^{-\epsilon})x - g(x') - |v(x') - v(x^0)| > 0$ for all x' such that $|x' - x^0| < \alpha$. This is possible since g and v are continuous.

Again, pick x_n in the support of G_n such that $|x_n - x^0| < \alpha$. Let u be the equilibrium utility of the buyer in the game $B^C(\Delta_n)$, if he invests x_n. Let u_x be the utility the buyer would enjoy if he had invested x and used the following strategy in the bargaining stage: For any $c \leq c'$ buy whenever the buyer with valuation $v(x_n)$ would have bought. For $c > c'$, never buy. Since no price below cost $v(x^0)$ is ever charged in equilibrium, $u < \sum_{c \leq c'} Prob\{C = c\} E\{[v(x_n) - p_n(c)]e^{-t_n(c)}\} + |v(x_n) - v(x^0)| - x_n$, where the expectation is over the possible random choice of the seller in period 0, given c, $t_n(c)$ is the time at which the buyer reaches agreement and $p_n(c)$ is the price at which agreement is reached conditional on x, c and the price sequence. Similarly, $u_x = \sum_{c \leq c'} Prob\{C = c\} E\{[v(x) - p_n(c)]e^{-t_n(c)}\} - x$. Therefore, for n sufficiently large, Step 1 yields, $u_x - u > \sum_{c \leq c'} Prob\{C = c\} E\{[v(x) - v(x_n)]e^{-t_n(c)}\} - |v(x_n) - v(x^0)| - x + x_n > g(x)e^{-\epsilon} - (1 - e^{-\epsilon})x - g(x_n) - |v(x_n) - v(x^0)|$. Since $|x^0 - x_n| < \alpha$, we conclude that $u_x - u > 0$. But this contradicts the fact that investing x_n is in the support of the buyer's strategy and hence, is optimal.

Thus $x^0 \geq x^*$. To prove $x^0 = x^*$ and $G(x^*) = 1$ and conclude the proof of Step 2, it suffices to show that the buyer will never invest more than x^*. The payoff to investing some $x > x^*$ is $u_x = \sum_{c \leq c'} Prob\{C = c\} E\{[v(x) - p_n(c)]e^{-t_n(c)}\} - x$, where c' is the largest c such that $c < v(x)$ and $Prob\{C = c\} > 0$. Again, the expectation is over the possible random choice of the seller in period 0, given c, $t_n(c)$ is the time at which the buyer reaches agreement and $p_n(c)$ is the price at which agreement is reached conditional on x, c and the price sequence. Suppose instead the buyer invests some x' such that $c' < v(x') < v(x)$ but follows the same buying strategy as the buyer who invests x. The expected utility associated with that decision is $u_{x'} = \sum_{c \leq c'} Prob\{C = c\} E\{[v(x') - p_n(c)]e^{-t_n(c)}\} - x'$. Hence, $u_{x'} - u_x = [v(x') - v(x)] \sum_{c \leq c'} Prob\{C = c\} E[e^{-t_n(c)}] - x' + x$. Since $v(x') - v(x) < 0$ and $E[e^{-t_n(c)}] < 1$, the last expression yields $u_{x'} - u_x \geq [v(x') - v(x)] \sum_{c \leq c'} Prob\{C =$

$c\} - x' + x = S(x') - S(x) > 0$. The last inequality follows from the strict quasi-concavity of S. Hence, investing x' yields a higher utility than investing x. Which proves that the buyer will never invest $x > x^*$. This concludes the proof of Step 2.

By Step 2, $G(x^*) = 1$, $x^0 = x^*$ and hence $G(x^* - \epsilon) = 0$. Given Step 1, this contradicts our assumption that the Proposition is false. In Proposition 4 we showed that the buyer's equilibrium payoff in $B(\Delta)$ is 0. The same argument yields the same conclusion for $B^C(\Delta)$. Hence, the outcome is efficient and the seller extracts all surplus. □

Proof of Proposition 7: Let c_0 denote $\min\{c \,|\, Prob\{C = c\} > 0\}$ and G denote the buyer's equilibrium investment strategy. Let $v(x)$ be the infimum of buyer valuations that purchase with positive probability, in equilibrium. If $v(x)$ is not well-defined, i.e., no buyer type purchases in equilibrium, we are done. Otherwise, $v(x) \geq c_0$ and a price below $v(x)$ is never charged.

If $c_0 > v(0)$, then, the expected payoff of the buyer, conditional on investing x is negative, a contradiction. Hence, no buyer type buys with positive probability in equilibrium and therefore the buyer invests 0 with probability 1.

So, suppose $c_0 = v(0)$. First, assume that there exist no $y > 0$ such that $G(y) = G(0)$. Then, in any sequential equilibrium, for any $k > 0$, conditional on $C = c_0$, with strictly positive probability the bargaining stage continues beyond period k. This follows from the fact that as long as there is some probability that the buyer has a valuation strictly greater than c_0, the seller can earn strictly positive profit by charging some price $p > c_0$ that is not accepted with probability 1, while charging a price $p \leq c_0$ earns her 0 profit. Hence, for any k there exists some set I of strictly positive buyer investment levels such that, I has strictly positive probability according to G and conditional on investing $y \in I$, the buyer purchases in period k if $C = c_0$ and does not purchase otherwise. Pick k so that $e^{-k\Delta} v'(0) < 1$. Then, any buyer type in I gains strictly higher payoff by investing 0 than by following his equilibrium purchasing strategy, a contradiction. So, there exists y such that $G(y) = G(0)$. Let z be the supremum of such y. If z is finite then, the seller will never charge a price below $v(z)$ until all buyer types with valuation at least $v(z)$ have purchased. This implies that any buyer with valuation close to z will obtain a strictly negative expected utility. Therefore, $z = \infty$; that is $G(0) = 1$ and hence both players receive 0 utility in equilibrium. $\square$

Proof of Proposition 9: Let p_n^* and p_n^0 denote prices in the support of the seller's initial period strategy, in some sequential equilibrium of $B^2(\Delta_n)$, conditional on the seller investing y^* and 0 respectively.

Let $a^* := [v^* - c^*]/[v^* - v^0]$, $a^0 := [v^* - c^0]/[v^* - v^0]$, $K(s) := s(s-1)/2$ and define $H(a, \Delta, S) := \sum_{s=1}^{S} a^s e^{\Delta K(s)}$. Let Ω be the set of all sequences (Δ_n, S_n^*, S_n^0) satisfying $(i) - (iii)$ below:

(i) $\Delta_n, S_n^*, S_n^0 > 0$, and $\lim_n \Delta_n = 0$.

(ii) $T^* := \lim \Delta_n S_n^* \leq \log[\frac{v^* - v^0}{x^*}]$ and $T^0 := \lim \Delta_n S_n^0$ ($T^0 = \infty$ is allowed).

(iii) $\limsup \frac{H(a^*, \Delta_n, S_n^*)}{H(a^0, \Delta_n, S_n^0)} \leq \frac{v^0 - c^0}{v^0 - c^*}$.

Define $\delta = [v^* - v^0] \inf_\Omega \lim_{n\to\infty} |e^{-\Delta_n S_n^0} - e^{-\Delta_n S_n^*}|$. To establish that $\delta > 0$, it is enough to show that for any sequence satisfying $(i) - (iii)$ above, $\lim_{n\to\infty}[e^{-\Delta_n S_n^0} - e^{-\Delta_n S_n^*}] \neq 0$. If δ were equal to 0, we could construct a sequence satisfying $(i) - (iii)$ such that $\lim[e^{-\Delta_n S_n^0} - e^{-\Delta_n S_n^*}] = 0$. Thus, it suffices to show that for any sequence satisfying (i) and (ii) above such that $\lim \Delta_n S_n^0 = \lim \Delta_n S_n^* = T^* < \infty$, $H(a^*, \Delta_n, S_n^*)/H(a^0, \Delta_n, S_n^0)$ converges to ∞. To prove the latter, define $\eta_n := \frac{S_n^*}{S_n^0}$ and note that since $a^* > 1$ and $a^0 > 1$,

$$\frac{H(a^*, \Delta_n, S_n^*)}{H(a^0, \Delta_n, S_n^0)} \geq \frac{(a^*)^{S_n^*} e^{\Delta K(S_n^*)}}{S_n^0 (a^0)^{S_n^0} e^{\Delta K(S_n^0)}}$$
$$= \frac{(A_n^*/A_n^0)^{S_n^0}}{S_n^0}$$

where $A_n^* = (a^*)^{\eta_n} e^{\frac{1}{2}\Delta_n \eta_n(\eta_n S_n^0 - 1)}$ and $A_n^0 = (a^0) e^{\frac{1}{2}\Delta_n (S_n^0 - 1)}$. Since η_n converges to 1, Δ_n converges to 0 and $\Delta_n S_n^*$, $\Delta_n S_n^0$ converge to $T^* < \infty$, A_n^*/A_n^0 converges to $a^*/a^0 = [v^* - c^*]/[v^* - c^0] > 1$. Therefore, $H(a^*, \Delta_n, S_n^*)/H(a^0, \Delta_n, S_n^0)$ converges to ∞, as desired.

Assume that $y^* > c^0 - c^* - \delta$. Next, for each investment level of the seller, we construct the unique consistent collection associated with the bargaining stage.

By Proposition 3, for any $\Delta > 0$, with probability 1, the game ends in finite time and $p_0' = v^0$ is the last price charged. A price below v^0 is never charged and in equilibrium the buyer accepts this v^0 whenever it is offered. Moreover, the low valuation buyer will only buy in the last period. Hence, the buyer with valuation v^* randomizes and is indifferent over the outcome of this randomization. Let the prices p'_s be indexed backward from the last period. The buyer's indifference implies $v^* - p'_{s+1} = [v^* - p'_s]e^{-\Delta}$. This equation together with the initial condition $p'_0 = v^0$ yields

$$p'_s = v^* - [v^* - v^0]e^{-\Delta s} \tag{1}$$

Let m_s denote the mass of buyers that purchase in period s. That is, m_s is the unconditional probability that the buyer purchases in period s. For $s \geq 0$, the profit of the seller conditional on period $s+1$ being reached, evaluted in period $s+1$ units is,

$$\pi_{s+1} = m_{s+1}(p'_{s+1} - c) + e^{-\Delta}\pi_s \tag{2}$$

where c is the cost of the seller and m_0 denotes the probability that the buyer invests 0. The equilibrium levels of m_s are determined by making the seller indifferent between charging this period's price and charging next period's price. That is:

$$\pi_{s+1} = m_{s+1}(p'_s - c) + \pi_s \tag{3}$$

Charging p'_s has the disadvantage that less profit is made on the mass m_{s+1} who would have purchased now at the higher price p'_{s+1}, but has the advantage that it avoids the time lost on the continuation profit. The m_{s+1} that solves (3) makes the seller indifferent between charging p'_{s+1} and p'_s. Solving (1), (2) and (3) yields, for $s > 1$,

$$m_s = a^s e^{\Delta K(s)} m_1 \tag{4}$$

where $a = [v^* - c]/[v^* - v^0]$ and $K(s) := s(s-1)/2$. Note that $\pi_0 = m_0(p_0 - c) = m_0(v^0 - c)$. Hence, solving for m_1 using (2) and (3) yields $m_1 = bm_0$ where $b := \frac{v^0 - c}{v^* - v^0}$. Let S be the smallest integer such that $\sum_0^S m_s \geq 1$. Then,

$$[bH(a, \Delta, S-1) + 1]m_0 = \sum_{s=0}^{S-1} m_s < 1 \leq \sum_{s=0}^{S} m_s = [bH(a, \Delta, S) + 1]m_0 \tag{5}$$

The mass of buyer's who do not buy at price p is $\sum_{s' \leq s(p)} m_{s'}$, where $s(p)$ is the largest value of s for which $p > p'_s$. We define $s(p)$ to be -1 if $p \leq p'_0$. Then, $q_\Delta(p) =$

$\max\{0, 1 - \sum_0^{s(p)} m_s\}$, where $\sum_0^{-1} m_s := 0$. Define r_Δ and Π_Δ as follows: for all $q^0 \in q(\mathbf{R}_+)$, $r_\Delta(q^0) = p'_{s-1}$ and $\Pi_\Delta(q^0) = \pi_s$ whenever $q^0 = 1 - \sum_{k=0}^{s} m_k$ and $\Pi_\Delta(0) = [1 - \sum_{k=0}^{S-1} m_k][p'_{S-1} - c] + \pi_{S-1}$.

It follows from (1) (2) and (3) that $q_\Delta, r_\Delta, \Pi_\Delta$ is a consistent collection for the bargaining stage and that p'_{S-1} (and also p'_S if $\sum_{k=0}^{S} m_s = 1$) is the only price that maximizes $[1 - q_\Delta(p)]p + e^{-\Delta}\Pi(q(p))$. Hence, it follows from Proposition 3 that $\Pi_\Delta(0)$ is the sequential equilibrium payoff for the seller and that only the prices p'_{S-1} or p'_S can be charged in the initial period. But, as Δ approaches 0, p'_S will approach p'_{S-1} and ΔS will converge to $\Delta(S-1)$. Hence, the outcome will be the same no matter how the seller randomizes over his two possible optimal strategies. Therefore, we will refer to p_S as the equilibrium price in for the initial period.

To conclude the proof, we will show that there exists no sequence Δ_n converging to 0 and a corresponding sequence of sequential equilibria of $B^2(\Delta_n)$ where the probability of the seller investing y^* is strictly positive along the entire sequence. Assume that such a sequence of equilibria σ_n, exists. Let $m_{0,n}$ denote the probability that the buyer invests 0, in the equilibrium σ_n and $\gamma_n^* > 0$ denote the probability that the seller invests y^* in the equilibrium σ_n. Using (1) – (5), solve for m_s, S_n, p'_s after setting $a = a^*, c = c^*$ and $m_0 = m_{0,n}$. Let $m^*_{s,n}, S^*_n, p'^*_{n,s}$ denote the corresponding values. Similarly, solve for $m^0_{s,n}, S^0_n, p'^0_{n,s}$ after setting $a = a^0, c = c^0$ and $m_0 = m_{0,n}$. Again, without loss of generality, assume that $\Delta_n S^*_n$, $\Delta_n S^0_n$, p'^*_{n,S^*_n} and p'^0_{n,S^0_n} converge to T^*, T^0, p^* and p^0 respectively, as n goes to infinity. This involves no loss of generality since all prices belong to the compact set $[0, v^*]$ and $T^* = \infty$ and/or $T^0 = \infty$ are permitted and therefore, a subsequence with the desired property can be found. Then, (1) – (5) and straightforward calculations yield the following:

$$\begin{gathered} p^* = v^* - [v^* - v^0]e^{-T^*}, \quad p^0 = v^* - [v^* - v^0]e^{-T^0} \\ T^* \leq T^0, \quad p^* \leq p^0 \end{gathered} \tag{6}$$

Since the payoff of the buyer is 0 and the probability of the buyer investing x^* is strictly positive, we have

$$\gamma_n^*[v^* - p_n^*] + (1 - \gamma_n^*)[v^* - p_n^0] = x^* \tag{7}$$

From (6) and (7) we obtain,

$$T^* \leq \log[\frac{v^* - v^0}{x^*}] \tag{8}$$

By the argument used in proving Proposition 5, for any $\epsilon > 0$, the probability of agreement being reached by time ϵ is at least $1 - \epsilon$ for sufficiently large n. Therefore, as n goes to infinity, the initial price minus cost will equal the seller's profit. Since investing y^* is an optimal action for the seller, we have,

$$p^* - c^* - y^* \geq p^0 - c^0 \tag{9}$$

From (4) and (5) we know that

$$\begin{aligned} [b^* H(a^*, \Delta_n, S_n^* - 1) + 1] m_{0,n} &< 1 \\ 1 &\leq [b^0 H(a^0, \Delta_n, S_n^0) + 1] m_{0,n} \end{aligned} \tag{10}$$

where $b^* = \frac{v^0 - c^*}{v^* - v^0}$ and $b^0 = \frac{v^0 - c^0}{v^* - v^0}$. Then, (10) implies,

$$\limsup H(a^*, \Delta_n, S_n^* - 1)/H(a^0, \Delta_n, S_n^0) \leq b^0/b^* \tag{11}$$

Thus, we have shown that the sequence $(\Delta_n, S_n^* - 1, S_n^0)$ is in Ω. Hence, $|p^0 - p^*| \geq \delta$. Since $p^0 - p^* \geq 0$ by (6), we conclude that $p^0 - p^* \geq \delta$. But this contradicts (9), since $y^* > c^0 - c^* - \delta$. □

References

Aghion, P., M. Dewatripont and P. Rey (1994): “Renegotiation Design with Unverifiable Information,” *Econometrica* 62: pp. 257–282.

Ausubel, L. and R. Deneckere (1989): “Reputation in Bargaining and Durable Goods Monopoly,” *Econometrica* 57: pp. 511–531.

Billingsley, P. (1986): *Probability and Measure.* John Wiley and Sons. New York.

Che, Y-K. and T-Y. Chung: “Incomplete Contracts and Cooperative Investment,” mimeo, University of Wisconsin.

Che, Y-K. and D. B. Hausch (1996): “Cooperative Investments and the Value of Contracting: Coase vs. Williamson,” mimeo, University of Wisconsin.

Fudenberg, D., D. Levine and J. Tirole (1985): “Infinite-Horizon Models of Bargaining with One-Sided Incomplete Information,” In *Game-Theoretic Models of Bargaining* ed. A. Roth. Cambridge University Press.

Fudenberg, D. and J. Tirole (1990): “Moral Hazard and Renegotiation in Agency Contracts,” *Econometrica* 58: pp. 1279–1320.

Gibbons, R. (1992): *Game Theory for Applied Economists*, Princeton University Press, Princeton, New Jersey.

Grossman, S. and O. Hart (1986):“The Costs and Benefits of Ownership: A Theory of Vertical and Lateral Investment,” *Journal of Political Economy* 94: pp. 691–719.

Grout, P. A. (1984): “Investment and Wages in the Absence of Binding Contracts: A Nash Bargaining Approach,” *Econometrica* 52: pp. 449–460.

Gul, F., H. Sonnenschein and R. Wilson (1986): “Foundations of Dynamic Monopoly and the Coase Conjecture.” *Journal of Economic Theory* 39: pp. 155–190.

Hermalin, B. and M. L. Katz (1991): “Moral Hazard and Verifiability: The Effects of Renegotiation in Agency,” *Econometrica* 59: pp. 1735–1754.

Ma, C.-T. A. (1991): “Adverse Selection in Dynamic Moral Hazard,” *Quarterly Journal of Economics* 106: pp. 255–276.

Ma, C.-T. A. (1994): “Renegotiation and Optimality in Agency Contracts,” *Review of Economic Studies* 61: pp. 109–130.

MacLeod, B. and J. Malcomson (1993): "Investments, Holdup, and the Form of Market Contracts," *American Economic Review* 83: pp. 811–837.

Matthews, S. A. (1995) "Renegotiation of Sales Contracts," *Econometrica* 63: pp. 567–590.

Rogerson, W. P. (1992) "Contractual Solutions to the Hold-Up Problem," *Review of Economic Studies* 59: pp. 777–793.