COMPUTABILITY AND LOGIC

FIFTH EDITION

HINTS

FOR ODD-NUMBERED PROBLEMS

CHAPTERS 1-10 & 12-17

Chapter 1

1.1 The converse assertion then follows from the first assertion by applying it to f^{-1} and its inverse f^{-1-1}.

1.3 For (a) consider the *identity* function $i(a) = a$ for all a in A. For (b) and (c) use the preceding two problems, as *per* the general hint above.

1.5 Show both sets are denumerable.

1.7 If we can fix for each i an enumeration of A_i

$$A_i = \{a_{i1}, a_{i2}, a_{i3}, \dots\}$$

Then we can enumerate $\cup A$, which is the set of all a_{ij} for all i and j in the same way we enumerated pairs (i, j) in Example 1.2.

However, when we assume that for each A_i there exists an enumeration of it, it follows that there exist many different such enumerations for each A_i; and when set theory is developed rigorously, in order to conclude that there is a way of fixing simultaneously for each i some one, specific enumeration out of all the many different enumerations that exist, we need a principle known as the axiom of choice. As this is not a textbook of set theory, we are not going to go into such subtleties.

Chapter 2

2.1 Imitate the proof for the set of positive integers.

2.3 You do *not* need to use trigonometry or give an analytical formula for the correspondence to do this problem; a simple geometric description of a correspondence will be enough.

2.5 There is a correction to the statement of this problem. While this can be done using the preceding two problems, as *per* the general hint, for students who remember trigonometry, a correspondence can also be defined directly using the tangent function.

2.7 Note that rational numbers whose denominator (when written in lowest terms) is a power of two have two binary representations, one ending in all 0's and the other in all 1's from some point on (as in 1/2 = .1000000… = .0111111…), while *in every other case the binary representation is unique and does not involve all* 0's *or all* 1's *from any point on*.

2.9 In addition to the immediately preceding problems, Problem 1.6 may be useful.

2.11 Read carefully through the sequence of preceding problems.

2.13 This is a philosophical rather than a mathematical question, and as such does not have a universally agreed answer, though there is a consensus that somehow *defining a set in terms of the notion of definability itself* is somehow to blame for the paradox.

Chapters 3

3.1 One state will be required in (a), two in (b).

3.3 Proceed as in Problem 3.1(b) but when reaching a blank in state 2 print a stroke and go into state 3. At this stage you will have a block of n strokes followed by a blank followed by a block of $m + 1 + k$ strokes. In state 3 on a stroke move right and go into state 4. In state 4 on a stroke erase it. You will now have blocks of n, $m + 1$, and $k - 1$ strokes. Take it from there.

3.5 Proceed in cycles, during each of which you erase the leftmost stroke of the first block and the rightmost stroke of the second block, and add a stroke to a third block to the right of them both. When one of the two original blocks has been completely erased, erase also the other. The trick is to keep track of when this happens.

Chapters 4

4.1 It is certainly not possible just *exploring* without *marking* the tape.

4.3 It is not possible to preserve the original block unaltered while making a copy.

4.5 A description of a function of the kind a universal machine would have to compute is implicit in the discussion of the diagonal function in the text.

Chapter 5

5.1 Subtraction is to the predecessor function as addition is to the successor function.

5.3 Use problem 5.1.

5.5 Keep subtracting y from x, while checking each time you do so that what is left is still $\geq y$.

5.7 Manœuvres of just this kind take place the simulation of abacus machines by Turing machines.

5.9 See preceding problems.

5.11 See the proof of Theorem 4.1.

Chapter 6

6.1 For instance, in (a), $g(x, y) = f(\mathrm{id}_2^2(x, y)\ ,\ \mathrm{id}_1^2(x, y)\ )$.

6.3 These can be done 'from scratch' or, generally more easily, by showing the indicated functions are compositions of functions already known to be primitive recursive.

6.5 Proposition 6.5 may be useful.

6.7 Each recursive function is denoted by some expression built up using Cn, Pr, and Mn from names for the zero, successor, and identity functions.

6.9 Use the following fact: There is a recursive function f such that $f(0) = 0$ but $f(x)$ is undefined for $x > 0$. (For instance, $f(x)$ = the least y such that $|x - y| + y = 0$.)

Chapter 7

7.1 Compare with Problem 6.1.

7.3 There is a correction to the statement of this problem. Use Corollary 7.8.

7.5 Consider the auxiliary function $g(n)$ = the least element of A that is $> n$.

7.7 Apply the preceding two problems to obtain a recursive function a and use it and the original f to define a suitable g.

7.9 First show that the auxiliary function $g(n) = J(f(n), f(n+1))$ is primitive recursive, where J is as in Problems 6.2 and 6.5.

7.11 First introduce a suitable auxiliary function, as in Example 7.20.

7.13 Suppose that c_i and d are the numbers associated with g_i and f respectively, so that

$$g_i(x_1, \ldots, x_n) < c_i \max(x_1, \ldots, x_n) + c_i,$$
$$f(y_1, \ldots, y_m) < d \max(y_1, \ldots, y_m) + d.$$

Show that $d(c + 1)$ will do as a number associated with h.

7.15 There is a correction to the statement of this problem. This is the problem that requires most familiarity with mathematical induction, according to which, in order to prove that all x and all y have some property it is enough to show that

(1) 0 and 0 have the property

(2) if 0 and j have the property, then 0 and $j + 1$ have the property

and that if i is such that i and j have the property for all j, then

(3) $i + 1$ and 0 have the property

(4) if $i + 1$ and k have the property, then $i + 1$ and $k + 1$ have the property.

7.17 First show that the auxiliary function

$$f(p, q) = \text{the least } s \text{ that covers } (p, q)$$

is a recursive total function.

Chapter 8

8.1 Remember that the right numeral is obtained by reading *backwards*, so that if $x_1 = 2$ and $x_2 = 3$, say, then the right numeral is 11110111.

8.3 Use the graph theorems.

8.5 Use the fact, noted just before the statement of Theorem 8.5 that the graph relation of the universal function F constructed in the proof of that theorem has the form $F(m, x) = y \leftrightarrow \exists t\, Qmxyt$ where Q is *primitive* recursive.

8.7 See the problems for chapter 7.

8.9 Let A be as in the proof of Corollary 8.8.

8.11 Show that if this claim failed for some f, then A would be recursive.

Chapter 9

9.1 For readers who have not previously studied logic, or whose memories of their previous study of logic are rusty, there will be one subtlety here, over how to represent 'All Ms are Ss'. For an indication of the manner in which this construction is treated in modern logic, see displayed formulas (9) and (10) in section 9.1.

9.3 There is a correction to the statement of this problem. Here 'in colloquial terms' would mean, for instance, saying 'grandparent' rather than 'parent of a parent'.

9.5 Use induction on complexity.

9.7 We do (c) as an example. If (F & B) is to be anything less than the whole of (F & G), then B) must be a left part of G, and hence by the Lemma 9.4(c) must have an excess of left over right parentheses. But this is impossible, since B, being a formula, has equally many parentheses of each kind, and therefore B) has one more right parenthesis than it has left parentheses.

Chapter 10

10.1 First show that substituting t for c in a closed term does not change the denotation of the term.

10.3 You will need to describe an interpretation, specifying its domain and the two-place relation on it that is to serve as the denotation of R. Reading R as 'greater than' may help suggest one.

10.5 In mathematics, 'All *As* are *Bs*' counts as 'vacuously' true if there are no *As*.

10.7 Compare with Example 10.3(d).

10.9 Compare with Example 10.5.

10.11 For (c), think of replacing A by B as a two-step process: introduce a new atomic C, and first replace A by C, then C by B.

10.13 For (a), the result for multiple variables is immediate from the result for a single replacement, on repeated application of the latter. To prove the result for a single variable, define a transformation * on formulas, eliminating bound occurrences of the variable y, by induction on complexity as follows. For an atomic formula G, let $G^* = G$. If $G = {\sim}F$, let $G^* = {\sim}F^*$, and if $G = (F_1 \mathbin{\&} F_2)$, let $G^* = (F_1{}^* \mathbin{\&} F_2{}^*)$, and similarly for $\vee$. If $G = \forall x F(x)$, where x is a variable other than y, let $G^* = \forall x F^*(x)$, while if $G = \forall y F(y)$, let $G^* = \forall z F^*(z)$, where z is the alphabetically first variable not already occurring, and similarly for $\exists$. It remains to prove G and G^* are equivalent for any sentence G.

Chapter 12

12.1 What does A tells us about the relative numbers of elements in the domain satisfying Px and satisfying $\sim Px$?

12.3 This can be done with a language having a one-place predicate Px and two one-place function symbols f and g. The trick is to find a sentence saying that there are as many elements in the domain altogether as there are *pairs* of elements satisfying Px.

12.5 Label the vertices in clockwise order A, B, C, D, and label the sides suggestively as $a = AB$, $b = BC$, $c = CD$, $d = DA$.

12.7 In the days before modern computers and calculators, a shortcut used with multiplication problems was to turn them into addition problems. How was this done?

12.9 If $\mathcal{M}$ is a model of Δ, and if j were an isomorphism from $\mathcal{M}$ to the standard model $\mathcal{N}$, what would be $j(c^{\mathcal{M}})$?

12.11 Combine the methods of the appropriate parts of the preceding problem.

12.13 Given a correspondence f from N to X_1, call one element a of X_1 less than another element b of X_1 if $f^{-1}(a)$ is less than $f^{-1}(b)$ in the usual order on natural numbers. Let $a_{0,0}$ be $f^{-1}(0)$, the least element of X_1. For each k let $a_{k+1,0}$ be the least element of X_1 not E_1-equivalent to any $a_{i,0}$ for $i \leq k$. For each m let $a_{k,m+1}$ be the least element of X_1 that is equivalent to $a_{k,0}$ and not identical to any $a_{k,i}$ for any $i \leq m$.

12.15 See Problem 10.6

12.17 Use the preceding problem and the observation that for any one, given denumerable nonstandard model or arithmetic, the set of sets of primes encrypted in that model is enumerable, since the set of elements of the domain available to encrypt sets of primes is.

12.19 Look at the problems to follow.

12.21 There is a correction to the statement of this problem. List the elements of the domain of j in increasing $<_A$ order as $a_0, a_1, \ldots, a_n$, and let $b_i = j(a_i)$, so that $b_0 < b_1 < \ldots < b_n$ in the usual order on natural numbers. What the problem asks you to show is that, given any new a in A there will be a rational number b such that b is related to the b_i in the usual order on rational numbers in the same way a is related to the a_i.

12.23 It will suffice to build a sequence of finite partial isomorphisms j_i as in Problem 12.22. Problem 12.21 can be used to get from j_i to j_{i+1}, but some care will be needed to arrange that every element of A gets into the domain of some j_i eventually.

12.25 Proceed as in Problem 12.23, but this time also take care to arrange that every rational number gets into the range of some j_i.

12.27 The preceding problems do not yet cover all the possibilities.

Chapter 13

13.1, **13.3**, **13.5**, **13.7** Hints are given in the text of section 13.5.

13.9 Imitate the proof of the isomorphism lemma, Proposition 12.5.

13.11 For (a) use the preceding problem; for (b) first note that if $B(c)$ implies A and c does not appear in A, then $\exists x\, B(x)$ implies A. (For if $\exists x B(x)$ does not imply A, then $\{\exists x B(x), \sim A\}$ is satisfiable, and then by Example 10.5(b) so is $\{B(c), \sim A\}$, and $B(c)$ does not imply A.)

13.13 See Problem 13.12.

13.15 See Problem 12.18.

Chapter 14

14.1 The compactness theorem is relevant.

14.3 Look how we got from (2) to (7) in Example 14.4.

14.5 Look how we got from (2) to (6) in Example 14.12.

14.7 Remember that you may use the results of earlier problems.

14.9 As in Example 14.13, all rides on making a suitable choice of formula $A(x)$ to apply (R8) to.

14.11 Imitate the proof of the inversion lemma for negation.

14.13 To show the effect of (R11) can be obtained using (R12), use the relevant inversion lemmas.

Chapter 15

15.1 The length is the number of digits in the decimal expansion of e that are < 8.

15.3 Apply Corollaries 12.17 and 15.7 to $T \cup \{A\}$ and $T \cup \{\sim A\}$ where neither A nor $\sim A$ is a theorem of T.

15.5 How many of the A_i would it take to deduce all the B_j?

15.7 The idea is just to 'check for each n through all possible models of size n', or more precisely, through a set of possible models containing at least one representative of each isomorphism type of models of size n. Generalize the preceding problem appropriately to show the set of isomorphism type representatives for a fixed n can be taken to be finite.

15.9 Let R be a recursive relation such that a is the code number of theorem of T if and only if $\exists n\, Ran$. Consider the set of sentences B such that for some A and n, B is the conjunction of n copies of A, and Ran holds, where a is the code number of A.

Chapter 16

16.1 Use Lemma 16.6.

16.3 Use Theorem 16.13.

16.5 See the proof of Corollary 15.6(a).

16.7 Use Proposition 7.17 and the remark following.

16.9 There is a correction to the statement of this problem. Again use Proposition 7.17.

16.11 Recall that we have proved $0 + y = y$ and $1 + y = y'$ have been proved in Examples 16.18 and 16.19.

16.13 For (*c*) first note that there is a least *n* with the property 'there is a sequence of length *n* with property *P*.'

16.15 To make (Q1)-(Q2) and (Q7)-(Q10) true, the denotation of **0** should be taken to be the least pair (in the $<_2$-order) and the denotation of **'** the function that given any pair as argument yields as value the least pair (in the $<_2$-order) among the pairs greater (in the $<_2$-order). It remains to devise a suitable addition function.

16.17 Use 'induction in the metalanguage,' proving the result first for $m = 0$, then for $m = n'$ assuming it holds for *n*.

16.19 For (a) again use 'induction in the metalanguage,' proving the result first for $b = 0$, then for $b = c'$.

16.21 The first half of the problem is to show how, using induction and the axioms of **Q**, to obtain the two axioms of **R** that are not axioms of **Q**. But one of these, (Q0), has already been done as Example 16.17, so it only remains to do (Q11). The other half of the problem is to show how, using induction and the axioms of **R**, to obtain the four axioms of **Q** that are not axioms of **R**. But two of these, (Q7) and (Q9), have already been done in section 16.4, so it only remains to do (Q8) and (Q10). For the first half of the problem note that according to Problems 16.10 and 16.11, we can get the commutative law for addition using induction and axioms common to **Q** and **R**.

Chapter 17

17.1 Use Theorem 16.16 and Problem 16.9.

17.3 Imitate the proof of the diagonal lemma, beginning as follows: For formulas $E_1(x, y)$ and $E_2(x, y)$ with code numbers e_1 and e_2, let the first and second double diagonals be

$\exists x\exists y(x = \mathbf{e}_1 \; \& \; y = \mathbf{e}_2 \; \& \; E_1(x, y))$, logically equivalent to $E_1(\mathbf{e}_1, \mathbf{e}_2)$ and
$\exists x\exists y(x = \mathbf{e}_1 \; \& \; y = \mathbf{e}_2 \; \& \; E_2(x, y))$, logically equivalent to $E_2(\mathbf{e}_1, \mathbf{e}_2)$.

17.5 The logical equivalence of $A(t)$ and $\exists y(y = t \; \& \; A(y))$ will be useful.

17.7 To begin with R be the Rosser sentence of T, $T_0 = T + \{\sim R\}$, $T_1 = T + \{R\}$. Let R_e for $e = 0$ or 1 be the Rosser sentence of T_e. Let $T_{00} = T_0 + \{\sim R_0\}$, $T_{01} = T_0 + \{R_0\}$, $T_{10} = T_1 + \{\sim R_1\}$, $T_{11} = T_1 + \{R_1\}$, and continue in this way.

17.9 Use the Craig reaxiomatization lemma, Problem 15.9.

17.11 To obtain $\mathcal{N}$, take the set of elements satisfying $N(x)$ as the domain $|\mathcal{N}|$. Let $R^{\mathcal{N}}$ hold for elements of the domain $|\mathcal{N}|$ if and only if $R^{\mathcal{M}}$ does, let $c^{\mathcal{N}} = c^{\mathcal{M}}$ for each constant c (as we may since it is given that $c^{\mathcal{M}}$ satisfies $N(x)$, that is, belongs to the domain $|\mathcal{N}|$), and let the value of $f^{\mathcal{N}}$ for elements of the domain $|\mathcal{N}|$ be the same as the value of $f^{\mathcal{N}}$ (as we may since it is given that if elements satisfy $N(x)$ so does the value of f for those elements).

17.13 It is enough to find a formula $N(x)$ that is satisfied in $\mathcal{Z}$ by an integer if and only if that integer is non-negative, for then by Problem 17.12, relativization will be a translation, and by Problem 17.10 the set T of sentences true in $\mathcal{Z}$ will be undecidable since the set S of sentences true in $\mathcal{N}$ is undecidable. If we have $<$, we can simply use $x = \mathbf{0} \vee \mathbf{0} < x$ for $N(x)$. To find an $N(x)$ that does *not* involve $<$ requires a major result of number theory, but one that has been mentioned more than once in this book.