Bidding Using Step Functions in Private Value Divisible Good Auctions

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Abstract

I analyze a model of a private value divisible good auction with different payment rules, standard rationing rule pro-rata on-the-margin and both with and without a restriction on the number of bids (steps) bidders can submit. I provide characterization of equilibrium bidding strategies in a model with restricted strategy sets and I show that these equilibria converge to an equilibrium of the model with unrestricted strategy sets as the restrictions are relaxed. However, not all equilibria in the unrestricted game can be achieved as limits of the equilibria of the restricted games. I demonstrate that the equilibrium conditions require that the Euler condition characterizing equilibrium in continuously differentiable strategies in the unrestricted model holds "on average" over the intervals defined by the length of each (price) step of the restricted strategy, where the average is taken with respect to the endogenous distribution of the market clearing price. More importantly, the characterization from the restricted model allows for a more natural interpretation of the involved trade-offs. Adapting the argument of Chao and Wilson (1987) I also prove that the foregone surplus of a bidder from using K steps rather than a continuous bid is proportional to $\frac{1}{K^2}$.

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1 Introduction

Accounting for the differences of real world applications and abstract models of economic theory is an important ingredient in any empirical work. The aim of this paper is to bridge the gap between the theoretical and empirical literature on divisible good auctions and to provide a more intuitive analysis of the involved trade-offs. The existing theory literature on divisible good auctions which started with the seminal work of Wilson (1979) usually assumes that the strategies may be continuously differentiable, which is not the case in most (if not all) real life applications. Using the results obtained using these theoretical models to draw inference from the bidding data on the primitives of the model such as the distribution of bidders' valuations can therefore be quite problematic. The goal of this paper is to provide a model which respects the defining features of the real world markets and to obtain equilibrium characterization which can then constitute basis for empirical work, the body of which has been growing in the recent years. I also provide asymptotic results which bound the loss of payoff a bidder might suffer due to using few steps and I relate the equilibria of the restricted model with those from the unrestricted model.

In particular, I consider a classic private value divisible good auction model with the standard rationing rule pro-rata on-the-margin. The new feature of my model is that bidders are restricted to use step functions as their bids, which is a common feature of virtually all auctions in practice. I provide characterization of equilibrium bidding strategies in both a discriminatory (DA) and a uniform price auction (UPA).[1] I further analyze the behavior in auctions with constrained strategy sets by proving two results: First, I prove that the

[1] In a discriminatory auction a bidder has to pay her full bid for all units she gets allocated. In a uniform price auction a bidder pays the same market clearing price for all such units.

necessary conditions for equilibrium of the restricted model require that the Euler condition characterizing equilibrium strategies in the unrestricted model holds "on average" over the intervals defined by the length of each (price) step of the restricted bidding strategy, where the average is taken with respect to the *endogenously determined* distribution of the market clearing price in the equilibrium of the *restricted* model. This is a result analogous to Wilson (1993)'s result about optimality of multipart tariffs, with the main difference being that in the price discrimination setting the distribution (of valuation types) over which the average is taken is *exogenous* and thus the same in the restricted and unrestricted models.[2] A corollary of this result is that as we increase the number of steps bidders are allowed to use without bounds, we approach the equilibrium characterization from a model without restrictions as characterized in Wilson (1979) for UPA and Hortaçsu (2002) for DA. Second, I provide a convergence result which bounds the loss of surplus of each bidder due to using only finitely many steps.

This paper is certainly not the first to analyze necessary conditions for bidding in multiunit auctions or to point out the importance of the discreteness of bidding. In recent work, de Castro and Riascos (2009) provide a very general characterization of optimal bidding in multiunit auctions with discrete quantity by focusing on necessary conditions with respect to optimal price bid. Instead, I analyze a model with continuous quantity and focus on optimality of quantity requests which results in conditions that have a very intuitive interpretation and allows for establishing a simple relationship with the models without discrete bidding. Other papers that examine necessary conditions for bidding in similar setting include Wilson (1979) and Hortaçsu (2002) who consider continuous quantity and Englebrecht-Wiggans and Kahn (2002) who focus on discrete quantity among others. As for dicrete bidding, in the context of the deregulated British electricity market, von der Fehr and Harbord (1993)

[2]The multipart tariffs arise when instead of offering the optimal cotinuous non-linear transfer schedule that would allow the continuously distributed private types to self-select, the monopolist instead offers only finitely many schedules (e.g., two-part tariffs).

point out that the equilibria of Klemperer and Meyer's (1989) model of supply function equilibria used in Green and Newbery (1992) to describe this market may not capture the bidding constraints appropriately, but many of the main insights generalize. Garcia-Diaz and Marin (2003) further extend the model of von der Fehr and Harbord (1993) to the case of deterministic and elastic demand. Neither of these papers, however, model the actual auction game as a uniform price divisible good auction and both assume bidders have no private information (for example, they abstract from the issue of private forward contracts). Importantly, they also assume that bidders submit one bid per generating unit (i.e., the quantities at which a bid is submitted are exogenously determined by the capacities of the generating units). In the context of treasury bill auctions Kremer and Nyborg (2004b) analyze a setting where bids have to be specified in terms of discrete quantity units and discrete prices. They show that if the distances between points on each grid are appropriately modified, the auctioneer can eliminate some equilibria which might be quite harmful to auction revenue otherwise - some of which have been identified in Back and Zender (2003) and LiCalzi and Pavan (2005). In contrast, I assume in the present paper that both quantity and price are continuous, but that bidding strategies can only consist of finitely many price-quantity pairs. I will argue that this alternative notion of discreteness is also very important for equilibrium characterization, and unlike von der Fehr and Harbord (1993), I will argue that some important results may change when this discreteness is taken into acount properly, for example bidders may submit bids higher than their values in a uniform price auction. It can also be shown that if price and quantity were continuous, rationing rule were pro-rata on-the-margin, and the auctioneer required bidding strategies to be step functions, the resulting set of possible equilibria would also not contain the undesirable underpricing equilibria - in other words the underpricing equilibria depend on the ability of bidders to submit strategies which create a sufficient slope in rivals' residual supplies which is possible only if either bids are differentiable or if the ratio of price ticks relative to quantity ticks violates the conditions provided in Kremer and Nyborg (2004b). Perhaps even more importantly with some non-degenerate supply uncertainty the strategies leading to the underpricing equilibria in the game with unrestricted strategy sets would cease to be even ε-equilibria of the game with restricted strategy sets.

The main reason for the growing interest in the empirical analysis of divisible good auctions is the long-standing debate about whether to use a discriminatory or uniform price auction in order to sell such commodities as government securities or electricity. In case of government securities, many influential economists have argued in favor of a uniform price auction as it seems to be less prone to informational barriers that might preclude some potential bidders from participating. Ausubel and Cramton (2002) established that these two auction mechanisms can be ranked ex ante neither on efficiency nor on revenue grounds. Therefore to be able to answer the question which mechanism might perform better in a given environment, the researcher needs to employ empirical techniques. On theoretical grounds progress has been made and the ranking of mechanisms in terms of extracted revenues was achieved for highly stylized models with no private information and restrictions on payoffs as in Fabra, von der Fehr and Harbord (2006) or with no private information and restricting attention to equilibria in linear strategies as in Rostek, Weretka and Pycia (2009). These papers are usually motivated by the seminal work of Klemperer and Meyer (1989) on supply function equilibria, where the behavior of the strategic agents leads them to use strategies that are ex post optimal. In an experimental setting, revenue ranking of different auction mechanisms has been examined in Brenner, Galai and Sade (2009). Several recent papers seek to answer this particular question also in a structural framework employing slight variations of the theoretical model proposed in the seminal work of Wilson (1979). Using a modification of the necessary conditions characterizing equilibrium bid functions of this model to restrict the bids to lie on a discrete grid, in an important paper Hortaçsu (2002) estimates marginal valuations of bidders in Turkish treasury bill auctions by providing a clever method for recovering an estimate of the residual supplies and noting that it is exactly this distribution that is needed to relate the bids and marginal values. However, as mentioned above, Wilson's model restricts the bidders to use only continuously differentiable bid functions so that elegant techniques of the calculus of variations can be applied to solve the model. Yet, in virtually all real world applications bidders are not allowed to use continuously differentiable bid functions. The number of points through which they can characterize their bid functions is finite and very often quite low (for example 10 in case of the Czech treasury bill auctions), moreover bidders almost never approach this upper bound. Kastl (2006) investigates a model of a uniform price auction of a perfectly divisible good in which each bidpoint (step) is costly to submit, and hence, in equilibrium, bidders submit finitely many points as their bids. He finds that necessary conditions and implied bidders' behavior can be quite different in such a setting as the need for a coarser characterization of bid functions forces the bidders to "bundle" bids for several units together and thus introduces new trade-offs. Due to this bundling effect and the resulting trade-off between gain or loss on the last unit in the bundle and higher or lower probability of winning the inframarginal units in the bundle, a rational bidder may submit a bid higher than his marginal valuation for that last unit in a UPA. This then makes empirical work more difficult as the researcher is not able to bound the ex-post revenue of a UPA by a hypothetical UPA with truthful bidding. In this paper I investigate these new trade-offs further. In particular, I show that the optimal demand at k^{th} step, q_k, in an auction with strategies being required to be step functions involves bidding in a way that the Euler condition characterizing the optimal demand $q(p)$ in a model with strategies being required to be continuously differentiable downward sloping functions holds "on average" over the length of each (price) step, i.e. the interval $[b_{k+1}, b_k]$. The probability measure with respect to which this average is taken is determined by the equilibrium of the restricted model.

The remainder of this paper is organized as follows. After setting up the general environment of the model in Section 2, I focus on equilibrium characterization in Section 3. I provide implicit characterization of equilibrium bidding strategies via a set of necessary conditions that rule out profitable local deviations in quantity demands for both a discriminatory and uniform price auctions and discuss equilibrium existence. In Section 4 I investigate the relationship between equilibria and the associated payoffs in auctions with restricted strategy sets and those in auctions with no restrictions on strategies. Finally, Section 5 concludes.

2 General Setup

I start with the basic share auction framework of Wilson (1979) with private information and private values. There are N potential bidders, who are bidding for a share of a perfectly divisible good and N is commonly known. Index 0 will be used for the auctioneer. Each bidder receives a private signal (possibly multidimensional of dimension M), S_i, which is the only private information about the underlying value of the auctioned good. The joint distribution of signals is denoted by $F(\mathbf{S})$.

Assumption 1 *Bidders signals, $S_0, S_1, ..., S_N$, are drawn from a common support $[0,1]^M$ according to an atomless joint d.f. $F(S_0, S_1, ..., S_N)$ with density f.*

Notice that a special case of this model - when signals are perfectly correlated - can be used to analyze models with no private information. The necessary conditions derived below require some conditions on the smoothness of the distribution of the residual supplies and hence if the smoothness is not provided by private information of bidders, one can alternatively assume an uncertain continuously distributed supply, as is common in the literature on electricity auctions. Similarly, sufficient conditions for existence in either auction type also require some limits on the dependence between private signals.

Winning share q of the unit good is valued according to a marginal valuation function $v_i(q, S_i)$. I impose the following assumptions on the marginal valuation function $v(\cdot, \cdot)$:

Assumption 2 *$v_i(q, S_i)$ is non-negative, measurable and bounded, strictly increasing in each component of S_i $\forall q$, and weakly decreasing and continuous in q $\forall S_i$.*

Note that the last assumption allows for flat marginal valuation schedule (for a given signal realization s_i). I denote by $V_i(q, S_i)$ the gross utility: $V_i(q, S_i) = \int_0^q v_i(u, S_i)\, du$.

As in practice in most auctions bidders are restricted in the number of points they can use to describe their bid functions, I start analyzing the model with an upper bound on the allowed number of bidpoints K. Later I consider relaxing this upper bound so that I can analyze the relationship between the model with restricted strategy sets and the traditional model with unrestricted strategy sets considered in Wilson (1979) and Hortaçsu (2002) for DA and UPA, respectively. I also assume that there is a bid l which loses no matter how the rivals behave and there is an upper bound on the maximal bid $\bar{b}$, which for example in the case of treasury bills could be the face value.[3] Let Q denote the size of the good to be divided between the bidders.[4] Q might itself be a random variable if it is not announced by the auctioneer ex ante and perhaps may depend on S_0 or if the auctioneer has the right to augment the supply after he collects the bids (or when supply is endogenous as in Back and Zender (2001)). In either case, I assume that the distribution of Q is common knowledge among the bidders and the upper bound of its support is $\overline{Q}$.

Assumption 3 *Each player $i = 1, ..., N$ has an action set:*

$$A_i = \left\{ \begin{array}{c} \left(\vec{b}, \vec{q}, K_i\right) : \dim\left(\vec{b}\right) = \dim\left(\vec{q}\right) = K_i \in \{0, ..., K\}, \\ b_{ik} \in B = \{l\} \cup \left[0, \bar{b}\right],\ q_{ik} \in \left[0, \overline{Q}\right],\ b_{ik} \geq b_{ik+1}, q_{ik} \leq q_{ik+1} \end{array} \right\}$$

[3] The assumption that bids are bounded is necessary only to guarantee existence. The bid l can be interpreted as the decision not to participate in a given auction.

[4] For example for a perfectly divisible unit good with certain supply, supply is usually normalized to one, i.e., $Q = 1$.

Notice that for unbounded K, assumption 3 does not impose any restriction on the actions other than that they be be non-increasing functions $y : \{l\} \cup [0, \bar{b}] \to [0, \overline{Q}]$.

Finally, since bidders may (for finite K must) use step functions, a situation may arise in which multiple prices would clear the market. If that is the case, I assume consistently with most real world auction mechanisms that the auctioneer selects the most favorable price from his perspective, i.e., the highest price.

Assumption 4 *If in any auction $\exists\ \underline{p}, \overline{p}$ such that $\forall p \in [\underline{p}, \overline{p}] : TD(p) = Q$, then $P^c = \max_p \{p : TD(p) = Q\}$, where $TD(p)$ denotes total demand at price p.*

Because bidders' strategies are step functions for any finite K, the residual supply is a step function and hence but for knife-edge cases any equilibrium involves rationing with probability one as the demand at the market clearing price will (but for knife-edge cases) exceed the available supply. Demands thus have to be rationed. Here I consider only the rationing rule pro-rata on-the-margin, under which the auctioneer proportionally adjusts the marginal bids so as to equate supply and demand, because this is the only rationing rule I am aware of being used in practice.[5]

Assumption 5 *The rationing rule is pro-rata on-the-margin, under which the rationing coefficient satisfies*

$$R(P^c) = \frac{Q - TD_+(P^c)}{TD(P^c) - TD_+(P^c)}$$

where $TD(P^c)$ denotes total demand at price P^c, and $TD_+(P^c) = \lim_{p \downarrow P^c} TD(p)$. Only the bids exactly at the market clearing price are adjusted.

It is important to notice that while rationing occurs with probability one (as the bidding functions are step functions), we have to distinguish between two different situations: (i)

[5] Kremer and Nyborg (2004a) show that an alternative rationing rule which does not give priority to bids above the market clearing price could also be used. They call such rationing rule "pro-rata" and they show that it may have some desirable properties in that it would eliminate low-revenue equilibria. The downside of implementing such a rationing rule is of course its potentially negative impact on efficiency of the allocation.

only one bidder is marginal, i.e., the residual supply cuts vertically this bidder's bid at the market clearing price and (ii) multiple bidders are marginal, i.e., the residual supply has a horizontal segment overlapping with each of these bidders' bids at the market clearing price. Notice that even though rationing occurs in both cases, in case (i) a slight perturbation in the quantity demanded by the rationed bidder at the market clearing price has no effect on his allocation conditional on him being rationed, while in case (ii) his allocation slightly increases. Throughout the paper I refer to the situation described in case (ii) as *a tie*.[6]

Finally, for some results I will also impose the following assumptions:

Assumption 6 *Supply Q is a random variable distributed on $\left[\underline{Q}, \overline{Q}\right]$ with strictly positive density conditional on S_i $\forall i$.*

While this assumption will not be necessary for the analysis of the uniform price auction, I will need it for discriminatory auction and for the results which examine the asymptotic behavior of equilibria as the restrictions (number of steps) are being relaxed. The reason is that for some results we need that for almost every bidder type and his equilibrium bid, the residual supplies are such that the distribution of the market clearing quantity allocated to this bidder type has no mass points at any quantity in the support of that bidder's demand. An alternative interpretation of this assumption is that it potentially induces a particular equilibrium selection in a discriminatory auction.

Bidders' pure strategies are mappings from private signals to bid functions: $\sigma_i : S_i \to A_i$ when strategy space is unrestricted or the set of step functions satisfying assumption 3 in the case of restricted strategies. A bid function for type s_i can thus be summarized by a function, $y_i(\cdot|s_i)$, which specifies for each admissible price $p \in P$, how big a share $y_i(p|s_i)$ of the securities offered in an auction (type s_i of) bidder i demands. When K_i is finite, we can summarize a bid function for type s_i also by a K_i-dimensional vector of price-quantity pairs, where k^{th} pair specifies the height and length of k^{th} step. To distinguish between equilibria

[6]Notice that rationing occurs even in case (i), but *a tie* does not occur.

in the restricted game and the unrestricted one, I use the term K-step equilibrium to denote a BNE of the restricted game.

Definition 1 *A K-step equilibrium is a Bayesian Nash Equilibrium of a game satisfying Assumption 3 with finite K.*

In the following sections I look at the characterization of Bayesian Nash Equilibria (and K-step equilibria) of this game with uniform price and discriminatory auction mechanisms with restricted strategy sets and relate these results to previous results which did not impose the restrictions on strategies.

3 Equilibrium Characterization

3.1 Discriminatory Auctions

The expected utility of a bidder i of type s_i who is employing a strategy $y_i(\cdot|s_i)$ in a discriminatory auction given that other bidders are using $\{y_j(\cdot|\cdot)\}_{j\neq i}$ can be written as:

$$EU_i(s_i) = E_{Q,S_{-i}|S_i=s_i}\left[\begin{array}{c} \int_0^{Q_i^c(Q,\mathbf{S},\mathbf{y}(\cdot|S))} v_i(u,s_i)\,du \\ -\sum_{k=1}^{K}\mathbf{1}\left(Q_i^c(Q,\mathbf{S},\mathbf{y}(\cdot|S))>q_k\right)(q_k-q_{k-1})\,b_k \\ -\sum_{k=1}^{K}\mathbf{1}\left(q_k\geq Q_i^c(Q,\mathbf{S},\mathbf{y}(\cdot|S))>q_{k-1}\right)\left(Q_i^c(Q,\mathbf{S},\mathbf{y}(\cdot|S))-q_{k-1}\right)b_k \end{array}\right]$$

where the random variable $Q_i^c(Q,\mathbf{S},\mathbf{y}(\cdot|S))$ is the (market clearing) quantity bidder i obtains if the state (bidders' private information and the supply quantity) is $(Q,\mathbf{S})$ and bidders submit bids specified in the vector $\mathbf{y}(\cdot|S)=[y_1(\cdot|S_1),...,y_N(\cdot|S_N)]$. $\mathbf{1}(\cdot)$ is an indicator function equal to 1 if the argument is true and 0 otherwise. A Bayesian Nash Equilibrium in this setting is thus a collection of functions such that almost every type s_i of bidder i is choosing his bid function so as to maximize his expected utility: $y_i(\cdot|s_i)\in\arg\max EU_i(s_i)$ for a.e. s_i and all bidders i.

It is straightforward to see that a bidder never submits a bid for any q that has a chance of being accepted and that is above his marginal valuation for that q in a discriminatory auction without restriction on strategies. All such bids are strictly dominated by bidding the marginal value for that q instead. When strategies are restricted, a similar result can be established and is stated as lemma 2 in the appendix.

Let the random variable $P^c(Q, \mathbf{S}, \mathbf{y}(\cdot|S))$ be the market clearing price associated with state $(Q, \mathbf{S})$. Hortaçsu (2002) shows that when strategy sets are restricted to include only continuously differentiable functions, every equilibrium strategy $y(p|s_i)$ has to obey the Euler equation:

$$v(y(p|s_i), s_i) = p + \frac{H(p, y(p|s_i))}{H_p(p, y(p|s_i))} \tag{1}$$

where $H(p, x)$ is the probability distribution of the market clearing price conditional on at least x units being awarded to bidder i at price p and all other bidders $j \neq i$ submit the equilibrium bid functions, i.e., $H(p, x) \equiv \Pr(P^c \leq p | y_i(p|s_i) = x) = \Pr\left(y_i(p|s_i) \leq Q - \sum_{j \neq i} y(p|s_j)\right)$ (H_p is the derivative of $H(\cdot, \cdot)$ with respect to the first argument, i.e., the density of the market clearing price.)

Now I focus on equilibria within the restricted class of strategies as specified in assumption 3 for finite K. I begin by stating a lemma which guarantees that under our assumptions no ties[7] such that at least one tying bidder would prefer not to tie occur with positive probability in a K-step equilibrium. While ties may occur in a discriminatory auction with restricted strategies, tying bidders must be indifferent between winning or not winning the marginal units and ties can occur only at the last step of each tying bidder's bid.

Lemma 1 *Under assumptions 1-5 ties occur with zero probability for a.e. s_i in any K-step equilibrium of a discriminatory auction, except possibly at the last step. If a tie occurs with positive probability at the last step, s_i must be indifferent between winning or losing all units*

[7] Recall that a tie occurs when demands of at least two bidders are rationed at the market clearing price.

between the lowest share he gets allocated after rationing in the event of a tie and the last infinitesimal unit he may be allocated in equilibrium, i.e., $b_K = v(q, s_i)\ \forall q \in \left[\underline{q}^{RAT}, \overline{q}\right]$*, where*
$\underline{q}^{RAT} \equiv \inf_{(q,s_{-i}) \in TIE(s_i)} Q_i^c(q, s_{-i}, s_i, \mathbf{y}(\cdot|S))$,
$TIE(s_i) \equiv \{(q, s_{-i}) : \exists j, m : b_{i,K_i}(s_i) = b_{j,m}(s_j) = P^c(q, s_{-i}, s_i, \mathbf{y}(\cdot|S))\}$,
and $\overline{q} \equiv \sup_{q,s_{-i}} Q_i^c(q, s_{-i}, s_i, \mathbf{y}(\cdot|S))$.

In order to grasp the intuition behind the last lemma, let us think about a type who is tying with positive probability. Being involved in a tie implies that the share allocated after rationing is strictly less than the share demanded. Whenever there is a positive probability of winning the last infinitesimal unit demanded at a given step, bidding above one's marginal valuation for that last infinitesimal unit is weakly dominated by bidding the marginal valuation itself.[8] Therefore, if the surplus enjoyed from obtaining the last infinitesimal unit allocated after rationing is strictly positive, then by continuity of $v(\cdot, s_i)$ so is the surplus for all units very close to this one. But then a slight deviation in i's bid up, which would break the tie, would result in a strict increase in i's payoff due to a strict increase in surplus from allocation and arbitrarily small increase in payment.

Therefore, were a bidder to tie with positive probability, the surplus on the last infinitesimal unit demanded would have to be weakly negative, as otherwise he would prefer to avoid the tie. Therefore, a tie could occur with positive probability only at the last step, because otherwise the bidder could just shift a small enough share demanded from the step at which tie has a positive probability of occurring to the next one at which he is bidding less. As in all states, in which he was allocated the share that he now shifted to the neighboring step, he had to pay the full bid, i.e., he earned no surplus on those (otherwise there would have been a simpler profitable deviation of increasing his bid by ε), and there is positive probability that the market clearing price will be weakly lower than his next bid (otherwise relabel the bids so that we call the last bid weakly below which market can actually clear the last one),

[8]For a formal statement see lemma 2 in the appendix.

this deviation would be strictly profitable (the expected quantity after rationing at k^{th} step is continuous in i's demand q_k and hence for a small enough deviation, the negative effect on the allocated quantity share after rationing can be made arbitrarily small).

Notice that lemma 1 does not rule out ties. In fact, with step functions, ties may occur in a discriminatory auction with positive probability, but if that is the case, lemma 1 guarantees that tying bidders are either exactly indifferent between tying and winning at b_{K_i} or in fact strictly prefer tying to winning at b_{K_i}. To see why ties may occur at the last step consider an example of a discriminatory auction of a unit divisible good where bidders are restricted to bidding just one step. Suppose there is one bidder with value $v_1(q) = \frac{5}{6} - q$ for $q \in [0,1]$ and two bidders with $v_2(q) = v_3(q) = \frac{1}{2}$ for $q \in [0,1]$. It is easy to see that an equilibrium exists in which all bidders submit a bid for the whole quantity, $q = 1$, at a price of $\frac{1}{2}$. Best responses of bidders 2 and 3 are verified trivially. The best response of bidder 1 to b_2, b_3 is also to submit a bid for the whole unit at $b_1 = \frac{1}{2}$ since in that case, the tie will be resolved by rationing and each bidder obtains $q = \frac{1}{3}$, which results in marginal value of the last infinitessimal unit won by bidder 1 being equal to the market clearing price $\frac{1}{2}$. Notice that bidder 1 strictly prefers tying at $\frac{1}{2}$ to winning at any other price and hence his best response is, in fact, unique. This equilibrium construction would remain valid also with non-degenerate private values if we assumed for example that values were $v_i + s_i$ where v_i is as above and private signals were perfectly correlated.

Notice that we can rewrite the expected utility of a bidder of type s_i in a discriminatory auction as:

$$\begin{aligned} EU(s_i) &= \sum_{k=1}^{K_i} \left[\Pr(b_k > P^c > b_{k+1} | s_i) V(q_k, s_i) - \Pr(b_k > P^c | s_i) b_k (q_k - q_{k-1}) \right] \\ &\quad + \sum_{k=1}^{K_i} \Pr(b_k = P^c | s_i) E_{Q, s_{-i} | s_i} \left[V(Q_i^c(Q, \mathbf{S}, \mathbf{y}(\cdot|S)), s_i) - b_k (Q_i^c(Q, \mathbf{S}, \mathbf{y}(\cdot|S)) - q_{k-1}) | b_k = P^c \right] \end{aligned} \tag{2}$$

where as before P^c is the (random) market clearing price, $Q_i^c(Q, \mathbf{S}, \mathbf{y}(\cdot|S))$ is the (random) quantity share allocated to i if the state of the world is $(Q, \mathbf{S})$ and bidders use strategies

$\mathbf{y}(\cdot|S)$, and $q_0 = b_{K_i+1} = 0$. Note that maximization of this expected utility with respect to quantity demanded at k^{th} step, q_k, results in expressions involving realizations of the market clearing price only in the interval $[b_{k+1}, b_k]$. In particular, as I emphasize later, the problem is slightly reminiscent of the solution to the maximization problem with unconstrained strategies - only there the optimality condition would involve only one realization of the market clearing price P^c at b_k rather than a whole interval.

Strategies in any equilibrium must be locally optimal, in other words, any local deviation must not be profitable. Using a local perturbation argument, for a discriminatory auction I obtain the following necessary conditions that the quantity requested in any step of a pure strategy that is a part of a K-step equilibrium has to satisfy.

Proposition 1 *Under assumptions 1-6 in any K-step Equilibrium of a discriminatory auction, for almost all s_i, every step $k < K_i$ in the equilibrium bid function $y_i(\cdot|s_i)$ has to satisfy*

$$\Pr(b_k > P^c > b_{k+1}|s_i)\left[v(q_k, s_i) - b_k\right] = \Pr(b_{k+1} \geq P^c|s_i)(b_k - b_{k+1}) \tag{3}$$

and at the last step K_i it has to satisfy $v(\overline{q}, s_i) = b_{K_i}$ where $\overline{q} = \sup_{q,s_{-i}} Q_i^c(q, s_{-i}, s_i, \mathbf{y}(\cdot|S))$.

The optimality condition with respect to the bid b_k can be derived in a straightforward manner by differentiating (2), but it cannot be simplified and interpreted as naturally as equation (3).[9] The intuition for the necessary condition (3) is nicely obtained once we think about in which states of the world varying i's demanded quantity can actually affect his payoff. Changing quantity demanded at k^{th} step affects i's payoff only in the states that he is not rationed in, as in the event he is rationed either he is the only bidder that is rationed and changing his demanded quantity will thus not affect his payoff, or if there is a tie, he must be indifferent between winning or not winning the last infinitessimal unit by lemma

[9] For empirical work in the spirit of the recent literature on non-parametric estimation of auction models pioneered by Guerre, Perrigne and Voung (2000), equation (3) is exactly what a researcher needs in order to obtain identification of marginal valuation at q_k using data on bids in a discriminatory auction.

1, and changing the quantity he wins marginally thus again has no effect on his payoff. Therefore the cost of quantity shading is losing the surplus on the last unit $v(q_k, s_i) - b_k$ which happens only in the case that the quantity demanded at this step is actually marginal, i.e., market clears strictly between b_k and b_{k+1} or in the (rather knife-edge) event that the residual supply is vertical at q_k at b_k. On the other hand by shading his demanded quantity by a unit, he saves the difference between his k^{th} and $(k+1)^{st}$ bid on this unit whenever the price will be weakly lower than the $(k+1)^{st}$ bid.

Notice an important distinction between the discretized version of the Euler equation from the model with unrestricted bidding given by equation (8) on page 10 in Hortaçsu (2002) and our optimality condition (3). In Hortaçsu's model, the indexes k and $k-1$ refer to adjacent prices on the discretized *price* grid, whereas in our optimality condition (3) they refer to bids at subsequent steps of the submitted bid curve. In both models, the bidders are assumed to choose the quantity demands optimally[10], but in Hortaçsu's setup the prices are *fixed* to lie on a K dimensional grid (potentially a very fine one) as in Nautz (1995), whereas in our setup the prices are determined by another set of necessary conditions. This distinction is important since Hortaçsu's equation (8) is a discretization of an equilibrium bidding function satisfying Wilson's optimality condition (which still has to hold as an equality when interpreted as a necessary condition for the choice of quantity at a given price) onto a grid of prices where the specification of the optimality condition in terms of the *observed* price bids $b_1, ..., b_K$ (which are not necessarily adjacent points in the grid) is achieved by assuming that the monotonicity constraint on the bid function is binding for intermediate prices on the grid. In other words, this assumes that a solution to the discretized Euler condition at that intermediate price would require a bid which would introduce a non-monotonicity into the bid function. Notice that the numerator in Hortaçsu's equation (8) which governs the choice

[10]Since the good is assumed to be perfectly divisible, and the marginal value is assumed to be continuous in quantity, the necessary conditions will hold with equality rather than being a system of inequalities. Also note that the necessary conditions with respect to price and quantity bid are identical in the setting where bids are differentiable functions.

of q_k, i.e., the quantity demanded at k^{th} grid step, involves the probability that the market clearing price is weakly lower than the bid at $(k+1)^{st}$ *step on the price grid*.[11] In our setup, the numerator involves the probability that the price is weakly lower than $(k+1)^{st}$ *bid* which is strictly lower than $(k+1)^{st}$ step on a sufficiently fine price grid.

3.1.1 Equilibrium Existence

Existence of equilibria in auctions and other potentially discontinuous games has been extensively studied in recent literature.[12] In particular, existence of equilibria in distributional strategies in multiunit auctions with private values has been established by Jackson and Swinkels (2005) for situations, where there is sufficient amount of uncertainty with respect to types of other bidders (full independence is not necessary). Immediate application of their result, however, requires that ties have zero probability in equilibrium, so that the payoff discontinuity is not relevant. By combining the results of lemma 1 and the technique of proving existence in Reny and Zamir (2003) (or alternatively, by slightly extending Jackson and Swinkels (2005)) existence of equilibrium in distributional strategies in case of discriminatory auctions with restricted strategy sets can be established.[13]

Proposition 2 *If A1-A5 hold and if the probability measure μ associated with the CDF $F(S_1, ..., S_N)$ on $\mathbf{S}$ is absolutely continuous with respect to $\prod_{i=1}^{N} \mu_i$ where μ_i is the marginal of μ on S_i, then an equilibrium in distributional strategies of a discriminatory auction exists $\forall K$.*

[11] This is later adjusted by assuming the monotonicity constraint on bids is binding at prices at which a bid is not observed (see page 34 in the appendix of Hortaçsu's paper).

[12] See e.g., Athey (2001), McAdams (2003, 2006), Kazumori (2003), Reny (1999), Reny and Zamir (2003), Jackson, Simon, Swinkels and Zame (2002) or Jackson and Swinkels (2005).

[13] See the earlier working paper version of this paper available online for a formal argument.

3.2 Uniform Price Auctions

Let us now turn to the case of a uniform price auction. The expected utility of a bidder i who is employing a strategy $y_i(\cdot|s_i)$ in a uniform price auction given that other bidders are using $\{y_j(\cdot|\cdot)\}_{j\neq i}$ can be written as:

$$\begin{aligned} EU_i(s_i) &= E_{Q,S_{-i}|S_i=s_i} u(s_i, S_{-i}) \\ &= E_{Q,S_{-i}|S_i=s_i}\left[\int_0^{Q_i^c(Q,\mathbf{S},\mathbf{y}(\cdot|S))} v_i(u, s_i, S_{-i})\, du - P^c(Q,\mathbf{S},\mathbf{y}(\cdot|S))\, Q_i^c(Q,\mathbf{S},\mathbf{y}(\cdot|S))\right] \end{aligned}$$

where as before $Q_i^c(Q,\mathbf{S},\mathbf{y}(\cdot|S))$ is the (market clearing) quantity bidder i obtains if the state (bidders' private information and the supply quantity) is $(Q,\mathbf{S})$ and bidders bid according to strategies specified in the vector $\mathbf{y}(\cdot|S) = [y_1(\cdot|S_1), ..., y_N(\cdot|S_N)]$, and similarly $P^c(Q,\mathbf{S},\mathbf{y}(\cdot|S))$ is the market clearing price associated with state $(Q,\mathbf{S})$. A Bayesian Nash Equilibrium in this setting is thus a collection of functions such that almost every type s_i of bidder i is choosing his bid function so as to maximize his expected utility: $y_i(\cdot|s_i) \in \arg\max EU_i(s_i)$ for a.e. s_i and all bidders i.

In most of the previous literature, starting with Wilson (1979), the set $\mathcal{Y}$ of admissible strategies was restricted to continuously differentiable functions so that calculus of variations techniques could be applied. These techniques enable us to show that in an IPV model and within this restricted class of strategies a symmetric BNE $y(\cdot|\cdot)$ has to satisfy the following necessary condition for all (p, s_i):

$$v(y(p|s_i), s_i) = p - y(p|s_i)\frac{H_y(p, y(p|s_i))}{H_p(p, y(p|s_i))} \tag{4}$$

where as above $H(p,x)$ is the probability that the market clearing price is weakly less than p conditional on $x = y_i(p|s_i)$ units being available at p to bidder i when all other bidders $j \neq i$ submit the equilibrium bid functions, i.e., $H(p,x) \equiv \Pr(P^c \leq p | y_i(p|s_i) = x) =$

$\Pr\left(y_i\left(p|s_i\right) \leq Q - \sum_{j\neq i} y\left(p|S_j\right)\right)$ (H_p and H_y are the derivatives of $H(\cdot,\cdot)$ with respect to the first and second argument respectively). As Wilson points out, the auction game might have multiple equilibria, some of which lead to low revenue for the auctioneer. Back and Zender (1993) also noted that a UPA might posses equilibria that are extremely "bad" for the auctioneer, because under rationing pro-rata on-the-margin any price above the reservation price could potentially be supported as a market clearing price at all states of the world, i.e., independent of the realization of the private information, by suitable choice of strategies (which are not continuously differentiable). Such equilibria, while achieved in a non-cooperative way, are usually called "seemingly collusive" and several authors (e.g., Kremer and Nyborg (2004b), LiCalzi and Pavan (2005) and McAdams (2007)) show how the auctioneer would eliminate at least some of these undesirable equilibria.

Let us now state the set of necessary conditions that pure strategies played by bidders in any equilibria have to satisfy with probability one. The argument is again based on a requirement that any local deviation must not be profitable. When bidders are restricted in the number of bids they are allowed to submit, Kastl (2006) proves the following result:

Proposition 3 *Under assumptions 1-5 in any K-step equilibrium of a Uniform Price Auction, for almost all s_i, every step k in the equilibrium bid function $y_i(\cdot|s_i)$ has to satisfy:*
(i) If $v(q_k, s_i) > b_k$

$$\Pr\left(b_k > P^c > b_{k+1}|s_i\right)\left[v\left(q_k, s_i\right) - E\left(P^c|b_k > P^c > b_{k+1}, s_i\right)\right] = q_k \frac{\partial E\left(P^c I\left[b_k \geq P^c \geq b_{k+1}\right]|s_i\right)}{\partial q_k} \tag{5}$$

(ii) If $v(q_k, s_i) \leq b_k$

$$\begin{aligned} &\Pr\left(b_k > P^c > b_{k+1}|s_i\right)\left[v\left(q_k, s_i\right) - E\left(P^c|b_k > P^c > b_{k+1}, s_i\right)\right] + \qquad (6) \\ &+ \Pr\left(\left(b_k = P^c \vee b_{k+1} = P^c|s_i\right) \ \wedge \ Tie\right) E\left(\left(v\left(Q^c, s_i\right) - P^c\right)\frac{\partial Q^c}{\partial q_k}|\left(b_k = P^c \vee b_{k+1} = P^c\right) \ \wedge \ Tie, s_i\right) \\ &= q_k \frac{\partial E\left(P^c I\left[b_k \geq P^c \geq b_{k+1}\right]|s_i\right)}{\partial q_k} \end{aligned}$$

Once again the necessary condition for each bid b_k can be derived in a straightforward manner, but it does not lend itself to a nice interpretation as (5).[14] The intuition for (5) is again quite simple. If a bidder bids below his marginal valuation of the last unit, he strictly prefers to win all units he requests at his bid, and thus cannot tie in equilibrium with positive probability. In this case, shading his demanded quantity by one unit results in a loss of his surplus on that unit, in the case that the market clearing price is between his k^{th} and $k+1^{st}$ bid and he thus wins exactly q_k units, which is captured on the LHS of (5). On the other hand, if his bid is marginal and sets the market clearing price, withholding the demand for the last unit can result in a decrease in the market clearing price, and this savings will be realized on all inframarginal units, and precisely this effect is captured on the RHS of (5). Whenever the restriction of strategy space prevents the bidder from a submitting a "separate" bid for any unit offered (which is of course trivially satisfied when q is continuous and only finitely many steps may be used), there is an additional trade-off: the surplus on the marginal unit is also compared to the effect of shading on the probability of winning the inframarginal units at that step in the bid function. This "bundling effect" may cause a bidder to bid above his marginal value. But when a bidder bids above his marginal valuation for some units, we can no longer guarantee that he does not tie with positive probability in equilibrium (in fact ex post if such bidder won some units for which he would be required to

[14] As equation (3) in the case of a discriminatory auction, in order to conduct non-parametric estimation, equation (5) provides the link between the (unobserved) primitives of the model, marginal valuation at q_k, and the observables - bids.

pay more than his bid, he would strictly prefer to be involved in a small tie so that rationing decreases his allocation), and hence the cost of shading his demand on the LHS of (6) now includes the effect on the surplus at the expected quantity after rationing in the event of a tie at one of his bids.

3.2.1 Equilibrium Existence

Since ties cannot be ruled out in the uniform price auctions with restricted strategy sets, even with independent types, the technique of obtaining existence as in Reny and Zamir (2003) or Jackson and Swinkels (2005) cannot be applied. In the next section I establish an asymptotic result which bounds the maximum loss of one bidder from using restricted strategies. Using this result together with an extension of the existence result of McAdams (2003) which establishes that with independent types and no restriction on the number of steps (i.e., as $K = \infty$) there exists an isotone pure strategy equilibrium when strategy spaces are lattices, I can establish existence of an ε-equilibrium. In particular, $\forall \varepsilon > 0 \ \exists \underline{K} : \forall K > \underline{K}$ there exists an ε-equilibrium of the game with strategies restricted to at most K steps.[15]

4 Properties of K-step equilibria

I begin the analysis of the properties of K-step equilibria by examining the relationship between K-step equilibrium optimality conditions derived in previous sections and the optimality conditions from the unrestricted model analyzed by Wilson (1979) and Hortaçsu (2002). Let $E^{DA}(b, q|s) \equiv v(q, s) - b - \frac{H(b,q)}{H_b(b,q)}$ be a function, which vanishes when the Euler condition for optimally bidding b for quantity q in a discriminatory auction in the unrestricted model holds. Let $E_{K,k}^{DA}$ be the corresponding function derived from the optimality conditions for quantity bids at k^{th} step in a K-step equilibrium. Similarly, let $E^{UPA}(b, q|s) \equiv v(q, s) - b + q\frac{H_q(b,q)}{H_b(b,q)}$ be the function vanishing at the optimal bid b for

[15]See the earlier working paper version of this paper available online for a formal statement and proof.

quantity q in a uniform price auction in the unrestricted model and $E^{UPA}_{K,k}$ be the function vanishing when the optimality condition for quantity bid in a K-step equilibrium holds. The optimality conditions for quantity demands in a K-step equilibrium are given by equations (3) and (5). The next proposition establishes that at demand q_k at the k^{th} step, the Euler condition from the continuous model has to hold on average over all prices covered by that step: $[b_{k+1}, b_k]$ such that q_k is available to bidder i, where the weight of each price in this interval corresponds to the likelihood that that price will clear the market which depends on the equilibrium strategies.

Proposition 4 *Under assumptions 1-6, the demand at step k, q_k, in both a uniform price and discriminatory auctions in the restricted model satisfies:*

$$0 = E^a_{K,k}\left(b_k, b_{k+1}, q_k | s_i\right) = \int_{b_{k+1}}^{b_k} E^a\left(p, q_k | s_i\right) dG\left(p, b_k, b_{k+1}, q_k\right)$$

for $a \in \{UPA, DA\}$ where $G\left(p, b_k, b_{k+1}, q_k\right)$ is the restriction of the cumulative distribution function of the market clearing price P^c to states where at least q_k units are awarded to bidder s_i, which is defined by

$$G\left(p, b_k, b_{k+1}, q_k\right) = \begin{cases} \Pr\left(P^c \leq b_{k+1}\right) & if \quad p \leq b_{k+1} \\ \Pr\left(P^c \leq b_{k+1}\right) + \int_{b_{k+1}}^{p} H_p\left(u, q_k\right) du & if \quad p \in \left(b_{k+1}, b_k\right) \\ \Pr\left(P^c \leq b_k \wedge Q^c_i \geq q_k\right) & if \quad p = b_k \end{cases}$$

The last proposition says that the demand at step k, q_k, in both a uniform price and discriminatory auctions in the restricted model satisfies the Euler condition from the unrestricted model at each step on average over the interval (b_{k+1}, b_k) for a.e. s_i (and at the boundaries of that interval when exactly q_k is allocated to s_i), where the average is taken with respect to the endogenous distribution of market clearing price, $H\left(p, x\right)|_{x=y(p|s_i)}$, from the restricted model.

A natural related question to investigate is what happens to equilibrium characterization as we increase the limit on the number of steps that bidders can use to characterize their bids. Fortunately, a corollary to the previous proposition confirms the intuition that as we increase the number of steps without bounds, our conditions for both auction mechanisms converge to the conditions derived in Wilson (1979) for a UPA and in Hortaçsu (2002) for a DA.

Corollary 1 *Under assumptions 1-6, as $K \to \infty$, any K-step equilibrium approaches an equilibrium from the differentiable model:*

$$\lim_{K \to \infty} E^a_{K,k}\left(b_k, b_{k+1}, q_k | s\right) = E^a\left(b_k, q_k | s\right) \;\; for \;\; a \in \{UPA, DA\} \; \forall k = 1, ..., K$$

Notice that if we were to require the prices to lie on a discrete grid with tick Δp and maintained continuity of quantity, which is the setup analyzed in Holmberg, Newbery and Ralph (2009), equilibrium existence would no longer be an issue in either format (even in the limit once an independent supply uncertainty with sufficient support is introduced, e.g., when $Q\left(s_0\right) \perp s_i, \forall i > 0$). The only difference between the necessary conditions of a model with a discrete price grid and those that I derived in previous sections is that there would be additional terms accounting for the choice of quantity (at a given price bid) on the allocation under rationing (as ties might occur with positive probability with discrete prices) as in (6). The existence of equilibrium in the limiting game as the price space becomes a continuum might be an issue in the uniform price auction *with a restriction on the number of steps*, without any restriction (which is the limiting game studied in Holmberg et al.) we can even establish existence of an isotone equilibrium in pure strategies whenever signals are not "too dependent" (see footnote 18 for a formal statement of this qualifier). Therefore the convergence of the necessary conditions established in the proof of corollary 1 implies also convergence of equilibria in the model with a discrete price grid as existence along the sequence as $K \to \infty$ and $\Delta p \to 0$ *and also* in the limit is not an issue and these necessary conditions hold in any equilibrium.[16] The potential problem with payoff discontinuity in the limit due to ties can be avoided by letting $K\Delta p \to \infty$, i.e., $K \to \infty$ faster than $\Delta p \to 0$, and then with sufficiently fine price grid, some bidder that were to tie with positive probability in a candidate equilibrium in the limiting game would have a profitable deviation already in the discretized game: either to avoid a tie or to beat a rival with whom he ties in the limit. This in turn implies that the limiting strategies continue to be mutual best-responses (as no discontinuity due to ties arises) and hence the limit of the sequence as $K \to \infty$ is an equilibrium. Let $\epsilon_{K,\Delta p}$ denote an equilibrium in which strategies are restricted to lie on a price grid with tick Δp (with the convention that $\Delta p = 0$ means continuous price) and bidders can use at most K steps and let $\mathcal{E}_{K,\Delta p}$ denote the whole set of equilibria, $\mathcal{E}_{K,\Delta p} := \cup \epsilon_{K,\Delta p}$. We can summarize the discussion above by the following corollary:

Corollary 2 *Under assumptions 1-6 where the support of bids in assumption 3 is modified to require a finite grid with tick Δp,*

$$\lim_{K\to\infty,\ \Delta p \to 0, K\Delta p \to \infty} \mathcal{E}_{K,\Delta p} \subset \mathcal{E}_{\infty,0}.$$

In other words, as $K \to \infty$, $\Delta p \to 0$ and $K\Delta p \to \infty$, the limit of the sequence of equilibria in games with restricted strategy sets is an equilibrium of a game with unrestricted strategy sets. However, there may be equilibria in the unrestricted game that would cease to be equilibria with even arbitrarily large restrictions on the number of steps. In particular, for example the equilibria identified in LiCalzi and Pavan (2005) and Back and Zender (2003)

[16]The limiting game with unrestricted strategies, however, typically has equilibria which are not limits of a sequence of equilibria of the restricted game. For example, the "collusive-seeming" equilibria identified in LiCalzi and Pavan (2005) are not present in the restricted game (at any point along the sequence as $\Delta p \to 0, K \to \infty$) , but exist in the limit. The equilibrium-set correspondence is thus not necessarily lower-hemicontinuous.

crucially depend on the differentiability so that the proper incentives not to break the tie at the low stop out price can be provided. Once quantity and price are assumed continuous, but submitted demands cannot be smooth, these equilibria do not exist: since these equilibria imply that there is a tie at the market clearing price, bidders would have an incentive to break the tie to increase their allocation. An infinitessimal deviation would then be profitable because of the impossibility of a differentiable residual supply.

Another interesting question to ask is therefore whether using only few steps is enough to capture almost the full expected surplus. This question is similar to analyzing a price discrimination problem, where the seller chooses few multipart tariffs. Wilson (1993, §8.3) established that multipart tariffs are approximately optimal, in the sense that the loss associated with using a n multipart tariffs rather than an optimal nonlinear tariff is of the order $\frac{1}{n^2}$. In related earlier work, Chao and Wilson (1987) showed a similar result in the context of a model with uncertain probability of service and few priority classes. The main difference from my setup is that in the price discrimination problem, the distribution of types is exogenously given, whereas in multiunit auctions, the distribution of the market clearing price is endogenously determined by equilibrium strategies of the participants. The following proposition establishes a similar result for bidding in multiunit auctions with restricted strategy sets.

Proposition 5 *Assume that assumptions 1-6 hold and that $v(q, s_i)$ has bounded second derivative w.r.t. q for a.e. s_i and fix a K-step equilibrium. Then s_i's loss in expected payoff from using a bid with K steps rather than a continuous bid is proportional to $\frac{1}{K^2}$, i.e., is $O\left(\frac{1}{K^2}\right)$.*

Note that I consider the asymptotics as the rivals' behavior is held fixed, which is motivated by the empirical fact that bidders actually never attain the upper bound of the number of steps they are allowed to use and thus could submit a finer bid unilaterally. Of course an interesting, but very different, question would be what happens to joint payoffs as all rivals increase their respective K_i. Knowing the answer to this question would be helpful in order to evaluate the effect of the choice of the upper bound K by the auctioneer on auction performance. Unfortunately, this question is too complicated to answer without imposing more structure, because of the potential multiplicity of equilibria. Nevertheless proposition 5 is sufficient to provide an explanation why we observe so few steps in bid functions submitted in practice. Since in practice almost no bidder attains the institutionally set upper bound on the number of steps, the assumption of holding rival's behavior fixed by fixing an equilibrium does not make the result uninteresting as there is room for a unilateral deviation to more steps. The result of proposition 5 would, of course, be much less relevant if in empirical applications all bidders were attaining the upper bound as the deviation by only one bidder to more steps would not then be feasible.

Assumption 6 is of particular importance for this proposition. The reason is that for the proof of the following proposition which relies on results on numerical approximation, we need that the second derivatives of $v(q, s_i)$ and the distribution of the quantity allocated to the type s_i with respect to q (at some price p) are bounded on the support of s_i's demand. While the boundedness of the former is a primitive assumption, the boundedness of the latter depends on the equilibrium and is thus endogenous. A sufficient, but not necessary, condition to achieve this is, for example, to assume that there is some arbitrarily small uncertainty about the total supply with continuous support and everywhere positive density, which is independent from $\mathbf{S}_{-i}$ as in assumption 6.[17] It would also be satisfied, for example, if an equilibrium involved strategies that were strictly monotone in private signals and signals were not perfectly correlated.[18]

[17]For example $Q \sim U[1-\varepsilon, 1+\varepsilon]$ and $Q \perp s_{-i}$.

[18] Formally, one would need that the measure μ induced by the CDF F on $\mathbf{S}$ is absolutely continuous with respect to $\prod_{i=1}^{N} \mu_i$ where μ_i is the marginal of μ on S_i, i.e., that it admits a density at least along the diagonal.

To put proposition 5 in better context, suppose that a bid function with just one step leads to 50% of the optimal expected surplus. Our result establishes that a bid function with just three steps already leads to about 95% of the optimal surplus. Therefore using only a few steps might indeed suffice for practical purposes. In light of the last proposition, it seems a little clearer why in practice bidders very often do not use all points that they are allowed to use. Moreover, the number of points actually submitted is quite often much lower than this upper bound. To rationalize this feature of the data Kastl (2006) assumes that bidding is costly. Using the estimates of marginal valuations he is then able to compute bounds on this implied cost of bidding, which in turn provide bounds on surplus that bidders could have extracted by using a larger number of points. His results suggest that the returns to the additional submitted point seem to be quite low. Similarly, Chapman, McAdams and Paarsch (2006) focus on ε-best responses and find that the loss of surplus due to not playing the exact best response (when costs of bidding are absent) seems to be surprisingly low.

5 Conclusion

I introduce a model of an auction of a perfectly divisible good, in which the bidders are restricted in the number of points through which they can characterize their bid functions. I provide equilibrium characterization for both uniform price and discriminatory auction mechanisms when values are private and discuss its existence. I further demonstrate that the equilibrium characterization from the restricted model limits to the equilibrium characterization of the unrestricted model as the number of points bidders are allowed to use grows large. I also show that the loss from using only K steps is of the order of $\frac{1}{K^2}$, which may provide an explanation why in practice we observe even fewer steps in submitted bid functions than the maximal number bidders are allowed to use. Even though bidders might not be best responding in many of the auctions as empirically documented in Chapman, McAdams and Paarsch (2006) and Kastl (2006), both of these papers find that the losses associated with not fine-tuning the bid a little more by submitting more steps seem to be very low.

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A Appendix

Some of the proofs that follow are complicated by the fact that I allow for flat marginal values and do not always require independence of signals or presence of some supply uncertainty. With these two assumptions, many of the proofs can be substantially simplified.

I begin with a preliminary lemma, which guarantees that no bidder submits a bid strictly above his marginal value in a discriminatory auction, whenever there is a positive probability of this bid being accepted.

Lemma 2 *In a discriminatory auction, if for a bidder of type s_i at some step k,* $\Pr\left(Q_i^c\left(Q, S_{-i}, S_i = s_i, \mathbf{y}\left(\cdot|S\right)\right) \geq q_k\right) > 0$, *then* $b_k \leq v\left(q_k, s_i\right)$.

Proof. Suppose for contradiction that $b_k > v\left(q_k, s_i\right)$ for some step k, such that $\pi \equiv \Pr\left(Q_i^c\left(Q, S_{-i}, s_i, \mathbf{y}\left(\cdot|S\right)\right) \geq q_k\right) > 0$. Let $\overline{q} = \sup\left\{q : v\left(q, s_i\right) > b_k\right\}$. Consider the following deviation: Change k^{th} step to $\left(b_k + \varepsilon, \overline{q} - \varepsilon^2\right)$. This deviation breaks a potential tie at b_k, but may result in a loss of surplus by demanding slightly less than $\overline{q}$. The upper bound on the loss from this deviation is: $\varepsilon\overline{q} + \varepsilon^2 v\left(\overline{q} - \varepsilon^2, s_i\right)$ where the first part is due to an increased payment and the second part due to a potential loss of surplus on quantities in $\left[\overline{q} - \varepsilon^2, \overline{q}\right]$. The lower bound on the gain from this deviation is $\pi\left(q_k - \overline{q}\right)\left(b_k - v\left(q_k, s_i\right)\right) > 0$, which is independent of ε. Therefore, for ε small enough the deviation is strictly profitable since $v\left(\cdot, s_i\right)$ is bounded by assumption. ■

The only case a bidder might submit a bid above his marginal value is at the last step (where we label the step as last when the probability of an allocation being larger than the one requested at this step is zero). Moreover, the only reason for submitting such a bid is if the bidder might tie with another bidder(s) and thus be rationed, in which case submitting a larger marginal demand might increase his allocation.

A.1 Proof of Lemma 1

Suppose that there exists an equilibrium, in which for a type s_i of bidder i a tie between at least two bidders can occur with positive probability $\pi > 0$. Since there can be only finitely many prices that can clear the market with positive probability, in order for a tie to be a positive probability event, it has to be the case that there exists a positive measure subset of types $\hat{S}_{-i} \in [0,1]^{N-1}$ such that for some bidder j, and all profiles of types $s_{-i} \in \hat{S}'_{-i} \subseteq \hat{S}_{-i}$ (another positive measure subset) and some steps k and l we have $b_{ik}(s_i) = b_{jl}(s_j) = P^c(Q, S_{-i}, s_i)$. Without loss suppose that this event occurs at the bid (b_{ik}, q_{ik}), and that the maximum quantity allocated to i after rationing is $\overline{q}_i^{RAT} < q_{ik}$. Let $\overline{S}_\pi^R$ denote the maximal level of the residual supply at b_{ik} in the states leading to rationing at b_{ik}.

Consider a deviation to a step $b'_{ik} = b_{ik} + \varepsilon$ and $q'_{ik} = q_{ik}$ where ε is sufficiently small. This deviation increases the probability of winning $q_{ik} - q_{ik-1}$ units. Most importantly in the states that led to rationing under the original bid, type s_i of bidder i will now obtain $q^* > \overline{q}_i^{RAT}$, where $q^* \geq \min\left\{q_{ik}, \overline{S}_\pi^R\right\}$. Notice that since we hypothesized a positive probability of a tie at b_{ik}, we need to have $q_{ik-1} < q_i^{RAT} < q_{ik}$ due to rationing pro-rata on-the-margin. Therefore, there is indeed room for a deviation. The probabilities of winning other units remain unchanged. Therefore the lower bound on the increase in s_i's expected gross surplus from such a deviation is $\pi\left(V(q^*, s_i) - V\left(\overline{q}_i^{RAT}, s_i\right)\right) \geq 0$

To continue, let us first focus on steps other than the last one, $k < K_i$ and suppose that marginal valuation function is strictly decreasing. Then we have that

$$\pi\left(V(q^*, s_i) - V\left(\overline{q}_i^{RAT}, s_i\right)\right) > 0$$

since $v(q^*, s_i) > v(q_k, s_i) \geq b_{ik}$ where the last inequality follows from lemma 2. The increased bid $b_k + \varepsilon$ also results in an increase in the payment for the share requested at this step. This increase, however, is bounded by $(q_k - q_{k-1})\varepsilon$. Comparing the upper bound on the change in expected payment with the lower bound on the change in expected gross utility, in order for this deviation to be strictly profitable we need to obtain

$$(q_k - q_{k-1})\varepsilon < \pi\left(V(q^*, s_i) - V\left(\overline{q}_i^{RAT}, s_i\right)\right) \tag{A-1}$$

Since

$$0 < \frac{\pi\left(V(q^*, s_i) - V\left(\overline{q}_i^{RAT}, s_i\right)\right)}{(q_k - q_{k-1})}$$

for ε small enough, the inequality (A-1) will hold, and thus the proposed deviation would indeed be strictly profitable for the type s_i. There can be only countably many types s_i with a profitable deviation otherwise bidder i could implement this deviation jointly and thus for a.e. type s_i ties have zero probability in equilibrium for all bidders i.

Suppose that marginal valuation function is not strictly decreasing and the above described deviation is not strictly profitable, i.e., $V(q^*, s_i) - V\left(\overline{q}_i^{RAT}, s_i\right) = 0$. Let $\overline{q} = \sup\{q : v(q, s_i) > b_k\}$. Then consider the following deviation instead: $(b_k + \varepsilon, \overline{q} - \varepsilon^2)$. The lower bound on the gain is: $\pi(q_k - \overline{q})(b_k - b_{k+1})$ due to a lower payment whenever the allocation under the original strategy was bigger than q_k (which is independent of ε) and the upper bound on the loss is: $\varepsilon\overline{q} + \varepsilon^2 v(\overline{q} - \varepsilon^2, s_i)$ due to a higher payment and loss of surplus. Again, for ε small enough, this deviation is strictly profitable.

Now focus on the last step, K_i. Suppose that s_i's expected gross surplus from the deviation described above is non-positive, i.e., that $b_k \geq v(q, s_i)$ for some $q \in \left[\overline{q}^{RAT}, q_k\right]$. Suppose that $\exists q' \in \left(\overline{q}^{RAT}, q_k\right)$ such that $v(q', s_i) > b_k$ and for some arbitrarily small $\delta > 0$, $v(q' + \delta, s_i) \leq b_k$.
Let $q'' = \sup_q \left\{q : v(q, s_i) > b_k \wedge q \in \left[\overline{q}_i^{RAT}, q_k\right]\right\}$. If $v(q'', s_i) > b_k$, then consider the following deviation instead: Submit the same bid with k^{th} step being replaced with $(b_k + \varepsilon, q'')$. By the same argument as above, such a deviation strictly increases s_i's expected gross utility since the loss from not winning $q \in (q'', q_k]$ is zero, which follows from definition of q'' and since the marginal valuation function is weakly decreasing. If $v(q'', s_i) = b_k$, then consider the following deviation: submit the same bid with k^{th} step being replaced with $(b_k + \varepsilon, q'' - \varepsilon^2)$. Now the loss from not winning units in $(q'' - \varepsilon^2, q'']$ is bounded by $\varepsilon^2 v(q'' - \varepsilon^2, s_i)$, the loss from increased payment is bounded as before by $(q_k - q_{k-1})\varepsilon$, and the gain from increased allocation in the event of a tie is as before (from an ex ante perspective) at least $\pi\left(V(q^*, s_i) - V\left(\overline{q}_i^{RAT}, s_i\right)\right)$ where $q^* \geq \min\left\{q'' - \varepsilon^2, \overline{S}_\pi^R\right\}$. As in the argument underlying the proof above, for sufficiently small ε, the proposed deviation results in a strict increase in s_i's payoff.

Now suppose that such q' does not exist, i.e., $v(q, s_i) \leq b_k\ \forall q \in \left[\overline{q}^{RAT}, q_k\right]$. Let $q'' = \inf_q \{q : v(q, s_i) \leq b_k \wedge q \in (q_{k-1}, q_k)\}$. If $\overline{q}^{RAT} > q'' > \underline{q}_i^{RAT}$ where $\underline{q}_i^{RAT}$ is the minimal allocation obtained by s_i in the event of a tie, then consider a deviation in k^{th} step to $(b_k + \varepsilon, q'' - \varepsilon^2)$. This results in a profitable deviation by the same argument as above, with some $0 < \pi' < \pi$ where π' is the probability of a tie which results in an allocation after rationing which is less than q''.

If instead $q'' \leq \underline{q}_i^{RAT}$, s_i wins all units on which he gains strictly positive surplus even when tying at k^{th} step. Therefore, a tie may occur with positive probability only at the last step and the bidder must not prefer winning any units in $\left[\underline{q}_i^{RAT},\overline{q}\right]$, where $\overline{q}\equiv\sup_{s_{-i},Q} y_i^c\left(Q,s_{-i},s_i\right)$, i.e., the maximal quantity s_i may be allocated in an equilibrium. Since units in $\left(\overline{q},q_{K_i}\right]$ are never won, the marginal value for those may be lower than the bid. QED

A.2 Proof of Proposition 1

I begin with steps $k<K_i$ so that lemma 1 can be applied. I will perturb the k^{th} step to $q'=q_k-\varepsilon$ and take the limit as $q'\to q_k$. Notice that all probabilities involved in the expressions below are differentiable under Assumption 6, since the supply uncertainty "smoothes" the distribution of residual supplies.

Let $\theta_1\left(q_k\right)$ denote set of states of the world in which $b_k<P^c$ (so that $Q^c<q_k$), similarly let $\theta_2\left(q_k\right)$ correspond to $b_k=P^c\wedge Q^c<q_k$, $\theta_3\left(q_k\right)$ to $b_k>P^c>b_{k+1}$ (so that $Q^c=q_k$), $\theta_4\left(q_k\right)$ to $b_{k+1}\geq P^c\wedge \textit{no tie at } b_{k+1}$, and $\theta_5\left(q_k\right)$ to $b_{k+1}=P^c\wedge \textit{tie at } b_k$. Notice that for $\theta_4\left(q_k\right)$ and $\theta_5\left(q_k\right)$ we have $Q^c>q_k$. Further let $\omega_2\left(q'\right)$ denote the set of states transferred from $\theta_2\left(q_k\right)$ to $\theta_3\left(q'\right)$ (as $q'<q_k$ market clearing price can only fall), $\omega_4\left(q'\right)$ set of states transferred from $\theta_2\left(q_k\right)$ to $\theta_4\left(q'\right)\cup\theta_5\left(q'\right)$ and finally $\omega_3\left(q'\right)$ states transferred from $\theta_3\left(q_k\right)$ to $\theta_4\left(q'\right)\cup\theta_5\left(q'\right)$.

Notice that

$$\begin{aligned}\theta_1\left(q'\right) &= \theta_1\left(q_k\right)\\ \theta_2\left(q'\right) &= \theta_2\left(q_k\right)-\omega_2\left(q'\right)-\omega_4\left(q'\right)\\ \theta_3\left(q'\right) &= \theta_3\left(q_k\right)+\omega_2\left(q'\right)-\omega_3\left(q'\right)\\ \theta_4\left(q'\right)\cup\theta_5\left(q'\right) &= \theta_4\left(q_k\right)\cup\theta_5\left(q_k\right)+\omega_3\left(q'\right)+\omega_4\left(q'\right)\end{aligned}$$

Since unlike in case of a uniform price auction, we cannot rule out that the residual supply and the bid curves have a common vertical overlap with positive probability, let $\nu_i=\lim_{q'\to q_k}\omega_i$, i.e., ν's contain the "knife-edge" states. Notice that for states in υ_2, residual supplies are vertical at q_k, go through b_k, but not through b_{k+1}. Residual supplies in ν_3 are vertical at q_k, go through b_{k+1} but not through b_{k+1} and finally residual supplies in ν_4 are vertical at q_k, go through both b_k and b_{k+1}.

Consider first the expected gross utility of a type s_i:

$$\begin{aligned}EV\left(s_i|q_k\right) &= \Pr\left(\theta_1\left(q_k\right)\right)E\left[V\left(Q_i^c\left(Q,\mathbf{S},\mathbf{y}\left(\cdot|S\right)\right),s_i\right)|\theta_1\right]+\Pr\left(\theta_2\left(q_k\right)\right)E\left[V\left(Q_i^c\left(Q,\mathbf{S},\mathbf{y}\left(\cdot|S\right)\right),s_i\right)|\theta_2\right]\\ &\quad +\Pr\left(\theta_3\left(q_k\right)\right)V\left(q_k,s_i\right)+\Pr\left(\theta_4\left(q_k\right)\cup\theta_5\left(q_k\right)\right)E\left[V\left(Q_i^c\left(Q,\mathbf{S},\mathbf{y}\left(\cdot|S\right)\right),s_i\right)|\theta_4\cup\theta_5\right]\end{aligned}$$

Perturbing q_k to q' and taking the difference, we obtain (dropping q_k from the argument of θ's):

$$\begin{aligned}
EV(s_i|q_k) - EV(s_i|q') &= \Pr(\theta_2)\{E[V(Q_i^c(Q,\mathbf{S},\mathbf{y}(\cdot|S)),s_i)|\theta_2] - E[V(Q_i^c(Q,\mathbf{S},\mathbf{y}'(\cdot|S)),s_i)|\theta_2]\} \\
&+ \Pr(\theta_3)\{V(q_k,s_i) - V(q',s_i)\} \\
&+ \Pr(\theta_4\cup\theta_5)\left\{\begin{array}{c} E[V(Q_i^c(Q,\mathbf{S},\mathbf{y}(\cdot|s)),s_i)|\theta_4\cup\theta_5] \\ -E[V(Q_i^c(Q,\mathbf{S},\mathbf{y}'(\cdot|s)),s_i)|\theta_4\cup\theta_5] \end{array}\right\} \\
&+ \Pr(\omega_2)\{E[V(Q_i^c(Q,\mathbf{S},\mathbf{y}'(\cdot|S)),s_i)|\omega_2,\theta_2] - V(q',s_i)\} \\
&+ \Pr(\omega_3)\{V(q',s_i) - E[V(Q_i^c(Q,\mathbf{S},\mathbf{y}'(\cdot|s)),s_i)|\omega_3,\theta_4\cup\theta_5]\} \\
&+ \Pr(\omega_4)\left\{\begin{array}{c} E[V(Q_i^c(Q,\mathbf{S},\mathbf{y}'(\cdot|s)),s_i)|\omega_4,\theta_2] \\ -E[V(Q_i^c(Q,\mathbf{S},\mathbf{y}'(\cdot|s)),s_i)|\omega_4,\theta_4\cup\theta_5] \end{array}\right\}
\end{aligned}$$

Dividing by $q_k - q'$ and taking the limit we obtain:

$$\begin{aligned}
\frac{\partial EV(s_i|q_k)}{\partial q_k} &= \Pr(\theta_2)E\left[v(Q_i^c(Q,\mathbf{S},\mathbf{y}(\cdot|S)),s_i)\frac{\partial Q_i^c(Q,\mathbf{S},\mathbf{y}(\cdot|S))}{\partial q_k}|\theta_2\right] \\
&+ \Pr(\theta_3\cup\nu_2)v(q_k,s_i) \\
&+ \Pr(\theta_4\cup\theta_5)E\left[v(Q_i^c(Q,\mathbf{S},\mathbf{y}(\cdot|S)),s_i)\frac{\partial Q_i^c(Q,\mathbf{S},\mathbf{y}(\cdot|S))}{\partial q_k}|\theta_4\cup\theta_5\right]
\end{aligned} \tag{A-2}$$

where I used that for states in ν_3 and ν_4 the allocation remains at q_k after perturbation and in state ν_2 the allocation decreases to q'.

Now consider the same argument applied to the expected payment for a type s_i:

$$\begin{aligned}
EPay(s_i|q_k) &= \Pr(\theta_1)Pay(\theta_1) + \Pr(\theta_2)[E(Q_i^c(Q,\mathbf{S},\mathbf{y}(\cdot|S))|\theta_2) - q_{k-1}]b_k + \Pr(\theta_3)[q_k - q_{k-1}]b_k \\
&+ \Pr(\theta_4)[E[Q_i^c(Q,\mathbf{S},\mathbf{y}(\cdot|S))|\theta_4] - q_k]b_{k+1} \\
&+ \Pr(\theta_5)[E[Q_i^c(Q,\mathbf{S},\mathbf{y}(\cdot|S))|\theta_4] - q_k]b_{k+1} \\
&+ \Pr(\theta_4\cup\theta_5)(q_k - q_{k-1})b_k + Pay(ind't\ of\ q_k)
\end{aligned}$$

Perturbing q_k to q' and taking the difference we obtain:

$$\begin{aligned}
EPay(s_i|q_k) - EPay(s_i|q') &= \Pr(\theta_2)[E(Q_i^c(Q,\mathbf{S},\mathbf{y}(\cdot|S))|\theta_2) - E(Q_i^c(Q,\mathbf{S},\mathbf{y}'(\cdot|S))|\theta_2)]b_k \\
&+ \Pr(\theta_3)[q_k - q']b_k \\
&+ \Pr(\theta_4\cup\theta_5)(q_k - q')(b_k - b_{k+1}) \\
&+ \Pr(\theta_4)[E[Q_i^c(Q,\mathbf{S},\mathbf{y}(\cdot|S))|\theta_4] - E[Q_i^c(Q,\mathbf{S},\mathbf{y}'(\cdot|S))|\theta_4]]b_{k+1} \\
&+ \Pr(\theta_5)[E[Q_i^c(Q,\mathbf{S},\mathbf{y}(\cdot|S))|\theta_5] - E[Q_i^c(Q,\mathbf{S},\mathbf{y}'(\cdot|S))|\theta_5]]b_{k+1} \\
&+ Terms\ involving\ \omega\text{'s}
\end{aligned}$$

Dividing by $q_k - q'$ and taking the limit we get:

$$\begin{aligned}\frac{\partial EPay\left(s_i|q_k\right)}{\partial q_k} &= \Pr\left(\theta_2\right) b_k E\left[\frac{\partial Q_i^c\left(Q,\mathbf{S},\mathbf{y}\left(\cdot|S\right)\right)}{\partial q_k}|\theta_2\right] \\ &+ \Pr\left(\theta_3 \cup \nu_2\right) b_k \\ &+ \Pr\left(\theta_4 \cup \theta_5\right)\left(b_k - b_{k+1}\right) \\ &+ \Pr\left(\theta_4\right) b_{k+1} E\left[\frac{\partial Q_i^c\left(Q,\mathbf{S},\mathbf{y}\left(\cdot|S\right)\right)}{\partial q_k}|\theta_4\right] \\ &+ \Pr\left(\theta_5\right) b_{k+1} E\left[\frac{\partial Q_i^c\left(Q,\mathbf{S},\mathbf{y}\left(\cdot|S\right)\right)}{\partial q_k}|\theta_5\right]\end{aligned} \tag{A-3}$$

where again all terms involving ω's except the one limiting to ν_2 vanish as above. Finally, combining (A-2) and (A-3), we can use lemma 1 to eliminate terms involving the derivative of the rationed quantity, as either (i) a tie is a positive probability event at b_{k+1} and then $E\left[v\left(Q_i^c\left(Q,\mathbf{S},\mathbf{y}\left(\cdot|S\right)\right),s_i\right)|\theta_5\right] = b_{k+1}$ or (ii) a tie is a zero probability event. Collecting terms, noting that under Assumption6, $\Pr\left(P^c = b_k \wedge q^c = q_k\right) = 0$ and rewriting θ's as the corresponding states of the market clearing price we obtain:

$$\Pr\left(b_k > P^c > b_{k+1}\right)\left[v\left(q_k,s_i\right) - b_k\right] = \Pr\left(b_{k+1} \geq P^c\right)\left(b_k - b_{k+1}\right)$$

as desired.

Finally, at the last step K, if $v\left(q,s_i\right)$ is locally continuous in q, then since there is no next step, we have $v\left(\overline{q},s_i\right) = b_K$ by lemma 1. QED

A.3 Proof of Proposition 4

Let us start with a discriminatory auction. The claim is that

$$\int_{b_{k+1}}^{b_k} E^{DA}\left(p,q_k|s_i\right) dG\left(p,b_k,b_{k+1},q_k\right) = E_{K,k}^{DA}\left(b_k,b_{k+1},q_k|s_i\right)$$

Recall that for $p \in \left(b_{k+1},b_k\right)$, we have $G\left(p,b_k,b_{k+1},q_k\right) = H\left(p,q_k\right) = H\left(p,y\left(p\right)\right)$ and at b_{k+1} we have: $G\left(b_{k+1},b_k,b_{k+1},q_k\right) = H\left(b_{k+1},q_k\right)$. Hence:

$$\int_{b_{k+1}}^{b_k}\left[v\left(q_k,s_i\right)-p-\frac{H\left(p,q_k\right)}{H_p\left(p,q_k\right)}\right]H_p\left(p,q_k\right)dp=$$
$$=\int_{b_{k+1}}^{b_k}\left[v\left(q_k,s_i\right)H_p\left(p,q_k\right)-pH_p\left(p,q_k\right)-H\left(p,q_k\right)\right]dp$$
$$=\Pr\left(b_k>P^c>b_{k+1}\right)v\left(q_k,s_i\right)-\Pr\left(P^c<b_k\right)b_k+\Pr\left(P^c\leq b_{k+1}\right)b_{k+1} \tag{A-4}$$

where the last equality follows by integrating by parts the term $pH_p(p,q_k)$ and by noting that under assumption 6, $\Pr\left(Q^c=q_k\wedge P^c=b_k\right)=0$ and hence $H\left(b_k,q_k\right)=\Pr\left(b_k>P^c\right)$. The desired equation (3) follows by noting that $\Pr\left(b_k>P^c\right)=\Pr\left(b_k>P^c>b_{k+1}\right)+\Pr\left(b_{k+1}\geq P^c\right)$ and re-arranging.

Similarly, for a uniform price auction we have:

$$\int_{b_{k+1}}^{b_k}\left[v\left(q_k(p),s_i\right)-p-q_k\frac{H_q\left(p,q_k\right)}{H_p\left(p,q_k\right)}\right]dH\left(p,q_k\right)=$$
$$\Pr\left(b_k>P^c>b_{k+1}\right)\left[v\left(q_k,s_i\right)-E\left(P^c|b_k>P^c>b_{k+1}\right)\right]-\int_{b_{k+1}}^{b_k}q_k\frac{H_q\left(p,q_k\right)}{H_p\left(p,q_k\right)}H_p\left(p,q_k\right)dp \tag{A-5}$$

The last term in (A-5) simplifies after integration by parts:

$$\begin{aligned}
&\int_{b_{k+1}}^{b_k}q_k\frac{H_q\left(p,q_k\right)}{H_p\left(p,q_k\right)}dH\left(p,q_k\right)=\\
&=q_k\frac{\partial}{\partial q_k}\int_{b_{k+1}}^{b_k}H\left(p,q_k\right)dp=\\
&=q_k\frac{\partial}{\partial q_k}\left[H\left(p,q_k\right)p|_{b_{k+1}}^{b_k}-\int_{b_{k+1}}^{b_k}pH_p\left(p,q_k\right)dp\right]=\\
&=-q_k\left[b_k\frac{\partial\Pr\left(b_k\leq P^c\wedge Q^c\geq q_k\right)}{\partial q_k}-b_{k+1}\frac{\partial\Pr\left(b_{k+1}\leq P^c\wedge Q^c\geq q_k\right)}{\partial q_k}\right]-\\
&-q_k\left[\Pr\left(b_k>P^c>b_{k+1}\right)\frac{\partial E\left[P^c|b_k>P^c>b_{k+1}\right]}{\partial q_k}+E\left[P^c|b_k>P^c>b_{k+1}\right]\frac{\partial\Pr\left(b_k>P^c>b_{k+1}\right)}{\partial q_k}\right]=\\
&=-q_k\frac{\partial}{\partial q_k}\left[E\left[P^c|b_k\geq P^c\geq b_{k+1}\right]*\Pr\left[b_k\geq P^c\geq b_{k+1}\right]\right]=\\
&=-q_k\frac{\partial E\left[P^c;b_k\geq P^c\geq b_{k+1}\right]}{\partial q_k}
\end{aligned}$$

where the first equality follows after interchanging the derivative and integral sign. The second equality follows after integrating by parts, the third by evaluating the terms and the last equality follows from the observation that $\frac{\partial \Pr(b_{k+1} \leq P^c \wedge Q^c \geq q_k)}{\partial q_k} = 0$ since q_k could be allocated to s_i at price b_{k+1} only if the residual supply was horizontal in which case an infinitessimal change in q_k obviously does not change the market clearing price. Therefore without loss we can add a term $2b_{k+1}\frac{\partial \Pr(b_{k+1} \leq P^c \wedge Q^c \geq q_k)}{\partial q_k} = 0$, noting that $\frac{\partial E(P^c|P^c = b_m)}{\partial q_k} = 0 \; \forall k, m$ and collecting terms then delivers the last equality.

When ties have positive probability, two additional terms would appear due to the effect of marginal demand on allocation as in equation 6. QED

A.4 Proof of Corollary 1

While this corollary follows immediately from the previous proposition, for the sake of clarity I show here explictly that as the number of bidpoints increases, the optimality condition limits to the one derived from the calculus of variations in Wilson (1979) for a uniform price auction and Hortaçsu (2002) for a discriminatory auction.

Define $H(p,q) = \Pr(P^c \leq p | q \geq Q^c) = \Pr\left(\sum_{j \neq i} q_j(p) + q \leq Q\right)$. Similarly, let $\hat{H}(p,q) = \Pr(P^c < p | q \geq Q^c)$ be the CDF of the market clearing price with strict inequality. Recall that equation (5) can be written as:

$$\Pr(b_k > P^c > b_{k+1})\, v(q_k, s_i) = E(P^c; b_k > P^c > b_{k+1}) + q_k \frac{\partial E(P^c; b_k \geq P^c \geq b_{k+1})}{\partial q_k}$$

In the proof of the previous proposition we established that :

$$\frac{\partial E(p; b_k \geq P^c \geq b_{k+1})}{\partial q_k} = \int_{b_{k+1}}^{b_k} \frac{H_q(p, q_k)}{H_p(p, q_k)} dH(p, q_k)$$

Hence after dividing both sides by $(b_k - b_{k+1})$ we can rewrite the first order condition as:

$$\begin{aligned} & \frac{\left[\hat{H}(b_k, q_k) - H(b_{k+1}, q_k)\right]}{b_k - b_{k+1}} v(q_k, s_i) \\ = & \frac{\int_0^{b_k} p\hat{H}_p(p, q_k)\, dp - \int_0^{b_{k+1}} pH_p(p, q_k)\, dp + q_k \left[\int_{b_{k+1}}^{b_k} H_q(p, q_k)\, dp\right]}{b_k - b_{k+1}} \end{aligned}$$

and take the limit as $b_{k+1} \to b_k$. First, notice that in the limit the distribution of the market clearing price does not have any masspoints as the probability of ties for any finite K is zero at all bids at which $b_k < v(q_k, s_i)$ and without restrictions on K, no bidder would submit a bid exceeding the marginal value. Therefore, $\hat{H} \to H$. Now after applying l'Hospital's rule

$E_{s_{-i}}\left(P^c; b_k > P^c > b_{k+1}\right) \rightarrow b_k H_p\left(b_k, q_k\right)$ since:

$$\lim_{b_{k+1}\rightarrow b_k} \frac{\int_0^{b_k} p\hat{H}_p\left(p, q_k\right) dp - \int_0^{b_{k+1}} pH_p\left(p, q_k\right) dp}{b_k - b_{k+1}} = \frac{-b_k H_p\left(b_k, q_k\right)}{-1}$$

Furthermore:

$$\lim_{b_{k+1}\rightarrow b_k} \frac{\int_{b_{k+1}}^{b_k} H_q\left(p, q_k\right) dp}{b_k - b_{k+1}} = \frac{-H_q\left(b_k, q_k\right)}{-1}$$

Finally:

$$\lim_{b_{k+1}\rightarrow b_k} \frac{\left[\hat{H}\left(b_k, q_k\right) - H\left(b_{k+1}, q_k\right)\right]}{b_k - b_{k+1}} = H_p\left(b_k, q_k\right)$$

Collecting terms and omitting subscript k, and using that in equilibrium $q = y\left(p|s_i\right)$, we obtain:

$$v\left(y\left(p|s_i\right), s_i\right) = p - y\left(p|s_i\right) \frac{H_q\left(p, y\left(p|s_i\right)\right)}{H_p\left(p, y\left(p|s_i\right)\right)}$$

as desired.

Now consider the necessary condition for a discriminatory auction and rewrite it using $\hat{H}$ and H defined as above:

$$\left[\hat{H}\left(b_k, q_k\right) - H\left(b_{k+1}, q_k\right)\right]\left[v\left(q_k, s_i\right) - b_k\right] = H\left(b_{k+1}, q_k\right)\left(b_k - b_{k+1}\right)$$

Dividing both sides by $b_k - b_{k+1}$ and take the limit as $b_{k+1} \rightarrow b_k$.

$$\lim_{b_{k+1}\rightarrow b_k} H\left(b_{k+1}, q_k\right) \frac{b_k - b_{k+1}}{b_k - b_{k+1}} = H\left(b_k, q_k\right)$$

And as above

$$\lim_{b_{k+1}\rightarrow b_k} \frac{\left[\hat{H}\left(b_k, q_k\right) - H\left(b_{k+1}, q_k\right)\right]}{b_k - b_{k+1}} = H_p\left(b_k, q_k\right)$$

Using $q_k = y\left(p|s_i\right)$ and $b_k = p$ we can rewrite the necessary condition as

$$v\left(y\left(p|s_i\right), s_i\right) = p + \frac{H\left(p, y\left(p|s_i\right)\right)}{H_p\left(p, y\left(p|s_i\right)\right)}$$

which is equation (1). QED

A.5 Proof of Proposition 5

By fixing a K-step equilibrium, we are fixing the bidding strategies of rivals, and thus we face a single agent maximization problem. I follow the strategy of Chao and Wilson (1987) of applying the results from numerical analysis to approximations of integrals by finite sums with equally space grids. The main insight here is to use appropriately re-defined variables to be able to apply an argument similar to Chao and Wilson. The main idea is to express the (expected) payoff as a product of the (expected) net surplus from every unit allocated and the probability that that unit will be won. Let $G(Y) = \Pr(q_i^c \geq Y)$, i.e., (one minus) the probability distribution of the market clearing quantity (given s_i's bid) - in other words this is the probability bidder i gets allocated at least Y. The potential surplus (expected utility) of bidder of type s_i in a discriminatory auction is:

$$U_\infty = \int_0^Q \left[v(q, s_i) - y^{-1}(q|s_i)\right] G(q)\, dq$$

With submitting an optimal bid under the restriction to K steps, let $\Delta_k = q_{k+1} - q_k$ be the difference in adjacent demands and let $q_0 = 0$ and $q_K \leq Q$ where Q solves $v(Q, s_i) = 0$. The realized surplus can be written as:

$$U_K = \sum_{k=1}^{K} \overline{v}_k \overline{G}_k \Delta_k$$

where for step k the average (expected) net surplus, $\overline{v}$, and the (average) probability of demand at k^{th} step being (at least partially) satisfied, $\overline{G}_k$, are:

$$\overline{v}_k = \frac{\int_{q_k}^{q_k+\Delta_k} \left[v(q, s_i) - y^{-1}(q|s_i)\right] dq}{\Delta_k}$$

and

$$\overline{G}_k = \frac{\int_{q_k}^{q_k+\Delta_k} G(q)\, dq}{\Delta_k}$$

Using Hermite interpolation formula (Ralston and Rabinowitz (1978), Chapter 3.7):

$$\overline{v}_k = v(q_k, s_i) - y^{-1}(q_k|s_i) + \frac{1}{2}\Delta_k \left(v'(q_k, s_i) - \left(y^{-1}(q_k|s_i)'\right)\right) + O\left(\Delta_k^2\right)$$
$$= v_k + \frac{1}{2}\Delta_k v_k' + O\left(\Delta_k^2\right)$$

where v_k denotes the net surplus at q_k, i.e., $v_k \equiv v(q_k, s_i) - y^{-1}(q_k|s_i)$, and

$$\overline{G}_k = \frac{\int_{q_k}^{q_k+\Delta_k} G(q)\, dq}{\Delta_k} = G_k + \frac{1}{2}\Delta_k G_k' + O\left(\Delta_k^2\right)$$

where $G_k = G(q_k)$. Notice that for the last approximation step to be valid, we need that the second derivatives of $v(q, s_i)$ and $G(q)$ are bounded and this is ensured by assumption 6.

Combining these, we can rewrite the expected surplus as:

$$\begin{aligned} U_K &= \sum_{k=1}^{K} \left(v_k + \frac{1}{2}\Delta_k v_k'\right)\left(G_k + \frac{1}{2}\Delta_k G_k'\right)\Delta_k + O\left(\Delta_k^3\right) \\ &= \sum_{k=1}^{K} v_k G_k \Delta_k + \frac{1}{2}\left(v_k G_k\right)' \Delta_k^2 + O\left(\Delta_k^3\right) \end{aligned}$$

Since the step sizes, Δ_k's, were chosen optimally, considering an evenly spaced grid, with steps $\Delta = \frac{Q}{K}$ apart, must result in surplus that is weakly lower:

$$U_K \geq U_{\hat{K}} = \left[\sum_{k=1}^{K} v_k G_k \Delta\right] + \frac{1}{2}\Delta\left[\sum_{k=1}^{K} \left(v_k G_k\right)' \Delta\right] + O\left(\Delta^3\right) \tag{A-6}$$

Applying the trapezoid rule for numerical integration (Gautschi (1997), Chapter 3.2.1), we get that:

$$U_\infty = \int_0^Q \left[v(q, s_i) - y^{-1}(q|s_i)\right] G(q)\, dq = \sum_{k=1}^{K} v_k G_k \Delta - \frac{1}{2}\Delta\left[v_0 G_0 + v_K G_K\right] + O\left(\Delta^2\right)$$

or

$$\sum_{k=1}^{K} v_k G_k \Delta = U_\infty + \frac{1}{2}\Delta\left[v_0 G_0 + v_K G_K\right] - O\left(\Delta^2\right)$$

and by a similar argument

$$\begin{aligned} \sum_{k=1}^{K} \left(v_k G_k\right)' \Delta &= \int_0^Q (vG)'\, dq + O(\Delta) \\ &= v_Q G_Q - v_0 G_0 + O(\Delta) \end{aligned}$$

Note that by definition of Q, $v_Q = v_K = 0$, as $v_K = v(Q, s_i) - y^{-1}(Q|s_i)$ and hence plugging these expressions into (A-6), the expressions $\frac{1}{2}\Delta v_0 G_0$ conveniently cancel and we get

$$U_K \geq U_{\hat{K}} = U_\infty - O\left(\Delta^2\right)$$

which was to be shown.

Finally, notice that in a uniform price auction the surplus can be written as:

$$U_\infty = \int_0^Q \left[v\left(q, s_i\right) - \int_q^Q y^{-1}\left(u|s_i\right) \frac{g\left(u\right)}{G\left(q\right)} du \right] G\left(q\right) dq$$

since $\int_q^Q y^{-1}\left(u|s_i\right) \frac{g(u)}{G(q)} du = \mathbb{E}\left(P^c|q^c > q\right)$. The proof for the discriminatory auction can thus be applied after the appropriate change of variables. QED