

# Liquidity, Business Cycles, and Monetary Policy

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## Abstract

This paper presents a model of monetary economy with differences in liquidity across assets. Our purpose is to study how aggregate production and asset prices fluctuate with shocks to productivity and liquidity. In so doing, we examine what role government policy might have through open market operations that change the mix of assets held by the private sector. We also show that certain apparent anomalies of asset markets are in fact normal features of a monetary economy in which the circulation of money is essential for a better allocation of resources.

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# 1 Introduction

In this paper, we provide a model of monetary economy with differences in liquidity across assets. Our purpose is to understand how aggregate production and asset prices fluctuate with recurrent shocks to productivity and liquidity. In so doing, we want to find out what role government policy might have through open market operations that change the mix of assets held by the private sector. The present paper takes fiat money to be one of the assets under consideration. We investigate under what the circumstances money is essential to a better allocation of resources. We show that certain apparent anomalies of the non-monetary economy are in fact normal features of an economy where money is essential. Among the well-known puzzles we have in mind are: the low risk-free rate puzzle; the excess volatility of asset prices; the anomalous savings behavior of certain households, and their low participation in asset markets. Before describing our monetary economy, we should start with some remarks about modeling strategy.

In broad terms, there are two ways of getting fiat money into a competitive macroeconomic model. One approach is to endow money with some special function – for example, cash-in-advance or sticky nominal prices.<sup>1</sup> The other approach is to starve agents of alternatives to money – as in an overlapping generations framework where money is the sole means of saving.<sup>2</sup> Although the first approach, in particular the cash-in-advance model and the dynamic sticky price model, has proved important to monetary economics and policy analysis, it is not well-suited to answering larger questions to do with liquidity. By endowing money with a special function in the otherwise frictionless economy with complete Arrow-Debreu security market, one is imposing rather than explaining the use of money, which precludes the possibility that other assets or media of denomination may substitute for money. And the second approach rules out any general discussion of liquidity if there are no alternative assets to money.

There are many noncompetitive models of money, leading with the random matching framework. In principle, such models are suited to analyzing liquidity. But they are necessarily special, and it is difficult to incorporate them into the rest of macroeconomics.<sup>3</sup> We believe there is a need for a work-

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<sup>1</sup>See Clower (1967), Lucas and Stokey (1987) and Woodford (2003).

<sup>2</sup>P.A.Samuelson (1958). Bewley (1980) also models money in a context where there are no alternative assets with which to save.

<sup>3</sup>See Kiyotaki and Wright (1989) and Duffie, Garleanu and Pedersen (2005) for exam-

horse model of money and liquidity, with competitive markets, which does not stray too far from the other workhorse, the real business cycle model.

In our framework, markets are competitive, money is not endowed with any special function, and there are other assets traded besides money. The basic model presented in Section 2 has two kinds of agents, entrepreneurs and workers, homogeneous general output, and three assets: fiat money, physical capital (or equity of physical capital), and human capital. The supply of fiat money is fixed. The supply of capital changes through investment and depreciation. A worker's human capital is inalienable, which means that he or she cannot borrow against future labour income: in any period, the only labour market is a spot market for that period's labour services. There is a commonly available technology for combining labour with capital to produce general output.

In each period a fraction of the entrepreneurs (but none of the workers) can invest in producing new capital from general output. The arrival of such an investment opportunity is randomly distributed across entrepreneurs through time. Because not all entrepreneurs can invest in each period, there is a need to transfer resources from those who don't have an investment opportunity (that period's savers) to those who do (that period's investors). To acquire general output as input for the production of new capital, investing entrepreneurs sell equity claims to the future returns from the newly produced capital. The crucial feature of the model is that, because the investing entrepreneur is still needed to run the project to produce output and he cannot precommit to work throughout its life, he is able to pledge only a fraction (say  $\theta$ ) of future returns from the new capital. As the investing entrepreneur can only issue new equity up to  $\theta$  fraction of his investment, he faces a borrowing constraint.

Because of the borrowing constraint, the investing entrepreneur needs to finance the investment cost partly by selling his holding of money and equity of the other agents (which he acquired in the past). Another important feature of our model is that the existing equity (the claim to the return of the existing capital stock) cannot be sold as quickly as money. Specifically, we assume that, in any given period  $t$ , an agent can sell only a fraction  $\phi_t$  of his equity holding. In contrast to the upper bound of new equity issue

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ples. For recent attempts to make matching models applicable for policy analysis, see Shi (1997), Lagos and Wright (2005), He, Huang and Wright (2005, 2008) and Aruoba, Waller and Wright (2007).

$\theta$ , the value of  $\phi_t$  is the limit that the investing entrepreneur can resell his equity holding before he misses the investment opportunity. Thus, we call  $\theta$  as borrowing constraint, call  $\phi_t$  as "resaleability constraint", and call both constraints together as "liquidity constraints". Here, we take both  $\theta$  and  $\phi_t$  as exogenous parameters and consider a stochastic shock to  $\phi_t$  as "liquidity shock".<sup>4,5</sup>

The question is to what extent does these liquidity constraints inhibit the efficient transfer of resources from savers to investors. There may be a role for money to lubricate the transfer of additional resources. Whether or not agents use money is determined endogenously. We show that for high enough values of  $\theta$  and average  $\phi_t$  money is not used and has no value in the neighborhood of the steady state. But for lower values of  $\theta$  and average  $\phi_t$ , money plays an essential role. In the latter case, we call the economy a monetary economy.

We find that a necessary feature of a monetary economy is that the investment of entrepreneurs is limited by liquidity constraints. He cannot raise the entire cost of investment externally, given that the borrowing constraint binds for the sale of new equity. That is, he has to make a downpayment for each unit of investment from his own internal funds. But in trying to raise funds to make this downpayment, he is constrained by how much of his equity holding can be sold in time: the resaleability constraint binds here. In this sense, an investing entrepreneur finds money more valuable than equity, because he can use all of his money to finance new investment whereas he can use only a fraction  $\phi_t$  of his equity: money is more liquid than equity.<sup>6</sup>

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<sup>4</sup>In Kiyotaki and Moore (2003, 2005b), we develop a framework in which the resaleability constraint arises endogenously due to adverse selection in resale market. Each new capital comprises a large number of parts, some of which will eventually fail (depreciate completely), although nobody knows which when the new capital is produced. Overtime, the insiders (producing entrepreneurs and those who bought the new equity) learn privately which parts will fail. If the fraction of failing parts is large enough, no outsider will buy second-hand equity for fear of being sold lemons.

In order to overcome the adverse selection problem, the investing entrepreneur can spend extra resource to bundle all the parts of the new capital together in such a way that they cannot later unbundled. Then the equity issued against bundled new capital will be resold freely.

<sup>5</sup>In the analysis of financial market, Brunnermeier and Perdersen (2007) use notion of "funding liquidity" to refer borrowing constraint and "market liquidity" to refer resaleability constraint.

<sup>6</sup>In practice, there are clearly differences between kinds of equities – e.g. between the share of a large publicly-traded company and stock of a small privately-held business.

We find that in a monetary economy, the expected rate of return on money is very low, less than the expected rate of return on equity. Nevertheless, a saving entrepreneur chooses to hold some money in his portfolio, because, in the event that he has an opportunity to invest in the future, he will be liquidity constrained, and money is more liquid than equity. The gap between the return on money and the return on equity is a liquidity premium. This may help explain the low risk-free rate puzzle.

We also find that both the returns on equity and money are lower than the rate of time preference. This means that agents such as workers, who don't anticipate having investment opportunities, will choose to hold neither equity nor money. They will simply consume their labour income, period by period. This may help explain why certain households do not save nor participate in asset markets. It is not that they don't have free access to asset markets nor that they are particularly impatient, but rather that the return on assets isn't enough to attract them.<sup>7</sup>

In most real business cycle models, there is no feedback from asset market to output. That is not true in our monetary economy. Consider a shock which reduces resaleability of equity  $\phi_t$  persistently. (This liquidity shock is meant to capture an aspect of the recent financial turmoil in which many assets - such as auction rate bonds - that used to be liquid suddenly have become only partially resaleable). Then, the amount the investing entrepreneurs can use as downpayment for investment shrinks. Moreover, anticipating a lower resaleability, the equity price falls, which can be thought as "a flight to liquidity". This raises the size of the required downpayment per unit of new investment. Altogether, investment suffers from the negative shock to the resaleability of equity. This feedback mechanism causes asset prices and investment to be vulnerable to liquidity shocks unlike standard general equilibrium asset pricing model without liquidity constraints.

In a later section of the paper we introduce government. Our interest is in seeing the effect of policy on the behavior of the private economy. We consider that the government holds equity and can costlessly change the supply of fiat money. In our framework with flexible prices, an once-for-all change of money supply through lump-sum transfer to the entrepreneurs (helicopter drop) does not have any effect on aggregate real variables. The open market operation to purchase equity by issuing money, however, will increase the

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<sup>7</sup>If workers face their own investment opportunity shocks, then workers would save but only in money. See discussion of the later section.

ratio of the value of liquid money to illiquid equity of the private sector, and will expand investment through a larger liquidity of investing entrepreneurs.<sup>8</sup> Using this framework, we analyze how government (or central bank) can use the open market operation to accommodate the effects of the shock to the productivity and how it can offset the effects of shock to the liquidity.

## 2 The Basic Model without Government

Consider an infinite-horizon, discrete-time economy with four objects traded: a nondurable general output, labour, equity and fiat money. Fiat money is intrinsically useless, and is in fixed supply  $M$  in the basic model of this section.

There are two populations of agents, entrepreneurs and workers, each with unit measure. Let us start with the entrepreneurs, who are the central actors in the drama. At date  $t$ , a typical entrepreneur has expected discounted utility

$$E_t \sum_{s=t}^{\infty} \beta^{s-t} u(c_s) \quad (1)$$

of consumption path  $\{c_t, c_{t+1}, c_{t+2}, \dots\}$ , where  $u(c) = \log c$  and  $0 < \beta < 1$ . He has no labour endowment. All entrepreneurs have access to a constant-returns-to-scale technology for producing general output from capital and labour. An entrepreneur holding  $k_t$  capital at the start of period  $t$  can employ  $l_t$  labour to produce

$$y_t = A_t (k_t)^\gamma (l_t)^{1-\gamma} \quad (2)$$

general output, where  $0 < \gamma < 1$ . Production is completed within the period  $t$ , during which time capital depreciates to  $\lambda k_t$ ,  $0 < \lambda < 1$ . We assume that the productivity parameter,  $A_t > 0$  which is common to all entrepreneurs, follows a stationary stochastic process. Given that each entrepreneur employs labour at competitive market with the real wage rate  $w_t$ , the gross profit is proportional to the capital stock as:

$$y_t - w_t l_t = r_t k_t, \quad (3)$$

where the gross profit per unit of capital  $r_t$  depends upon productivity, aggregate capital stock and labour supply condition as will be seen shortly.

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<sup>8</sup>This idea can be traced back to Metzler (1951).

The entrepreneur may also have an opportunity to produce new capital stock. Specifically, at each date  $t$ , with probability  $\pi$  he has access to a constant-returns technology that produces  $i_t$  units of capital from  $i_t$  units of general output good. The arrival of such an investment opportunity is independently distributed across entrepreneurs and through time, and is independent of aggregate shocks. Again, investment is completed within the period  $t$  – although newly-produced capital does not become available as an input to the production of general output until the following period  $t+1$ :

$$k_{t+1} = \lambda k_t + i_t. \tag{4}$$

We assume there is no insurance market against having an investment opportunity.<sup>9</sup> We also make a regularity assumption that the subjective discount factor is larger than the fraction of capital left after production (one minus the depreciation rate):

$$\textit{Assumption 1} : \beta > \lambda.$$

This mild restriction is not essential, but will make the distribution of capital and asset holdings across of individual entrepreneurs well-behaved.

In order to finance the cost of investment, the entrepreneur who has an investment opportunity can issue equity claim to the future returns from the newly produced capital. Normalize one unit of equity at date  $t$  to be claim to the future returns from the one unit of investment of date  $t$ : it pays  $r_{t+1}$  output at date  $t+1$ ,  $\lambda r_{t+2}$  at date  $t+2$ ,  $\lambda^2 r_{t+3}$  at date  $t+3$ , and so on.

We make two critical assumptions. First, we assume that the entrepreneur who produces new capital cannot precommit to work through the lifetime even though he is needed to produce full output described by the production function; thus an investing entrepreneur can pledge at most  $\theta$  fraction of future returns from his new capital. As he can issue new equity only up to  $\theta$  fraction of new capital, the parameter  $\theta$  represents the tightness

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<sup>9</sup>This assumption can be justified in a variety of ways. For example, it may not be possible to verify that someone has an investment opportunity; or verification may take so long that the opportunity has gone by the time the claim is paid out. A long-term insurance contract based on self-reporting does not work here because the people are able to trade assets covertly. Each of these justifications warrants formal modelling. But we are reasonably confident that even if partial insurance were possible our broad conclusions would still hold. So rather than clutter up the model, we simply assume that no insurance scheme is feasible.

of the borrowing constraint an investing entrepreneur faces.<sup>10</sup> Because an entrepreneur who finds an investment opportunity faces this borrowing constraint, he must finance the cost of investment partly from selling his holding of equity and money.

The second critical assumption is that entrepreneurs cannot sell their equity holding as quickly as money. More specifically, before the investment opportunity disappears, the investing entrepreneur can sell only  $\phi_t$  fraction of his equity holding within a period even though he can use all his money holding. It is tantamount to a peculiar transaction cost per period: zero for the first fraction  $\phi_t$  of equity sold, and then infinite. We take  $\phi_t$  as an exogenous parameter of liquidity of the equity, and call  $\phi_t$  as "resaleability constraint". We consider that the aggregate productivity  $A_t$  and the liquidity of equity  $\phi_t$  jointly follow a stationary Markov process in the neighborhood of the constant unconditional mean  $(A, \phi)$ . A shock to  $A_t$  is productivity shock, and shock to  $\phi_t$  is considered as "liquidity shock".

In general, an entrepreneur has three kinds of asset in his portfolio: money, equity of the other entrepreneur, and unmortgaged capital stock (a fraction of own capital stock against which the entrepreneur has not issued the equity: own capital stock minus own equity issued).

Balance sheet	
money	own equity issued
equity of others	
own capital stock	net worth

It turns out to be difficult to analyze the aggregate fluctuations of the economy with these three assets, because there is a rich dynamic interaction between the distribution of assets and aggregate production. Thus, we make a simplifying assumption: at every period, an entrepreneur can remortgage up to a fraction  $\phi_t$  fraction of his unmortgaged capital stock. Then, equity of the other entrepreneurs and unmortgaged capital stock become perfect substitute as means of saving: both pays the same returns stream of  $r_{t+1}$  at date t+1,  $\lambda r_{t+2}$  at date t+2,  $\lambda^2 r_{t+3}$  at date t+3, and so on per unit; and the holder can sell up to  $\phi_t$  fraction of his holding of both. Because equity held by the agents other than the producer is sometimes called "outside equity" and unmortgaged capital stock is called "inside equity", we call both together as

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<sup>10</sup>See Hart and Moore (1994) which explains how the borrowing constraint arises from inalienability of human capital of the entrepreneur.



"equity".

Let  $n_t$  be the quantity of equity and let  $m_t$  be money held by an individual entrepreneurs at the start of period  $t$ . The liquidity constraints (borrowing constraint and resaleability constraint) are expressed as

$$n_{t+1} \geq (1 - \theta)i_t + (1 - \phi_t)\lambda n_t, \text{ and} \quad (5)$$

$$m_{t+1} \geq 0. \quad (6)$$

The entrepreneur who invests  $i_t$  can issue at most  $\theta i_t$  equity and can resell at most  $\phi_t$  fraction of outside and inside equity holding after depreciation during this period. Taken together, (5) implies the minimum holding of equity at the beginning of period  $t+1$  is equal to  $1 - \theta$  fraction of investment plus  $1 - \phi_t$  fraction of depreciated equity holding of this period. (6) implies his fiat money holding cannot be negative - the only agent who can have a negative position of fiat money is government (or central bank) in the next section.

Let  $q_t$  be the price of equity in terms of general output. It is also equal to Tobin's  $q$ : the ratio of stock market value to the replacement cost of capital, (noting that production cost of capital is unity per unit). Let  $p_t$  be the price of money in terms of general output. (Warning!  $p_t$  is customarily defined as the inverse: the price of general output in terms of money. But, a priori, money may not have value, so we prefer not to make it the numeraire.) The entrepreneur's flow of funds constraint at date  $t$  is then given by

$$c_t + i_t + q_t(n_{t+1} - i_t - \lambda n_t) + p_t(m_{t+1} - m_t) = r_t n_t. \quad (7)$$

The left-hand side (LHS) is his expenditure on consumption, investment and net purchases of equity and money. The right-hand side (RHS) is his dividend income, which is proportional to the holding of equity at the start of this period.

Turn now to the workers. At date  $t$ , a typical worker has expected discounted utility

$$E_t \sum_{s=t}^{\infty} \beta^{s-t} U \left[ c'_s - \frac{\omega}{1 + \nu} (l'_s)^{1+\nu} \right], \quad (8)$$

of consumption path  $\{c_t, c_{t+1}, c_{t+2}, \dots\}$  given his labour supply path  $\{l_t, l_{t+1}, l_{t+2}, \dots\}$ , where  $\omega > 0, \nu > 0$  and  $U[\cdot]$  is increasing and strictly concave. The flow-of-funds constraint of the worker is

$$c'_t + q_t(n'_{t+1} - \lambda n'_t) + p_t(m'_{t+1} - m'_t) = w_t l'_t + r_t n'_t. \quad (9)$$

The consumption expenditure and net purchase of equity and money in the LHS is financed by wage and dividend income. Workers do not have investment opportunities, and cannot borrow against their future labour income.

$$n'_{t+1} \geq 0, \text{ and } m'_{t+1} \geq 0. \quad (10)$$

An equilibrium process of prices  $\{p_t, q_t, w_t\}$  is such that: entrepreneurs choose labour demand  $l_t$  to maximize the gross profit (3) subject to production function (2) for a given start-of-period capital stock, and choose consumption, investment, capital stock and start-of-next-period equity and money holdings  $\{c_t, i_t, k_{t+1}, n_{t+1}, m_{t+1}\}$ , to maximize (1) subject to (4) - (7); workers choose consumption, labour supply, equity and money holding  $\{c'_t, l'_t, n'_{t+1}, m'_{t+1}\}$  to maximize (8) subject to (9) and (10); and the markets for general output, labour, equity and money all clear.

Before we characterize equilibrium, it helps to clear the decks a little by suppressing reference to the workers. Given that their population has unit measure, it follows from (8) and (9) that their aggregate labour supply equals  $(w_t/\omega)^{1/\nu}$ . Maximizing the gross profit of a typical entrepreneur controlling capital  $k_t$ , we find his labour demand,  $k_t [(1 - \gamma)A_t/w_t]^{1/\gamma}$  which is proportional to  $k_t$ . So if the aggregate stock of capital controlled by entrepreneurs at the start of date  $t$  is  $K_t$ , labour-market clearing requires that

$$(w_t/\omega)^{1/\nu} = K_t [(1 - \gamma)A_t/w_t]^{1/\gamma}.$$

Substituting back the equilibrium wage  $w_t$  into the LHS of (3), we find that the individual entrepreneur's maximized gross profit equals  $r_t k_t$  where

$$r_t = a_t (K_t)^{\alpha-1}, \quad (11)$$

and the parameters  $a_t$  and  $\alpha$  are derived from  $A_t, \gamma, \omega$  and  $\nu$ :

$$\begin{aligned} a_t &= \gamma \left( \frac{1 - \gamma}{\omega} \right)^{\frac{1-\gamma}{\gamma+\nu}} (A_t)^{\frac{1+\nu}{\gamma+\nu}} \\ \alpha &= \frac{\gamma(1 + \nu)}{\gamma + \nu}. \end{aligned} \quad (12)$$

Note from (12) that  $\alpha$  lies between 0 and 1, so that  $r_t$  – which is parametric for the individual entrepreneur – declines with the aggregate stock of capital  $K_t$ , because the wage increases with  $K_t$ . But for the entrepreneurial sector

as a whole, gross profit  $r_t K_t$  increases with  $K_t$ . Also note from (12) that  $r_t$  is increasing in the productivity parameter  $A_t$  through  $a_t$ . Later we will show that in the neighborhood of the steady state monetary equilibrium, a worker will choose to hold neither equity nor money. That is, the worker simply consumes his labour income at each date:

$$c'_t = w_t l'_t. \quad (13)$$

We are now in a position to characterize the equilibrium behavior of the entrepreneurs. Consider an entrepreneur holding equity  $n_t$  and money  $m_t$  at the start of date  $t$ . First, suppose he has an investment opportunity: let this be denoted by a superscript  $i$  on his choice of consumption, and start-of-next-period equity and money holdings,  $(c_t^i, n_{t+1}^i, m_{t+1}^i)$ . He has two ways of acquiring equity  $n_{t+1}^i$ : either produce it at unit cost 1, or buy it in the market at price  $q_t$ . (See the LHS of the flow-of-funds constraint (7), where, recall, investment  $i_t$  corresponds to investment.) If  $q_t$  is less than 1, the agent will not invest. If  $q_t$  equals 1, he will be indifferent. If  $q_t$  is greater than 1, he will invest by selling as much equity as he can subject to the constraint (5). The entrepreneur's production choice is similar to Tobin's  $q$  theory of investment.

As the aggregate productivity and liquidity of equity  $(A_t, \phi_t)$  follow a stochastic process in the neighborhood of constant  $(A, \phi)$ , we have the following claim in the neighborhood of the steady state equilibrium (All the proofs are in Appendix):

**Claim 1** *Suppose that  $\theta$  and  $\phi$  satisfy*

$$\text{Condition 1 : } (1 - \lambda)\theta + \pi\lambda\phi > (1 - \lambda)(1 - \pi).$$

*Then in the neighborhood of the steady state:*

- (i) the allocation of resource is the first best*
- (ii) Tobin's  $q$  is equal to unity:  $q_t = 1$ ;*
- (iii) money has no value:  $p_t = 0$ ;*
- (iv) the gross dividend is roughly equal to the time preference rate plus the depreciation rate:  $r_t \simeq \frac{1}{\beta} - \lambda$ .*

If the investing entrepreneurs can issue new equity relatively freely and existing equity is relatively liquid to satisfy Condition 1, then the equity market transfers enough resources from the savers to the investing entrepreneurs

to achieve the first best allocation.<sup>11</sup> There would be no extra advantage of having investment opportunity; Tobin's  $q$  is equal to 1 (or the market value of capital is equal to the replacement cost) and both investing entrepreneurs and savers earn the same net rate of return on equity which is approximately equal to the time preference rate. (Note that the usual risk premium would be fairly negligible in the first best with our logarithmic utility function). Because the economy achieves the first best allocation without money, money has no value in the equilibrium.

In the following we want to restrict attention to an equilibrium in which  $q_t$  is greater than 1. We also want money to have value in equilibrium. Let us assume that  $\theta$  and  $\phi$  satisfy:

*Assumption 2* :  $0 < \Phi(\theta, \phi)$ , where

$$\begin{aligned} \Phi(\theta, \phi) \equiv & \pi\lambda\beta^2(1-\pi)(1-\phi)[(1-\lambda)(1-\pi) - (1-\lambda)\theta - \pi\lambda\phi] \\ & + [(\beta-\lambda)(1-\pi) - (1-\lambda)\theta - \pi\lambda\phi] [1-\lambda + \pi\lambda - (1-\lambda)\theta - \pi\lambda\phi] \\ & \cdot [\lambda(1-\beta)(1-\pi) + (1-\lambda)\theta + \lambda(\beta + \pi - \pi\beta)\phi]. \end{aligned}$$

Because both  $\theta$  and  $\phi$  are between 0 and 1, we observe all the terms in the RHS are positive, except for the terms  $(1-\lambda)(1-\pi) - (1-\lambda)\theta - \pi\lambda\phi$  and  $(\beta-\lambda)(1-\pi) - (1-\lambda)\theta - \pi\lambda\phi$ . Thus a sufficient condition for Assumption 2 is

$$(1-\lambda)\theta + \pi\lambda\phi < (\beta-\lambda)(1-\pi),$$

and a necessary condition is

$$(1-\lambda)\theta + \pi\lambda\phi < (1-\lambda)(1-\pi).$$

We observe that if the Condition 1 for Claim 1 is satisfied, then the necessary condition is not satisfied and there would be no equilibrium with valued fiat money. Under Assumption 2, however, upper bound on  $\theta$  and  $\phi$  is tight enough to ensure that the following claim holds.

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<sup>11</sup>In the steady state, the aggregate saving (which is equal to aggregate investment) is equal to the depreciation of capital. The RHS of Condition 1 is equal to the ratio of aggregate saving of non-investing entrepreneurs (who are  $1-\pi$  fraction of total entrepreneurs) to aggregate capital stock in the first best allocation. The LHS is the ratio of maximum equity sold by the investing entrepreneurs to aggregate capital stock:  $\theta(1-\lambda)$  corresponds to the new equity issued and  $\pi\lambda\phi$  corresponds to the existing equity sold by the investing entrepreneurs (which is  $\pi$  fraction of total entrepreneurs). Thus Condition 1 says that the maximum equity sold by the investing entrepreneurs is enough to shift the aggregate saving of the non-investing entrepreneurs.

**Claim 2** *Under Assumption 2, in the neighborhood of the steady state:*

- (i) *the price of money,  $p_t$ , is strictly positive;*
- (ii) *the price of capital,  $q_t$ , is strictly greater than 1;*
- (iii) *an entrepreneur with an investment opportunity faces the binding liquidity constraints and will not choose to hold money:  $m_{t+1}^i = 0$ .*

We will be in a position to prove the claim once we have laid out the equilibrium conditions - we use a method of guess-and-verify in the following. For values of  $\theta$  and  $\phi$  which do not satisfy Assumption 2 nor the condition for Claim 1, we can show that money has no value even though the liquidity constraint (5) still binds. To streamline the paper, we have chosen not to give an exhaustive account of the equilibria throughout the parameter space.

There is a caveat to Claim 2(i). Fiat money can only be valuable to someone if other people find it valuable, hence there is always a non-monetary equilibrium in which the price of fiat money is zero. Thus when there is a monetary equilibrium in addition to the non-monetary equilibrium, we restrict attention to the monetary equilibrium:  $p_t > 0$ . Claim 2(iii) says that the entrepreneur prefers investment with the maximum leverage to holding money, even though the return is in the form of equity which at date  $t+1$  is less liquid than money. (Incidentally, even though the investing entrepreneurs don't want to hold money for liquidity purposes, the non-investing entrepreneurs do – see below. This is why Claim 2(i) holds.)

Thus, for an investing entrepreneur, the liquidity constraints (5) and (6) are both binding. His flow of funds constraint (7) can be rewritten

$$c_t^i + (1 - \theta q_t) i_t = (r_t + \lambda \phi_t q_t) n_t + p_t m_t. \quad (14)$$

In order to finance investment cost  $i_t$ , the entrepreneur issues equity as much as  $\theta i_t$  at price  $q_t$ . Thus the second term in the LHS is the investment cost that has be financed internally - the downpayment for investment. The LHS is the total liquidity needs of the investing entrepreneur. The RHS corresponds to the maximum liquidity supplied from dividend, sales of resaleable fraction of equity after depreciation and the value of money. When we solve this flow-of-funds constraint with respect to the equity of the next period, we have

$$c_t^i + q_t^R n_{t+1}^i = r_t n_t + [\phi_t q_t + (1 - \phi_t) q_t^R] \lambda n_t + p_t m_t, \quad (15)$$

$$\text{where } q_t^R \equiv \frac{1 - \theta q_t}{1 - \theta} < 1, \text{ as } q_t > 1. \quad (16)$$

The value of  $q_t^R$  is the effective replacement cost of equity to the investing entrepreneur: because he needs downpayment  $1 - \theta q_t$  for every unit of investment in order to retain  $1 - \theta$  inside equity, he needs  $(1 - \theta q_t)/(1 - \theta)$  in order to acquire one unit of inside equity. The RHS of (15) is his net worth: gross dividend, the value of his depreciated equity  $\lambda n_t$  - of which resaleable  $\phi_t$  fraction is valued by the market price and the non-resaleable  $1 - \phi_t$  fraction is valued by the effective replacement cost -, and the value of money.

Given the discounted logarithmic preferences (1), the entrepreneur saves a fraction  $\beta$  of his net worth, and consumes a fraction  $1 - \beta$ :<sup>12</sup>

$$c_t^i = (1 - \beta) \{ r_t n_t + [\phi_t q_t + (1 - \phi_t) q_t^R] \lambda n_t + p_t m_t \}. \quad (17)$$

And so, from (14), we obtain an expression for his investment in period t:

$$i_t = \frac{(r_t + \lambda \phi_t q_t) n_t + p_t m_t - c_t^i}{1 - \theta q_t}. \quad (18)$$

The investment is equal to the ratio of liquidity available after consumption to the downpayment per unit of investment.

Next, suppose the entrepreneur does not have an investment opportunity: denote this by a superscript  $s$  to stand for a pure saver. The flow-of-funds constraint (7) reduces to

$$c_t^s + q_t n_{t+1}^s + p_t m_{t+1}^s = r_t n_t + q_t \lambda n_t + p_t m_t. \quad (19)$$

For the moment, let us assume that constraints (5) and (6) do not bind. Then the RHS of (19) corresponds to the entrepreneur's net worth. It is the same as the RHS of (15), except that now his depreciated equity is valued at the market price,  $q_t$ . From this net worth he consumes a fraction  $1 - \beta$ :

$$c_t^s = (1 - \beta)(r_t n_t + q_t \lambda n_t + p_t m_t). \quad (20)$$

Note that consumption of entrepreneur who does not have investment opportunity is larger than consumption of the investing entrepreneur if both hold the same equity and money at the start of period. The remainder is split across a savings portfolio of  $m_{t+1}$  and  $n_{t+1}$ .

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<sup>12</sup>Compare (1) to a Cobb-Douglas utility function, where the expenditure share of present consumption out of total wealth is constant and equal to  $1/(1 + \beta + \beta^2 + \dots) = 1 - \beta$ .

To determine the optimal portfolio, consider the choice of sacrificing one unit of consumption  $c_t$  to purchase either  $1/p_t$  units of money or  $1/q_t$  units of equity, which are then used to augment consumption at date  $t+1$ . The first-order condition is

$$\begin{aligned}
u'(c_t) &= E_t \left\{ \frac{p_{t+1}}{p_t} \beta [(1 - \pi) u'(c_{t+1}^s) + \pi u'(c_{t+1}^i)] \right\} \\
&= (1 - \pi) E_t \left\{ \frac{r_{t+1} + \lambda q_{t+1}}{q_t} \beta u'(c_{t+1}^s) \right\} \\
&\quad + \pi E_t \left\{ \frac{r_{t+1} + \lambda \phi_{t+1} q_{t+1} + \lambda (1 - \phi_{t+1}) q_{t+1}^R}{q_t} \beta u'(c_{t+1}^i) \right\}.
\end{aligned} \tag{21}$$

The RHS of the first line of (21) is the expected gains from holding  $1/p_t$  additional units of money at date  $t+1$ : money will always yields  $p_{t+1}$  which will increase utility by  $u'(c_{t+1}^s)$  times as much when he will not have an investment opportunity with probability  $1 - \pi$  and will increase utility by  $u'(c_{t+1}^i)$  times as much when he will have an investment opportunity with probability  $\pi$  at date  $t+1$ . The second line is the gain in expected discounted utility from holding  $1/q_t$  additional units of equity at date  $t+1$ . Per unit, this additional equity yields  $r_{t+1}$  dividend, plus its depreciated value. With probability  $1 - \pi$  the entrepreneur will not have an investment opportunity and the depreciated equity will be valued at the market price,  $q_{t+1}$ , and these yields will increase utility by  $u'(c_{t+1}^s)$  times as much. With probability  $\pi$  the entrepreneur will have an investment opportunity at date  $t+1$ , in which case he will value depreciated equity by the market price  $q_{t+1}$  for resaleable fraction and by the effective replacement cost  $q_{t+1}^R$  for non-resaleable fraction, and these yield will increase utility by the marginal utility conditional on investment  $u'(c_{t+1}^i)$ . Because the effective replacement cost is lower than the market price in monetary economy and the equity is only partially resaleable, the equity will have lower contingent return in the state in which the entrepreneur needs fund most with an arrival of investment opportunity and the marginal utility of consumption is high (as  $c_{t+1}^i < c_{t+1}^s$ ).

Equity is "risky" to the saving entrepreneur not only because the rate of return is correlated with aggregate consumption (aggregate risk) but also because the contingent rate of return is low due to limited resaleability when the entrepreneur's marginal utility is high (idiosyncratic risk). Money is "free" from idiosyncratic risk of having investment opportunity, because its rate of return is independent of whether or not the entrepreneur has an

investment opportunity in the next period.

We are now in a position to consider the aggregate economy. The great merit of the expressions for an investing entrepreneur's consumption and investment choices,  $c_t^i$  and  $i_t$ , and a non-investing entrepreneurs' consumption and savings choices,  $c_t^s$ ,  $n_{t+1}$  and  $m_{t+1}$ , is that they are all linear in start-of-period equity and money holdings  $n_t$  and  $m_t$ .<sup>13</sup> Hence aggregation is easy: we do not need to keep track of the distributions. Notice that, because workers do not choose to save, the aggregate holdings of equity and money of the entrepreneurs are equal to aggregate capital stock  $K_t$  and money supply  $M$ . Since investment opportunities are independently distributed, we can work with the total capital and money holdings in the economy,  $K_t$  and  $M$ . At the start of date  $t$ , a fraction  $\pi$  of  $K_t$  and  $M$  is held by entrepreneurs who have an investment opportunity. From (18), total investment,  $I_t$ , in new capital therefore satisfies

$$(1 - \theta q_t) I_t = \pi \left\{ \beta [(r_t + \lambda \phi_t q_t) K_t + p_t M] - (1 - \beta)(1 - \phi_t) \lambda q_t^R K_t \right\}. \quad (22)$$

Goods market clearing requires that total output (net of labour costs, which equals the consumption of workers),  $r_t K_t$ , equals investment plus consumption of entrepreneurs— which, using (17) and (20), yields

$$\begin{aligned} r_t K_t &= a_t K_t^\alpha = I_t + (1 - \beta) \cdot \\ &\quad \left\{ [r_t + (1 - \pi + \pi \phi_t) \lambda q_t + \pi (1 - \phi_t) \lambda q_t^R] K_t + p_t M \right\}. \end{aligned} \quad (23)$$

It remains to find the aggregate counterpart to the portfolio equation (21). During period  $t$ , the investing entrepreneurs sell a fraction  $\theta$  of their investment  $I_t$ , together with a fraction  $\phi_t$  of their depreciated equity holdings  $\pi \lambda K_t$ , to the non-investing entrepreneurs. So the stock of equity held by the group of non-investing entrepreneurs at the end of the period is given by  $\theta I_t + \phi_t \pi \lambda K_t + (1 - \pi) \lambda K_t \equiv N_{t+1}^s$ . And, by claim 2(iii), we know that this group also hold all the money stock,  $M$ . We also know that the marginal utility of consumption is proportional to the inverse of consumption (that is proportional to the net worth) due to logarithmic utility function. The

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<sup>13</sup>From (19) and (20), the value of savings,  $q_t n_{t+1}^s + p_t m_{t+1}^s$  is linear in  $n_t$  and  $m_t$ , and (the reciprocal of) the portfolio equation (21) is homogeneous in  $(n_{t+1}^n, m_{t+1}^n)$ , noting that  $u'(c) = 1/c$  with logarithmic utility function.



group's savings portfolio  $(N_{t+1}^s, M)$  satisfies (21), which can be simplified to:

$$\begin{aligned} & (1 - \pi) E_t \left[ \frac{(r_{t+1} + \lambda q_{t+1})/q_t - p_{t+1}/p_t}{(r_{t+1} + q_{t+1}\lambda)N_{t+1}^s + p_{t+1}M} \right] \\ &= \pi E_t \left[ \frac{p_{t+1}/p_t - [r_{t+1} + \phi_{t+1}\lambda q_{t+1} + (1 - \phi_{t+1})\lambda q_{t+1}^R]/q_t}{[r_{t+1} + \phi_{t+1}\lambda q_{t+1} + (1 - \phi_{t+1})\lambda q_{t+1}^R]N_{t+1}^s + p_{t+1}M} \right]. \end{aligned} \quad (24)$$

Equation (24) lies at the heart of the model. When there is no investment opportunity at date  $t+1$ , so that the partial liquidity of equity doesn't matter, the return on capital,  $(r_{t+1} + \lambda q_{t+1})/q_t$  exceeds the return on money,  $p_{t+1}/p_t$ : the LHS of (24) is positive. However, when there is an investment opportunity, the contingent rate of return on equity,  $[r_{t+1} + \phi_{t+1}\lambda q_{t+1} + (1 - \phi_{t+1})\lambda q_{t+1}^R]/q_t$  is less than the return on money: the RHS of (24) is positive. These return differentials have to be weighted by the respective probabilities and marginal utilities. The liquidity premium of equity over money in the LHS can be substantial and time-varying, because the equity holder faces the idiosyncratic cost of limited resaleability in the RHS in case of having an investment opportunity (in addition to usual aggregate risk).<sup>14</sup>

Aside from the liquidity shock  $\phi_t$  and the technology parameter  $A_t$  which follow an exogenous stationary Markov process, the only state variable in this system is  $K_t$ , which evolves according to

$$K_{t+1} = \lambda K_t + I_t. \quad (25)$$

Restricting attention to stationary price process, the competitive equilibrium can be defined recursively as function  $(I_t, p_t, q_t, K_{t+1})$  of aggregate state  $(K_t, A_t, \phi_t)$  that satisfy (11), (22) – (25), together with the law of motion of  $A_t$  and  $\phi_t$ . From these four equations to characterize the equilibrium, we observe that there are rich interaction between quantities  $(I_t, K_{t+1})$  and

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<sup>14</sup>Holmstrom and Tirole (1998, 2001) develop models of three-period production economy with financial intermediaries in which pledgeable future returns are limited. One of the main differences from ours is that the liquidity needs of each individual entrepreneur is contractible and that there is no constraint on resaleability. In Holmstrom and Tirole (2001), the liquidity premium of each asset depends upon the covariance between its rate of return and the aggregate liquidity needs. Because of limited insurance, our approach is perhaps closer to Luttmer (1996, 1999) which examine the implications of transaction costs and short-sales constraints for consumption and asset prices.

Atkeson and Kehoe (2008) argue for the need for the time-varying risk premium to analyzing monetary policy.

asset prices  $(p_t, q_t)$ . In this sense, our economy is similar to Keynes (1936). In fact, perhaps the closest ancestor of our model is Tobin (1969) where he considers Tobin's  $q$  as the key variable to analyze the interaction between goods market and asset market. From methodological point of view, our model is similar to more modern macroeconomics because we derive all the behavioral relationship from individual optimization under the constraints of technology and liquidity.

In steady state, when  $a_t = a$  (the RHS of (12) with  $A_t = A$ ) and  $\phi_t = \phi$ , capital stock  $K$ , investment  $I$ , and prices  $p$  and  $q$ , satisfy  $I = (1 - \lambda)K$  and

$$\pi\beta r + \pi\beta l = \left[ 1 - \lambda + \pi\lambda(1 - \beta) \frac{1 - \phi}{1 - \theta} \right] (1 - \theta q) - \pi\beta\lambda\phi q \quad (26)$$

$$\begin{aligned} \beta r - (1 - \beta)l &= \left[ 1 - \lambda + \pi\lambda(1 - \beta) \frac{1 - \phi}{1 - \theta} \right] \\ &+ (1 - \beta) \left( 1 - \pi \frac{1 - \phi}{1 - \theta} \right) \lambda q \end{aligned} \quad (27)$$

$$r - (1 - \lambda)q = \pi\lambda \frac{1 - \phi}{1 - \theta} (q - 1) \frac{q + (l/\chi)}{r + \lambda \frac{1 - \phi}{1 - \theta} + \lambda \frac{\phi - \theta}{1 - \theta} q + (l/\chi)}, \quad (28)$$

where  $r = aK^{\alpha-1}$ ,  $l = pM/K$ , and  $\chi \equiv \theta(1 - \lambda) + (1 - \pi + \pi\phi)\lambda$  (the steady-state fraction of equity held by non-investing entrepreneurs at the end of a period).

Equations (26), (27) and (28) can be viewed as a simultaneous system in three unknowns: the price of capital,  $q$ ; the gross profit rate on capital,  $r$ ; and the value of the money stock as a fraction of total capital,  $l$ . (26) and (27) can be solved for a  $r$  and  $l$ , each as affine functions of  $q$ , which when substituted into (28) yield a quadratic equation in  $q$  with a unique positive solution. Assumption 2 is sufficient to ensure that this solution lies strictly above 1 (but below  $1/\theta$ ). We can also show that Assumption 2 is the necessary and sufficient condition for money to have value:  $p > 0$ .

As a prelude to the dynamic analysis that we undertake later on, notice that the technology parameter  $A$  only affects the steady-state system through the gross profit term  $r = aK^{\alpha-1}$ . That is, a rise in the steady state value of  $A$  increases the capital stock,  $K$ , but does not affect  $q$ , the price of capital. The price of money,  $p$ , increases to leave  $l = pM/K$  unchanged.

It is interesting to compare our economy, in which the liquidity constraints (5) and (6) bind for investing entrepreneurs, to a "first-best" economy without such constraints. Consider steady states. In the first-best economy, the price of capital would equal its cost, 1; and the capital stock,  $K^*$  say, would equate the return on capital,  $aK^{\alpha-1} + \lambda$ , to the agents' common subjective return,  $1/\beta$ . (See Claim 1). We can show that, in our constrained economy, the level of activity – measured by the capital stock  $K$  – is strictly below  $K^*$ . Hence, by continuity, the same is true in the neighborhood of the steady state. Because of the partial liquidity of equity, the economy fails to transfer enough resources to the investing entrepreneurs to achieve the first-best level of investment.

The liquidity constraint creates the wedges between the marginal product of capital and the expected rate of returns on equity, and turns out to keep both the expected rates of return on equity and that of money below the time preference in the neighborhood of the steady state. Intuitively, the rate of returns on assets to savers are below their time preference rate so that the savers will not save enough to escape the liquidity constraint when they find an opportunity to invest in future.

**Claim 3** *In the neighborhood of the steady state monetary economy,*

(i) *the stock of capital,  $K_{t+1}$  is less than in the first-best (unconstrained) economy:*

$$K_{t+1} < K^* \Leftrightarrow E_t (a_{t+1}K_{t+1}^{\alpha-1} + \lambda) > \frac{1}{\beta},$$

(ii) *the expected rate of return on equity (if the saver does not have investment opportunity at date  $t+1$ ) is lower the time preference rate:*

$$E_t \frac{a_{t+1}K_{t+1}^{\alpha-1} + \lambda q_{t+1}}{q_t} < \frac{1}{\beta},$$

(iii) *the expected rate of return on money is lower than the expected rate of return on equity:*

$$E_t \frac{p_{t+1}}{p_t} < E_t \frac{a_{t+1}K_{t+1}^{\alpha-1} + \lambda q_{t+1}}{q_t},$$

(iv) *the expected rate of return on equity contingent on having an investment opportunity in the next period is lower than the expected rate of return*

on money:

$$E_t \frac{a_{t+1} K_{t+1}^{\alpha-1} + \phi_{t+1} \lambda q_{t+1} + (1 - \phi_{t+1}) \lambda q_{t+1}^R}{q_t} < \frac{p_{t+1}}{p_t}.$$

Claim 3(iii) follows directly from (28), given that in steady state  $q > 1$ . This difference between the expected return on equity and money reflects a liquidity premium. It equals the nominal interest rate on equity.<sup>15</sup> Because entrepreneurs are constrained when they have an investment opportunity, they have to be compensated for holding less liquid equity in their savings portfolio. If there were no binding liquidity constraints, money would have no value.

In our monetary economy, there are a spectrum of interest rates: from the highest, we have the expected marginal product of capital, the time preference rate, the expected rate of return on equity, the expected rate of return on money, and the expected rate of return on equity contingent on the saver having an investment opportunity in the next period. There spreads across different interest rates come from the very feature of monetary economy in which the circulation of money is essential for resource allocation. Thus in our economy the impact of the asset markets on aggregate production cannot be summarized by the expectations of a single real interest rate from present to future as in some popular models such as Woodford (2003). It is equally misleading to use the average real rates of returns on money and equity for the time preference rate to calibrate our economy.

The fact that the expected rates of return on equity and money are both lower than the time preference rate justifies our earlier assertion that workers will not choose to save by holding capital or money.<sup>16</sup> (Of course, if workers could borrow against their future labour income they would do so. But we

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<sup>15</sup>By the Fisher equation, the nominal interest rate on equity equals the net real return on equity plus the inflation rate. But minus the inflation rate equals the net real return on money. Hence the nominal interest rate on equity equals the real return on equity minus the return on money, i.e. the liquidity premium. Because our money is broad money - asset readily resaleable, our nominal interest rate is similar to the interest rate in Keynes (1936); the gap between the rate of return on partially resaleable assets and broad money.

<sup>16</sup>Workers would save if workers were to face their own investment opportunity shocks. Suppose, instead, that each worker randomly meets a "health shock" with which the worker has to spend immediately some fixed amount  $\zeta$  of co-payment in order to maintain his human capital. (The health insurance covers some costs, but the patient has to cover the co-payment from his own pocket here). Then, if the resaleability of equity is low, we can show that worker save only in money up to the value enough to cover the co-payment  $\zeta$ .

have ruled this out.) In steady state, workers enjoy a constant consumption equal to their wages.

The reason why an entrepreneur saves, and workers do not, is because the entrepreneur is preparing for his next investment opportunity. And the entrepreneur saves using money as well as equity, despite money's particularly low return, because he anticipates that he will be liquidity constrained at the time of investment. Along a typical time path, he experiences episodes without investment, during which he consumes part of his saving. As the return on saving – on both capital and money – is less than his time preference rate, the value of his net worth gradually shrinks, as does his consumption. He only expands again at the time of investment. In the aggregate picture, we do not see all this fine grain. But it is important to realize that, even in steady state, the economy is made up of a myriad of such individual histories.

### 3 Dynamics and Numerical Examples

In order to examine the dynamics of our economy, let us present numerical examples by specifying the law of motion of productivity and liquidity  $(A_t, \phi_t)$ . Suppose that  $(A_t, \phi_t)$  follows independent AR(1) processes so that  $a_t = \gamma \left(\frac{1-\gamma}{\omega}\right)^{\frac{1-\gamma}{\gamma+\nu}} (A_t)^{\frac{1+\nu}{\gamma+\nu}}$  (in (12)) and  $\phi_t$  follow AR(1) as

$$a_t - a = \rho_a (a_{t-1} - a) + \varepsilon_{at}, \quad (29)$$

$$\phi_t - \phi = \rho_\phi (\phi_{t-1} - \phi) + \varepsilon_{\phi t}, \quad (30)$$

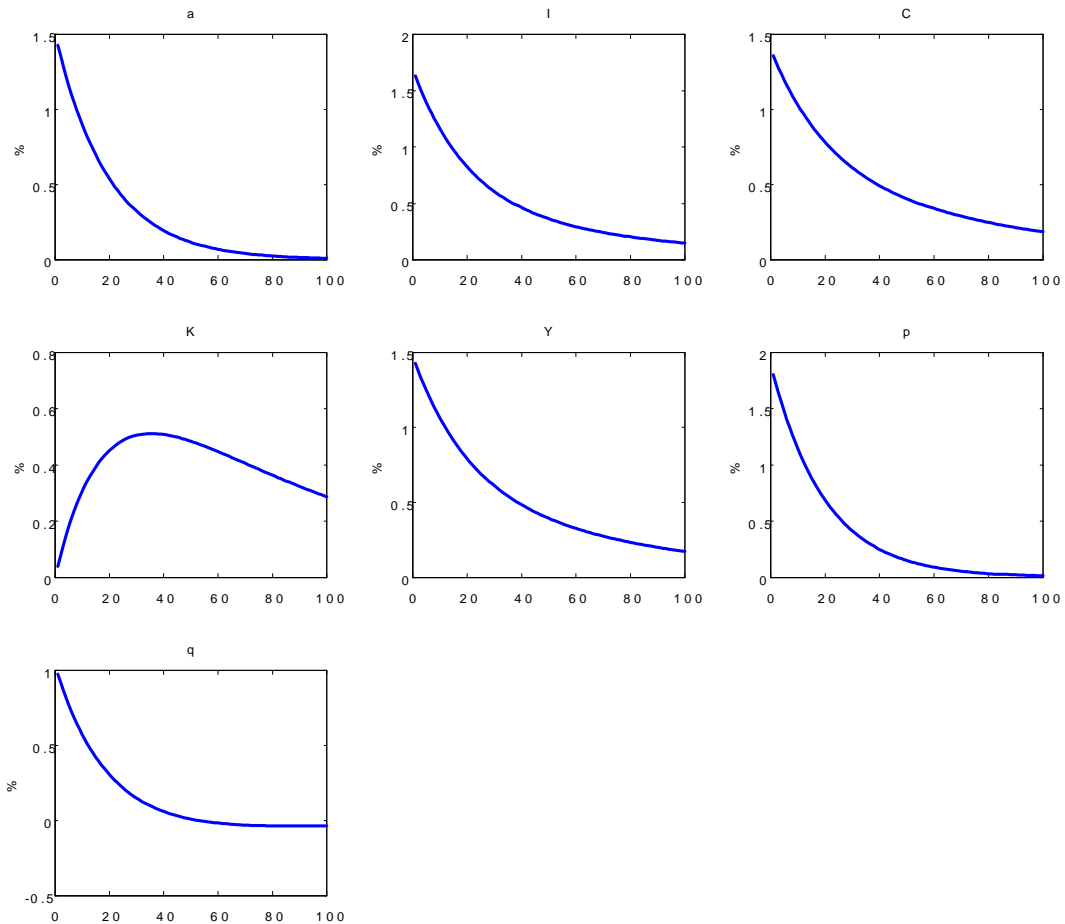
where  $\rho_a$  and  $\rho_\phi \in (0, 1)$  and we set  $\rho_a = \rho_\phi = 0.95$  for calibration. The variables  $\varepsilon_{at}$  and  $\varepsilon_{\phi t}$  are iid. innovation of the levels of productivity and liquidity, which are mutually independent with mean zero. We present our numerical examples to illustrate mainly qualitative features of our model. We follow Del Negro et. al. (2011) for choosing parameters. In particular, we consider one period is quarterly and use  $\pi = 0.05$  (arrival rate of investment opportunity),  $\theta = 0.19$  (mortgageable fraction of new investment),  $\phi = 0.19$  (resaleable fraction of equity in the steady state),  $\gamma = 0.4$  (share of capital),  $\nu = 1$  (inverse of Frisch elasticity of labor supply),  $\beta = 0.99$  (utility discount

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Even though the rate of return on equity is higher than money, the worker would require to save more in equity than money in order to compensate the resaleability constraint, which would be more costly, given that the rate of return on equity is lower than the time preference rate. See Kiyotaki and Moore (2005a) for the detail.

factor),  $\lambda = 0.975$  (one minus depreciation rate). *Figure 1* shows the impulse response function to 1% increase in  $A_t$ , which increases  $a_t$  by  $\frac{1+\nu}{\gamma+\nu} = 1.43\%$ :

Figure 1. Impulse Responses of Basic Economy to Productivity Shock

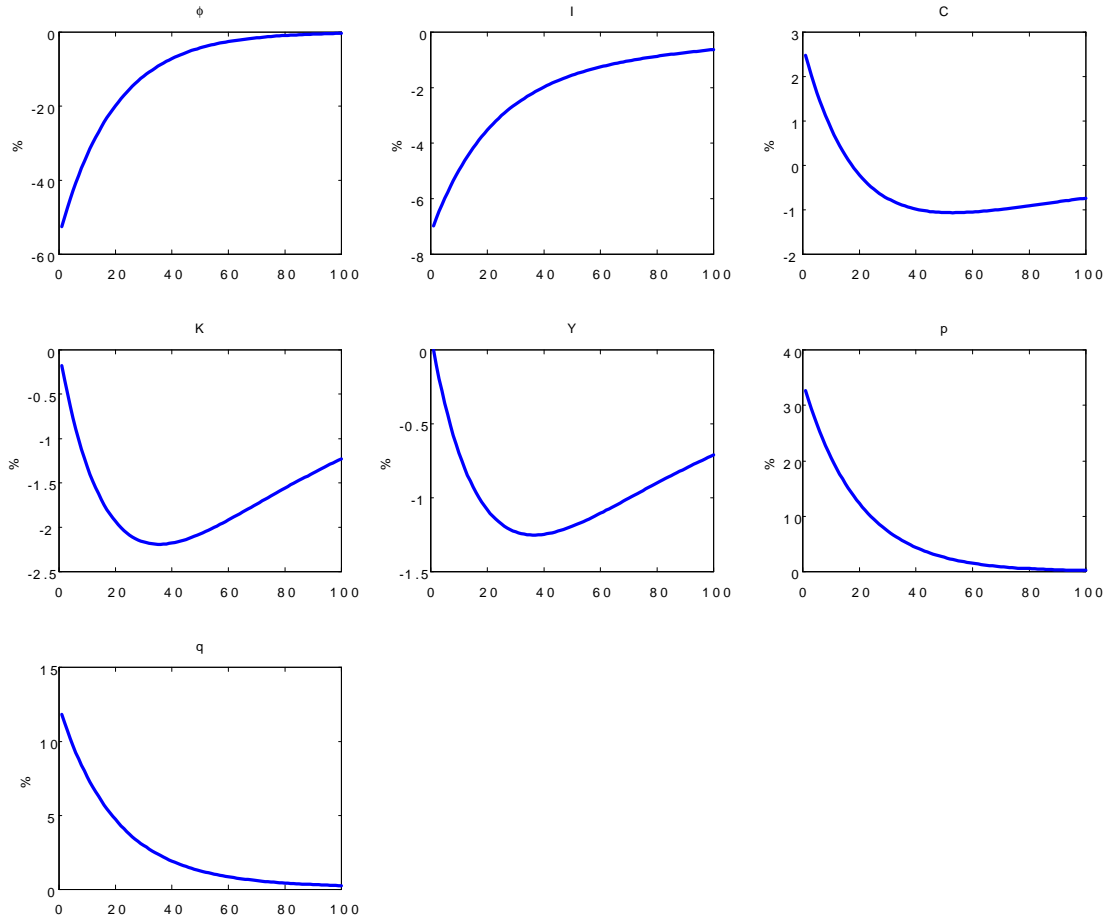


Because capital stock is pre-determined, output increases by 1.43% (the same proportion as  $a_t$ ) by labor market equilibrium condition. Then from goods market equilibrium condition (23), we observe the asset prices ( $p_t, q_t$ ) have to increase together with productivity in order to increase consumption and investment in line with a larger output. Although investment is more sensitive to the asset prices and thus increases more than consumption in proportion, aggregate consumption of entrepreneurs and worker increases

substantially (especially because workers' consumption is equal to their wage income in our economy). This is different from the first best allocation under *Condition 1*, in which consumption is much more smooth than investment because consumption depends upon permanent income rather than current income without the binding liquidity constraints. The co-movement of quantities and asset prices is also a unique features of the monetary equilibrium with binding liquidity constraints. In contrast, under the first best allocation, Tobin's  $q$  is always equal to 1 and the value of money is always equal to zero.

Now, let's consider liquidity shocks. *Figure 2* shows the impulse response of quantities and asset prices against 50% fall in the resaleability of the equity.

Figure 2. Impulse Responses of Basic Economy to Liquidity Shock



When the resaleability of equity falls persistently, the investing entrepreneurs can finance only a smaller downpayment from selling his equity holding and investment decreases substantially. Then capital stock and output gradually decrease with persistently lower investment. Also the entrepreneurs without investment opportunity find money more attractive than equity as means of saving (if the expected rate of returns were unchanged), because he can resale only a smaller fraction of equity holding if he has an investment



opportunity in the following period and he has to revalue the non-resaleable fraction of equity by the effective replacement costs (which is significantly lower than the market price). (See (24)). Thus, the value of money increases compared to the equity price in order to restore the asset market equilibrium. This can be thought of "a flight to liquidity". However, the equity price tends to rise with the fall in the liquidity in our economy, because the gap between Tobin's  $q$  and unity—a measure of tightness of liquidity constraint—increases as the resaleability of equity falls and the liquidity constraints are tightened. In addition, because output is not affected initially with full employment, consumption increases at impact to restore goods market equilibrium. The negative co-movement between investment, asset prices and consumption is the short-coming of our paper, which is common to many macroeconomic model with flexible prices.<sup>17</sup>

In contrast to our monetary economy, the first best allocation will not react to the liquidity shock as the liquidity constraint is not binding.

## 4 Full Model with Storage and Government

In order to overcome the short-coming of negative co-movement between investment, consumption and the stock price, we introduce an alternative liquid means of saving—storage. Specifically, the individual agent can store  $\sigma_t z_{t+1}$  units of goods at date  $t$  to obtain  $z_{t+1}$  units of goods at date  $t+1$ , where  $z_{t+1}$  must be non-negative. Although the storage technology is constant returns to scale at the individual level, it is decreasing returns to scale at the society level as  $\sigma_t$  is an increasing function of the aggregate quantity of storage  $Z_{t+1}$ ,

$$\sigma_t = \sigma(Z_{t+1}) = \left( \frac{Z_{t+1}}{\zeta_0} \right)^\zeta, \text{ where } \zeta_0, \zeta > 0.$$

We can think storage represents various means of short-term saving besides money, such as consumer durables and net foreign asset (when domestic residents can save in foreign assets but cannot borrow from foreigners).

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<sup>17</sup>Shi (2011) points out forcefully that our framework is difficult to generate positive comovement of aggregate investment and stock price with liquidity shock. Barro and King (1984) show that the typical co-movement of aggregate quantity fluctuation during business cycle requires the real wage to move procyclically under time-separable preference, which is difficult to achieve with many shocks except for productivity shock.

We also introduce government. Our goal here is simply to explore the effects on equilibrium of an exogenous government policy rule. We make no attempt to explain government behavior. At the start of date  $t$ , suppose the government holds  $N_t^g$  equity. Unlike entrepreneurs, the government cannot produce new capital. However, it can engage in open market operations, to buy (sell) equity by issuing (taking in) money – it has sole access to a costless money-printing technology. Any sale of equity by the individual entrepreneur is subject to the same constraint as (5) irrespective of the buyer being government or private agents.<sup>18</sup> Finally, the government can purchase goods, or transfer to the workers (which can be negative if it is lump-sum tax). Let  $G_t$  denote the total government purchase of general output and real lump-sum minus transfer to workers. We assume  $G_t$  does not affect utility of entrepreneurs. This leaves intact our analysis of entrepreneurs' behavior. We assume that  $N_t^g$  and  $G_t$  are not so large that the private economy switches regimes. That is, we are still in an equilibrium in which the liquidity constraints bind for investing entrepreneurs, and money is valuable.

If  $M_t$  is the stock of money privately held by entrepreneurs at the start of date  $t$ , then the government's flow-of-funds constraint is given by

$$G_t + q_t (N_{t+1}^g - \lambda N_t^g) = r_t N_t^g + p_t (M_{t+1} - M_t) = r_t N_t^g + (\mu_t - 1) L_t, \quad (31)$$

where  $L_t \equiv p_t M_t$  is real balances, and  $\mu_t \equiv \frac{M_{t+1}}{M_t}$  is money supply growth rate. That is, government purchase of general output and equity must equal dividend of equity plus seigniorage revenues. Since government is a large agent, at least relative to each of the atomless private citizens, open market operations will affect the prices  $p_t$  and  $q_t$ .

We consider government follows a rule of open market operation and fiscal policy:

$$\frac{N_{t+1}^g}{K} = \psi_a \frac{a_t - a}{a} + \psi_\phi \frac{\phi_t - \phi}{\phi} \quad (32)$$

$$G_t = \xi [(r_t + \lambda q_t) N_t^g - (L_t - L)], \quad (33)$$

where  $\psi_a$ ,  $\psi_\phi$  and  $\xi$  are policy parameters, and  $K$  and  $L$  are capital stock and real money balance in the non-stochastic steady state. The first equation is government feedback rule for the open market operation, in which government chooses the size of open market operation (ratio of government equity

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<sup>18</sup>The government also is subject to the same resaleability constraint as the entrepreneur:  $N_{t+1}^g \geq (1 - \phi_t) \lambda N_t^g$ .

holding to total equity supply in the steady state) as a function of proportional deviations of productivity and liquidity from the steady state levels. This rule implies government equity holding is zero in the steady state. The second equation is fiscal policy rule, in which government adjusts the transfer to workers in proportional to the deviation of its net asset from the steady state at the beginning of period  $t$ . For all the following experiments, we choose  $\xi = 0$  so that  $G_t = 0$ . Thus all the fiscal adjustment is done through money supply growth rate  $\mu_t$ .

All of our earlier analysis goes through, but with obvious adjustments. Total supply of equity (which is equal to aggregate capital stock by the way of defining the equity) is equal to the sum of the government holding and aggregate holding of the entrepreneurs (denoted as  $N_{t+1}$ )

$$K_{t+1} = N_{t+1}^g + N_{t+1}. \quad (34)$$

The tax changes workers' consumption (as they consume all the disposable income), but, given the form of their preferences in (8), does not affect their labour supply. Equations (22), (23) and (24) are modified to:

$$(1 - \theta q_t) I_t = \pi \{ \beta [(r_t + \lambda \phi_t q_t) N_t + L_t + Z_t] - (1 - \beta)(1 - \phi_t) \lambda q_t^R N_t \} \quad (35)$$

$$a_t K_t^\alpha + Z_t = I_t + \sigma (Z_{t+1}) Z_{t+1} + G_t + (1 - \beta) \cdot \{ [r_t + (1 - \pi + \pi \phi_t) \lambda q_t + \pi (1 - \phi_t) \lambda q_t^R] N_t + L_t + Z_t \} \quad (36)$$

$$(1 - \pi) E_t \left[ \frac{(r_{t+1} + \lambda q_{t+1})/q_t - L_{t+1}/(\mu_t L_t)}{(r_{t+1} + q_{t+1} \lambda) N_{t+1}^s + L_{t+1} + Z_{t+1}} \right] \\ = \pi E_t \left[ \frac{L_{t+1}/(\mu_t L_t) - [r_{t+1} + \phi_{t+1} \lambda q_{t+1} + (1 - \phi_{t+1}) \lambda q_{t+1}^R]/q_t}{[r_{t+1} + \phi_{t+1} \lambda q_{t+1} + (1 - \phi_{t+1}) \lambda q_{t+1}^R] N_{t+1}^s + L_{t+1} + Z_{t+1}} \right] \quad (37)$$

$$(1 - \pi) E_t \left[ \frac{(r_{t+1} + \lambda q_{t+1})/q_t - (1/\sigma(Z_{t+1}))}{(r_{t+1} + q_{t+1} \lambda) N_{t+1}^s + L_{t+1} + Z_{t+1}} \right] \\ = \pi E_t \left[ \frac{(1/\sigma(Z_{t+1})) - [r_{t+1} + \phi_{t+1} \lambda q_{t+1} + (1 - \phi_{t+1}) \lambda q_{t+1}^R]/q_t}{[r_{t+1} + \phi_{t+1} \lambda q_{t+1} + (1 - \phi_{t+1}) \lambda q_{t+1}^R] N_{t+1}^s + L_{t+1} + Z_{t+1}} \right], \quad (38)$$

where  $N_{t+1}^s = \theta I_t + \phi_t \pi \lambda N_t + (1 - \pi) \lambda N_t + \lambda N_t^g - N_{t+1}^g$ . In (35), investing entrepreneurs use money, storage and resaleable portion of equity holdings to finance the downpayment for investment after consuming a fraction. In

(36), sum of output (net of the worker's consumption) and storage return in the LHS is equal to sum of investment, new storage, government purchase and entrepreneurs' consumption in the RHS. If storage is considered as net foreign asset, the accumulation of net foreign asset,  $Z_{t+1} - Z_t$ , is considered as current account. In asset market equilibrium, (37) describes the trade-off between stock and money in portfolio. Because we now have another liquid asset, storage, we have trade-off in portfolio between stock and storage which is described in (38).

Restricting our attention to stable price process, the competitive equilibrium is defined recursively as functions  $(I_t, r_t, L_t, q_t, Z_{t+1}, K_{t+1}, N_{t+1}, N_{t+1}^g, G_t, \mu_{t+1})$  of the aggregate state  $(K_t, Z_t, N_t^g, a_t, \phi_t)$  that satisfy (11), (25), (31) – (38) together with the exogenous law of motion of  $(a_t, \phi_t)$ .<sup>19</sup>

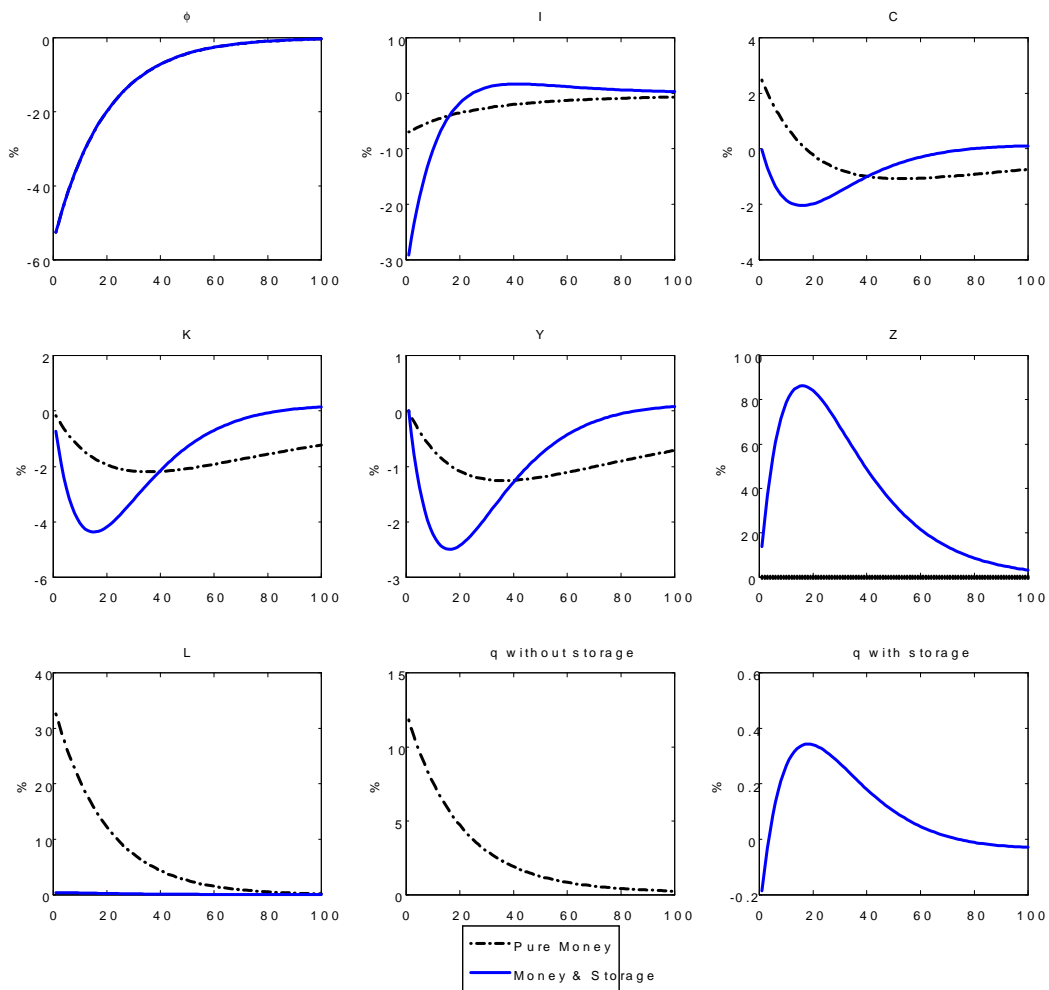
How does the presence of alternative means of liquid saving (storage) alter the impulse responses? *Figure 3* compares the impulse responses to the liquidity shock with and without storage. We choose the storage technology which is very close to constant returns to scale ( $\zeta = 0.0001$ ) and the steady state size of storage ( $\zeta_0 = 0.5$ ) is modest compared to capital stock ( $K = 9.49$  in the steady state). As the resaleability of equity falls, storage increases sharply, and investment falls more significantly than the economy without storage, leading to a more significant fall in output. Consumption also falls with a significant fraction of output is stored. In addition, the real balance and storage are very close substitute with similar rate of returns in the asset market. Then because the rate of return on storage rate of return is very close to one, the value of money is stable and the real balance hardly increases with liquidity shock. Then "flight to liquidity" induces the stock price to fall at least initially (despite of the modest magnitude). Therefore, the presence of alternative liquid means of saving overcomes some of the major shortcomings of the basic model. Many quantities and asset prices move together, by having the storage serving as a buffer stock to absorb some output and provide additional liquidity with relatively stable value of

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<sup>19</sup>If there were lump-sum transfer of money to the entrepreneurs (helicopter drop), then the aggregate quantities do not change in our economy because the prices are flexible. The consumption and investment of the individual entrepreneur, however, is affected by the helicopter drop, because there is redistribution from rich to poor entrepreneurs through lump-sum money transfer and inflation.

money.

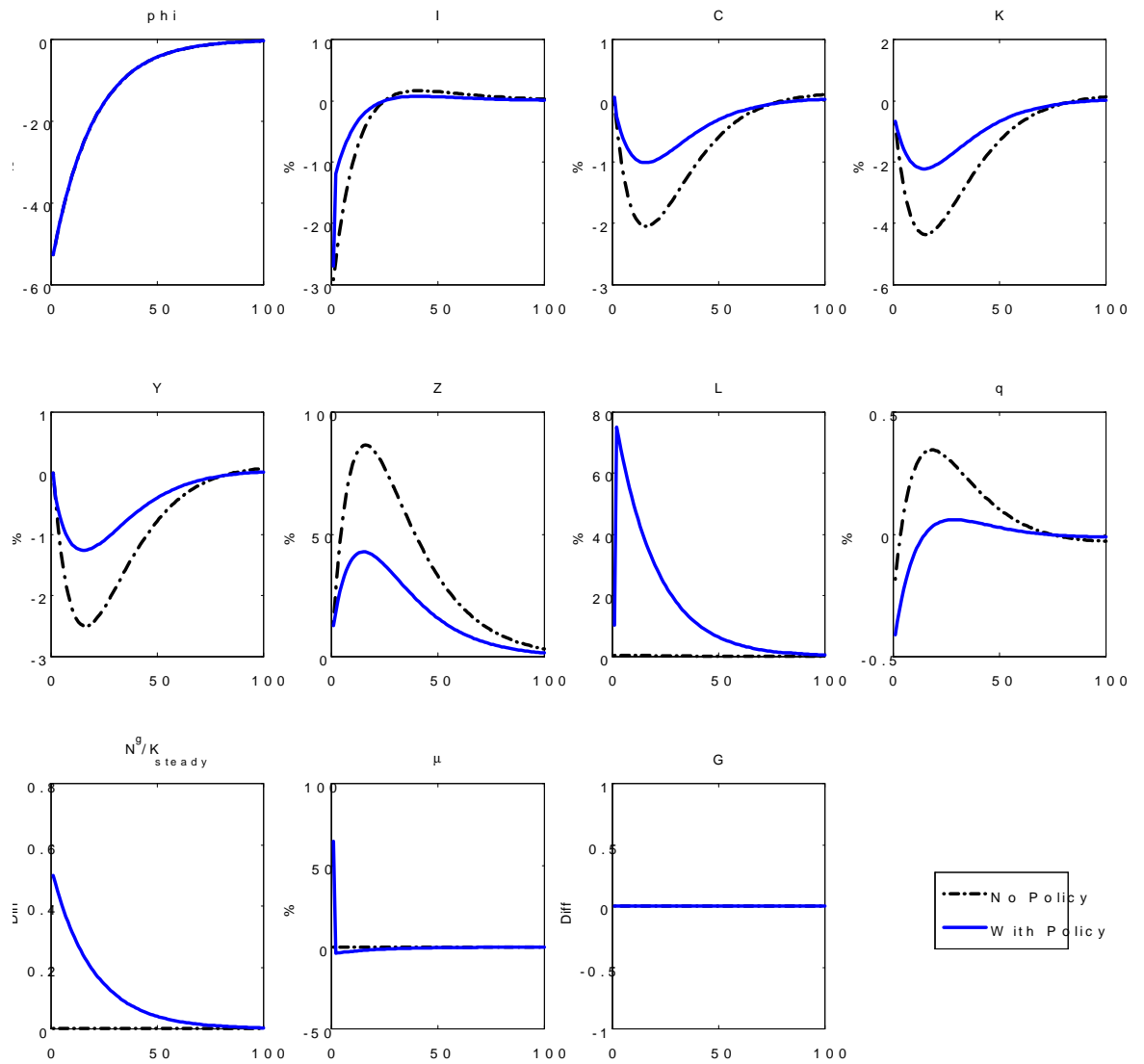
Figure 3. Impulse Responses of Economy with Storage to Liquidity Shock



How would government and central bank conduct open market operation in response to the liquidity shock? If we use the first best allocation as a benchmark, the efficient allocation does not depend upon the liquidity shock. Then, the central bank can use the open market operation in order to partially offset the effect of liquidity shock. By setting the feedback rule coefficient  $\psi_\phi$  to be negative in (32), the central bank can do the open market purchase operation to increase the liquidity of the investing entrepreneurs

in response to a negative liquidity shock. *Figure 4* compares the impulse responses of the economy with and without this policy rule ( $\psi_\phi = -0.1$ , or 0, and  $\xi = 0$ ).

Figure 4. Impulse Responses of Full Model to Liquidity Shock

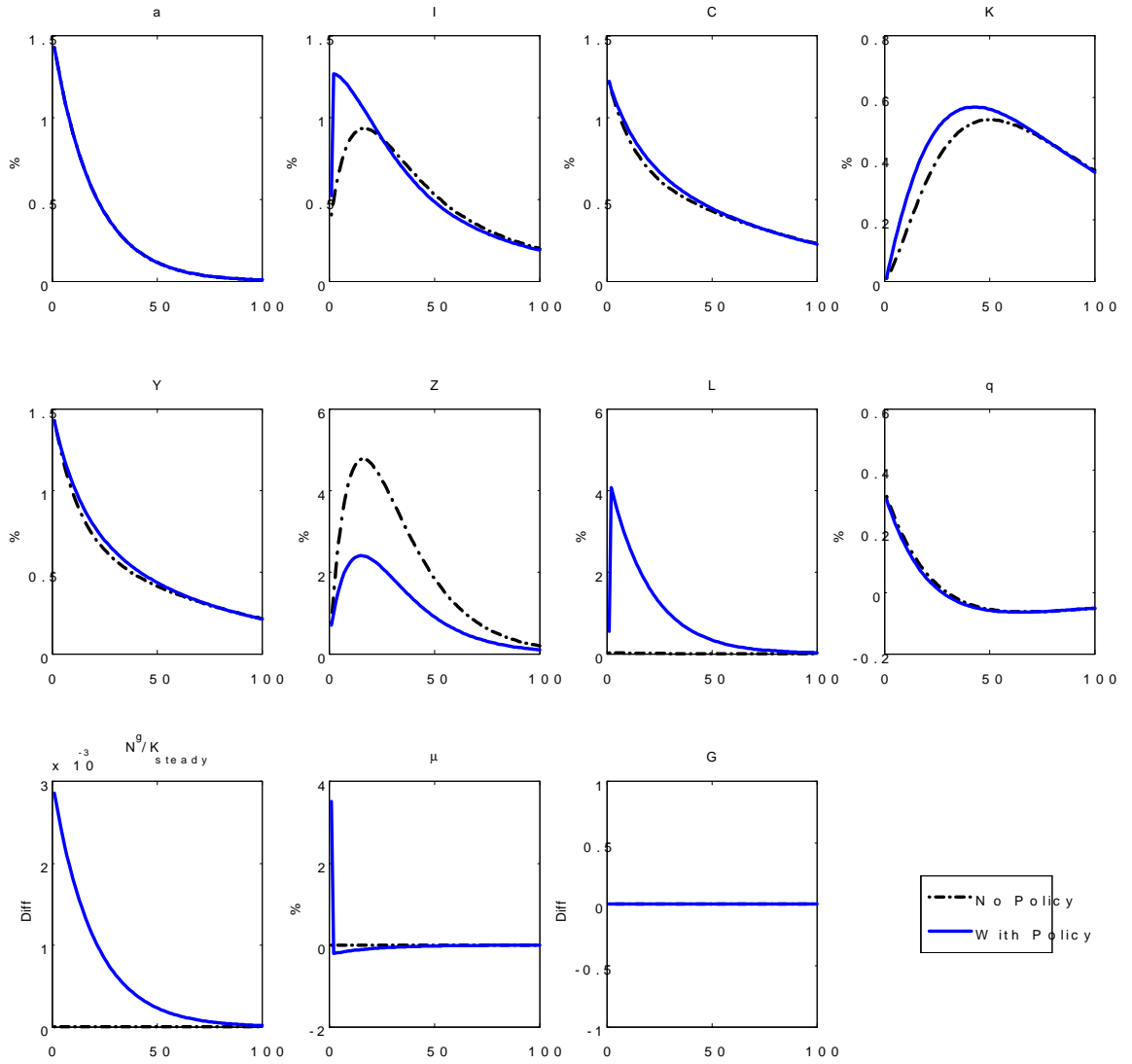


With 50% fall in the resaleability of equity  $\phi_t$ , the central bank purchases equity with money. The real balance increases sharply (despite of the relatively stable price of money in terms of goods), and storage increases less than the economy without the policy intervention. Investment falls initially by 30%, almost as much as case of no policy, because investment entrepreneur's portfolio is predetermined. In the following period, however, investing entrepreneurs (a majority of whom were savers in previous period) have a larger proportion of liquidity asset and smaller proportion of illiquid stock, and recover their investment to a level of 10% below the steady state. (From the RHS of (35), precisely because the equity is less liquid than money  $\phi_t < 1$ , we see investment will recover if the value of money increases as much as the value of private equity holdings decreases by the open market purchase operation). Thus capital stock and output do not fall as much as the economy without intervention. After the initial purchase of stock, government tends to run surplus because stock on its asset tends to have a higher rate of returns than their liability (money), and uses the surplus to reduce the money supply by setting  $\mu_t < 1$ , (when there is no adjustment of government transfer or purchase). Because this deflationary policy rewards the money holder, the flight to liquidity is more pronounced and stock price fall more with the intervention (despite the magnitude of the stock price fall is small).

How would the central bank use the open market operation in response to the fluctuation of productivity shock? When we consider the first best allocation as the benchmark, one of the problems of laissez-faire monetary economy is that investment do not react enough to productivity shocks and consumption is not smooth enough with the fluctuation of productivity. Here government can conduct open market operation and provide liquidity procyclically in order to accommodate productivity shocks, by setting the feedback coefficient of  $\psi_a$  to be positive in (32). *Figure 5* compares the impulse response functions of the laissez-faire monetary economy with the accommodating monetary policy with no fiscal adjustment ( $\psi_a = 0.2$  and  $\xi = 0$ ). As productivity rises by 1.43%, the central bank buys equity to provide liquidity by 4% despite of the relatively stable price level. The private entrepreneurs hold a larger amount of liquid money and hold a smaller amount of partially resaleable equity, and thus the entrepreneurs investment more. Investment increases by 1.3% in the immediately following periods in stead of increasing gradually in the economy without intervention. Then storage and consumption increase less than the case of no intervention. Through the accommodating open market operation, investment, capital stock and output respond

more to the productivity shock.

Figure 5, Impulse Responses of the Full Model to Productivity Shock



Here the open market operation must be to purchase the asset which has partial resaleability and a substantial liquidity premium. If the liquidity



premium of the short-term government bond is very low (as in Japan since the late 1990s), then the traditional open market operation only changes the composition of broad money and has limited effects. The recent unorthodox policy of the Federal Reserve Bank and the Bank of England, such as Term Security Lending Facility, is an attempt of increasing the liquidity by supplying more treasury bills against partially resaleable securities, such as mortgage backed securities.<sup>20</sup>

In order to analyze the effect of the open market operation over the business cycles, we assume that the government and central bank can pre-commit to conduct a particular policy. If we were to extend our analysis to explain monetary and fiscal policy in the long-run, including policy to increase the return to money holders (such as Friedman's rule or paying a higher interest on broad money), then perhaps we have to explain why government may be able to commit to a particular future action more than private agents. Also we have to take into account how government enforces people to pay tax and how the enforcement of taxation may crowd out people's pledgeable future returns to the other private agents.<sup>21</sup> These are topics for future research.

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<sup>20</sup>Perhaps one of the main reasons that the Fed lends treasury bills instead of selling treasury bills in exchange of mortgage backed securities is that the Fed concerns about the adverse selection problem in the mortgage backed securities that the current holders may have better information about the quality. (See footnote 4 and the reference within).

<sup>21</sup>A related question would be: If government has a superior power to enforce private agents to pay, why no government directly finances people's liquidity needs?

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## 6 Appendix

### 6.1 Proof of Claim 1

We construct a competitive equilibrium which satisfies Claim 1 under Condition 1. Suppose that inequalities (5), (6) and (10) are not binding. Then from (7), we need  $q_t = 1$ . (If  $q_t$  were strictly larger than one, investing entrepreneurs would invest arbitrary large amount and (5) would be binding. If  $q_t$  were strictly smaller than one, investing entrepreneurs would not invest at all, which is not consistent with the equilibrium with positive gross investment in the neighborhood of the steady state). Then the choice of entrepreneurs and workers in the competitive equilibrium imply:

$$\begin{aligned}
 1 &= E_t \left[ \beta \frac{c_t}{c_{t+1}} (r_{t+1} + \lambda) \right] = E_t \left[ \beta \frac{c'_t}{c'_{t+1}} (r_{t+1} + \lambda) \right], \quad (\text{A1}) \\
 r_t &= \gamma \left( \frac{l'_t}{K_t} \right)^{1-\gamma} \\
 w_t &= (1 - \gamma) \left( \frac{K_t}{l'_t} \right)^\gamma = \omega(l'_t)^\nu \\
 A_t(K_t)^\gamma (l'_t)^{1-\gamma} &= C_t + c'_t + K_{t+1} - \lambda K_t,
 \end{aligned}$$

where  $K_t$  is aggregate capital stock,  $l'_t$  is aggregate labor, and  $C_t$  and  $c'_t$  are aggregate consumption of entrepreneurs and workers. Because these are the conditions for the first best allocation, the competitive equilibrium achieves the first best allocation if (5), (6) and (10) are not binding. In this competitive equilibrium, we get  $p_t = 0$ . (If  $p_t$  were strictly positive with non-binding (6), we would have  $1 = E_t \left( \beta \frac{c_t}{c_{t+1}} \frac{p_{t+1}}{p_t} \right)$  which would not be satisfied in the neighborhood of the steady state).

Consider this competitive economy in which workers do not save so that  $c'_t = w_t l'_t$  and aggregate investment is equal to aggregate saving of entrepreneurs:

$$I_t = K_{t+1} - \lambda K_t = r_t K_t - C_t.$$

Using (7) with  $q_t = 1$  and  $p_t = 0$ , the inequality (5) for the investing entrepreneur becomes

$$r_t n_t - c_t = n_{t+1} - \lambda n_t \geq (1 - \theta) i_t - \phi_t \lambda n_t.$$

Aggregating this inequality for all the investing entrepreneurs, observing the arrival of the investment opportunity is iid. across entrepreneurs and over time, we have

$$\pi (r_t K_t - C_t) = \pi I_t \geq (1 - \theta)I_t - \phi_t \lambda \pi K_t. \quad (A2)$$

In the steady state, Condition 1 implies

$$\pi(1 - \lambda)K > (1 - \theta)(1 - \lambda)K - \phi \lambda \pi K.$$

Then in the neighborhood of the steady state (in which  $I = (1 - \lambda)K$ ), inequality (A2) is satisfied. Because  $p_t = 0$ , (6) and the second inequality of (10) are not binding. Therefore under Condition 1, we can find a competitive equilibrium in which the inequalities (5), (6) and (10) are not binding and  $c'_t \simeq w_t l'_t$  and  $n'_t \simeq 0$  in the neighborhood of the steady state. Claim 1(iv) follows from (A1). *QED.*

## 6.2 Derivation of Consumption and Portfolio Equations

Let  $V_t(m_t, n_t)$  be the value function of the entrepreneur who holds money and equity  $(m_t, n_t)$  at the beginning of the period  $t$  before meeting an opportunity to invest with probability  $\pi$ . The Bellman equation can be written as

$$\begin{aligned} V_t(m_t, n_t) = & \pi \cdot \underset{\substack{c_t^i, i_t, m_{t+1}^i, n_{t+1}^i \\ st.(5)(6)(7)}}{Max} \left\{ \ln c_t^i + \beta E_t [V_t(m_{t+1}^i, n_{t+1}^i)] \right\} \\ & + (1 - \pi) \cdot \underset{\substack{c_t^s, m_{t+1}^s, n_{t+1}^s \\ st.(5)(6)(7)}}{Max} \left\{ \ln c_t^s + \beta E_t [V_t(m_{t+1}^s, n_{t+1}^s)] \right\}. \end{aligned}$$

Solving the flow-of-funds condition (14) and (15) for consumption  $c_t^s$  and  $c_t^i$ , the Bellman equation is

$$\begin{aligned} & V_t(m_t, n_t) \\ = & \pi \cdot \underset{m_{t+1}^i, n_{t+1}^i}{Max} \left\{ \ln \left( [r_t + \phi_t \lambda q_t + (1 - \phi_t) \lambda q_t^R] n_t + p_t m_t - q_t n_{t+1}^i - p_t m_{t+1}^i \right) \right. \\ & \left. + \beta E_t [V_t(m_{t+1}^i, n_{t+1}^i)] \right\} \\ & + (1 - \pi) \cdot \underset{m_{t+1}^s, n_{t+1}^s}{Max} \left\{ \ln \left( (r_t + \lambda q_t) n_t + p_t m_t - q_t n_{t+1}^s - p_t m_{t+1}^s \right) \right. \\ & \left. + \beta E_t [V_t(m_{t+1}^s, n_{t+1}^s)] \right\}. \end{aligned}$$

Let  $R_{mt+1}$  be the rate of return on money from date  $t$  to date  $t+1$  and let  $R_{t+1}^{hh'}$  be the implied rate of returns on equity for the entrepreneur when her type is  $h$  ( $h = i$  for investing and  $h = s$  for saving or non-investing) at date  $t$  and  $h'$  at date  $t+1$ , i.e.,

$$\begin{aligned} R_{mt+1} &= \frac{p_{t+1}}{p_t}, \\ R_{t+1}^{ss} &= \frac{r_{t+1} + \lambda q_{t+1}}{q_t}, \quad R_{t+1}^{si} = \frac{r_{t+1} + \lambda \phi_{t+1} q_{t+1} + \lambda (1 - \phi_{t+1}) q_{t+1}^R}{q_t}, \\ R_{t+1}^{is} &= \frac{r_{t+1} + \lambda q_{t+1}}{q_t^R}, \quad R_{t+1}^{ii} = \frac{r_{t+1} + \lambda \phi_{t+1} q_{t+1} + \lambda (1 - \phi_{t+1}) q_{t+1}^R}{q_t^R}. \end{aligned}$$

Then the first order conditions are

$$1 = E_t \left[ \pi \frac{\beta c_t^i}{c_{t+1}^{ii}} R_{t+1}^{ii} + (1 - \pi) \frac{\beta c_t^i}{c_{t+1}^{is}} R_{t+1}^{is} \right] \quad (\text{A3})$$

$$1 > E_t \left[ \pi \frac{\beta c_t^i}{c_{t+1}^{ii}} R_{mt+1} + (1 - \pi) \frac{\beta c_t^i}{c_{t+1}^{is}} R_{mt+1} \right] \quad (\text{A4})$$

$$1 = E_t \left[ \pi \frac{\beta c_t^s}{c_{t+1}^{si}} R_{t+1}^{si} + (1 - \pi) \frac{\beta c_t^s}{c_{t+1}^{ss}} R_{t+1}^{ss} \right] \quad (\text{A5})$$

$$1 = E_t \left[ \pi \frac{\beta c_t^s}{c_{t+1}^{si}} R_{mt+1} + (1 - \pi) \frac{\beta c_t^s}{c_{t+1}^{ss}} R_{mt+1} \right] \quad (\text{A6})$$

where  $c_t^h$  is date- $t$  consumption of the entrepreneur of type  $h$  and  $c_{t+1}^{hh'}$  is date  $t+1$  consumption of entrepreneur when her type is  $h$  at date  $t$  and  $h'$  at date  $t+1$ .

We guess that

$$\begin{aligned} c_t^i &= (1 - \beta) \{ [r_t + \phi_t \lambda q_t + (1 - \phi_t) q_t^R] n_t + p_t m_t \}, \\ c_t^s &= (1 - \beta) \{ (r_t + \lambda q_t) n_t + p_t m_t \}, \\ n_{t+1}^i &= \beta \{ [r_t + \phi_t \lambda q_t + (1 - \phi_t) q_t^R] n_t + p_t m_t \} / q_t^R, \\ m_{t+1}^i &= 0, \\ n_{t+1}^s &= \beta f_t \{ [r_t + \phi_t \lambda q_t + (1 - \phi_t) q_t^R] n_t + p_t m_t \} / q_t, \\ m_{t+1}^s &= \beta (1 - f_t) \{ [r_t + \phi_t \lambda q_t + (1 - \phi_t) q_t^R] n_t + p_t m_t \} / p_t \end{aligned}$$

$$\begin{aligned}
c_{t+1}^{ii} &= (1 - \beta) [r_{t+1} + \phi_{t+1} \lambda q_{t+1} + (1 - \phi_{t+1}) q_t^R] n_{t+1}^i, \\
c_{t+1}^{is} &= (1 - \beta) (r_{t+1} + \lambda q_{t+1}) n_{t+1}^i, \\
c_{t+1}^{si} &= (1 - \beta) \{ [r_{t+1} + \phi_{t+1} \lambda q_{t+1} + (1 - \phi_{t+1}) q_t^R] n_{t+1}^s + p_{t+1} m_{t+1}^s \}, \\
c_{t+1}^{ss} &= (1 - \beta) \{ (r_{t+1} + \lambda q_{t+1}) n_{t+1}^s + p_{t+1} m_{t+1}^s \}.
\end{aligned}$$

where  $f_t \equiv q_t n_{t+1}^s / (p_t m_{t+1}^s + q_t n_{t+1}^s) \in (0, 1)$  is the share of equity in portfolio for the non-investing entrepreneur. Under this guess we learn

$$\frac{c_{t+1}^{ii}}{c_t^i} = \beta R_{t+1}^{ii}, \quad \text{and} \quad \frac{c_{t+1}^{is}}{c_t^i} = \beta R_{t+1}^{is},$$

and thus (A3) is always satisfied. (A4) can be written as

$$1 > E_t \left[ \pi \frac{R_{mt+1}}{R_{t+1}^{ii}} + (1 - \pi) \frac{R_{mt+1}}{R_{t+1}^{is}} \right]. \quad (\text{A7})$$

We also learn that

$$\begin{aligned}
\frac{c_{t+1}^{si}}{c_t^s} &= \beta [f_t R_{t+1}^{si} + (1 - f_t) R_{mt+1}], \\
\frac{c_{t+1}^{ss}}{c_t^s} &= \beta [f_t R_{t+1}^{ss} + (1 - f_t) R_{mt+1}].
\end{aligned}$$

Thus (A5) and (A6) become

$$1 = E_t \left[ \pi \frac{R_{t+1}^{si}}{f_t R_{t+1}^{si} + (1 - f_t) R_{mt+1}} + (1 - \pi) \frac{R_{t+1}^{ss}}{f_t R_{t+1}^{ss} + (1 - f_t) R_{mt+1}} \right] \quad (\text{A8})$$

$$1 = E_t \left[ \pi \frac{R_{mt+1}}{f_t R_{t+1}^{si} + (1 - f_t) R_{mt+1}} + (1 - \pi) \frac{R_{mt+1}}{f_t R_{t+1}^{ss} + (1 - f_t) R_{mt+1}} \right] \quad (\text{A9})$$

Then  $f_t$  times the RHS of (A8) plus  $1 - f_t$  times RHS of (A9) is always equal to 1, and one of (A8) and (A9) is not independent. Subtracting (A9) from (A8) side by side and arranging, we get

$$\pi E_t \left[ \frac{R_{mt+1} - R_{t+1}^{si}}{f_t R_{t+1}^{si} + (1 - f_t) R_{mt+1}} \right] = (1 - \pi) \left[ \frac{R_{t+1}^{ss} - R_{mt+1}}{f_t R_{t+1}^{ss} + (1 - f_t) R_{mt+1}} \right]. \quad (\text{A10})$$

This is equivalent to (24) in the text. Because  $q_t > 1 > q_t^R$  in our equilibrium, we always have

$$R_{t+1}^{ii} > R_{t+1}^{si} \quad \text{and} \quad R_{t+1}^{is} > R_{t+1}^{ss}.$$

In the neighborhood of the steady state equilibrium, we have

$$R_{t+1}^{ii} > R_{t+1}^{ss} > R_{mt+1}.$$

Thus, comparing (A9) and (A7), we learn the inequality (A7) holds in the neighborhood of the steady state.

### 6.3 Proof of Claim 2

Claim 2(iii) directly follows from inequality (A4) or (A7) which we proved above.

Let  $l$  be the ratio of real money balance to capital stock in the steady state. From (26, 27, 28) in the text, the steady state value of  $(r, q, l)$  solves

$$r = 1 - \lambda + (1 - \beta) [r + \lambda(1 - \pi + \pi\phi)q + \lambda\pi(1 - \phi)q^R + l], \quad (\text{A11})$$

$$(1 - \lambda)(1 - \theta q) = \pi[\beta(r + \lambda\phi q) - \lambda(1 - \beta)(1 - \phi)q^R + \beta l], \quad (\text{A12})$$

$$(1 - \pi) \frac{\frac{r}{q} + \lambda - 1}{(r + \lambda q)\chi + l} = \pi \frac{1 - \frac{r + \lambda\phi q + \lambda(1 - \phi)q^R}{q}}{[r + \lambda\phi q + \lambda(1 - \phi)q^R]\chi + l} \quad (\text{A13})$$

where  $\chi = \theta(1 - \lambda) + (1 - \pi + \pi\phi)\lambda$  and  $q^R = \frac{1 - \theta q}{1 - \theta}$  as in the text.

From (A11) and (A12), we have

$$\begin{pmatrix} 1 & -(1 - \beta) \\ \pi & \pi\beta \end{pmatrix} \begin{pmatrix} \beta r \\ l \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \kappa + \begin{pmatrix} \lambda(1 - \beta)(1 - \pi\eta) \\ -[\widehat{\theta} + \lambda\pi\beta(1 - \eta)] \end{pmatrix} q$$

where  $\eta = \frac{1 - \phi}{1 - \theta}$ ,  $\kappa = 1 - \lambda + \lambda\pi(1 - \beta)\eta$  and  $\widehat{\theta} = (1 - \lambda + \lambda\pi\eta)\theta$ . Thus

$$\pi\beta r = (1 - \beta + \pi\beta)\kappa + (1 - \beta) [\lambda\pi\beta(1 - \pi)\eta - \widehat{\theta}] q \quad (\text{A14})$$

$$\pi l = (1 - \pi)\kappa - [\lambda\pi - \lambda\pi(\beta + \pi - \pi\beta)\eta + \widehat{\theta}] q. \quad (\text{A15})$$

Because  $\lambda\pi - \lambda\pi(\beta + \pi - \pi\beta)\eta + \widehat{\theta} = \lambda\pi(1 - \pi)(1 - \beta)\eta + \lambda\pi\phi + \theta(1 - \lambda) > 0$ , we have

$$l > 0, \text{ iff } q < \frac{(1 - \pi)\kappa}{\lambda\pi - \lambda\pi(\beta + \pi - \pi\beta)\eta + \widehat{\theta}} \equiv \widehat{q}. \quad (\text{A16})$$



Then there is a monetary equilibrium with financing constraint only if  $\widehat{q} > 1$ ,  
or

$$(1 - \lambda)\theta + \lambda\pi\phi < (1 - \lambda)(1 - \pi),$$

i.e., Condition 1 in the text for the first best allocation is violated.

(A13) can be written as

$$[r - (1 - \lambda)q] \left[ r - (1 - \lambda)q + q + \frac{l}{\chi} \right] = \lambda\eta(q-1) \left[ r - (1 - \lambda)q + \pi \left( q + \frac{l}{\chi} \right) \right] \quad (\text{A17})$$

Together with (A14) and (A15), we have the condition for steady state  $q$  as

$$\begin{aligned} 0 &= \Psi(q) \\ &\equiv \lambda\pi\eta(q-1) \left( \begin{array}{c} \left\{ (1 - \beta + \pi\beta)\kappa + q \left[ (1 - \beta) \left( \lambda\pi\beta\eta - \widehat{\theta} \right) - \pi\beta\kappa \right] \right\} \left[ \lambda(1 - \pi\eta) + \widehat{\theta} \right] \\ + \pi\beta(1 - \pi) \left[ \kappa + q \left( \lambda\pi\beta\eta - \widehat{\theta} \right) \right] \end{array} \right) \\ &\quad - \left\{ (1 - \beta + \pi\beta)\kappa + q \left[ (1 - \beta) \left( \lambda\pi\beta\eta - \widehat{\theta} \right) - \pi\beta\kappa \right] \right\} \cdot \\ &\quad \left( \begin{array}{c} \left\{ (1 - \beta + \pi\beta)\kappa + q \left[ (1 - \beta) \left( \lambda\pi\beta\eta - \widehat{\theta} \right) - \pi\beta\kappa \right] \right\} \left[ \lambda(1 - \pi\eta) + \widehat{\theta} \right] \\ + \beta(1 - \pi) \left[ \kappa + q \left( \lambda\pi\beta\eta - \widehat{\theta} \right) \right] \end{array} \right). \quad (\text{A18}) \end{aligned}$$

Then we learn

$$\Psi(1) = -(1 - \beta) \left( \kappa + \lambda\pi\beta\eta - \widehat{\theta} \right)^2 \{ (1 - \beta) [\lambda(1 - \pi\eta)(1 - \theta) + \theta] + \beta(1 - \pi) \} < 0.$$

Therefore the necessary and sufficient condition for the existence of monetary equilibrium with  $q \in (1, \widehat{q})$  is

$$0 < \Psi(\widehat{q}). \quad (\text{A19})$$

Using (A16), (A19) becomes

$$\begin{aligned} 0 &< \beta\lambda\eta(1 - \pi - \chi) [\chi - \beta(1 - \pi) + \pi\beta(1 - \pi)] - [\chi - \beta(1 - \pi)] \chi\kappa \\ &= \pi\beta^2\lambda\eta(1 - \pi)(1 - \pi - \chi) - [\chi - \beta(1 - \pi)](1 - \lambda + \lambda\pi\eta) [\chi - \beta\lambda(1 - \pi)(1 - \phi)]. \end{aligned}$$

Multiplying both sides with  $1 - \theta$ , we get the condition

$$\begin{aligned} 0 &< \pi\beta^2\lambda(1 - \pi)(1 - \phi)[(1 - \pi)(1 - \lambda) - (1 - \lambda)\theta - \pi\lambda\phi] \\ &\quad + [(\beta - \pi)(1 - \lambda) - (1 - \lambda)\theta - \pi\lambda\phi] [(1 - \lambda)(1 - \theta) + \lambda\pi(1 - \phi)] \\ &\quad \cdot [\lambda(1 - \pi)(1 - \beta) + \theta(1 - \lambda) + \phi\lambda(\pi + \beta - \pi\beta)]. \end{aligned}$$

This is equivalent to Assumption 2 in the text. Therefore, under Assumption 2, we have a competitive equilibrium in which fiat money has a positive value (Claim 3(i)  $p_t > 0$ ) and the investing entrepreneur faces a binding liquidity constraint (Claim 3(iii)) in the neighborhood of the steady state. *QED*.

## 6.4 Proof of Claim 3

Claim 3(i):

Under Assumption 2, we have  $(1 - \lambda)\theta + \pi\lambda\phi < (1 - \lambda)(1 - \pi)$ . Thus we have

$$\begin{aligned} \frac{\partial \text{RHS of (A11)}}{\partial q} &= \lambda(1 - \beta) \left( 1 - \pi + \pi\phi - \pi \frac{1 - \phi}{1 - \theta} \theta \right) \\ &> \lambda(1 - \beta) \frac{\pi\phi}{(1 - \lambda)(1 - \theta)} > 0. \end{aligned}$$

Given  $q > 1$  and  $l \geq 0$ , we have from (A11)

$$r > 1 - \lambda + (1 - \beta)(r + \lambda), \text{ or}$$

$$r + \lambda > \frac{1}{\beta}. \text{ QED.}$$

Claim 3(iv):

Suppose that 3(iv) is not true. From (A13) and (A11) respectively, we have

$$\frac{r}{q} \leq 1 - \lambda$$

and

$$\frac{r}{q} - (1 - \lambda) \geq (1 - \beta) \left[ \frac{r}{q}(1 - \pi) + \lambda(1 - \pi) + \pi \right] + \frac{1 - \beta}{q} l.$$

Together these imply  $l < 0$ . This is contradiction. *QED*.

Claim 3(iii):

From equation (A13), Claim 3(iv) implies  $\frac{r}{q} + \lambda > 1$ . *QED*.

Claim 3(ii):

Using  $q^R < 1 < q$  in (A13), we have

$$(1 - \pi) [r - (1 - \lambda)q] > \pi [q - r - \lambda\phi q - \lambda(1 - \phi)q^R], \text{ or}$$

$$r - (1 - \lambda)q > \pi\lambda\eta(q - 1) > 0.$$

Hence, given  $l > 0$ , from (A17) it follows that

$$\frac{\Delta(\Delta + 1)}{\Delta + \pi} < \lambda\eta \left(1 - \frac{1}{q}\right),$$

where  $\Delta \equiv \frac{r}{q} + \lambda - 1 > 0$  by Claim 3(iii). But from (A14),

$$\frac{1}{q} = \frac{1}{(1 - \beta + \pi\beta)\kappa} \left\{ \pi\beta(\Delta + 1 - \lambda) - (1 - \beta)[\lambda\pi\beta(1 - \pi)\eta - \widehat{\theta}] \right\}.$$

Substituting this to the above inequality, we get

$$\frac{\Delta(\Delta + 1)}{\lambda\eta(\Delta + \pi)} + \frac{\pi\beta\Delta}{(1 - \beta + \pi\beta)\kappa}$$

$$< 1 - \frac{1}{(1 - \beta + \pi\beta)\kappa} \left\{ \pi\beta(1 - \lambda) - (1 - \beta)[\lambda\pi\beta(1 - \pi)\eta - \widehat{\theta}] \right\}.$$

The LHS of this inequality is increasing in  $\Delta$ .

Suppose Claim 3(ii) is not true, i.e.,  $\Delta \geq \frac{1 - \beta}{\beta}$ . Then

$$\frac{1 - \beta}{\lambda\eta\beta(1 - \beta + \pi\beta)} + \frac{\pi(1 - \beta)}{(1 - \beta + \pi\beta)\kappa}$$

$$< 1 - \frac{1}{(1 - \beta + \pi\beta)\kappa} \left\{ \pi\beta(1 - \lambda) - (1 - \beta)[\lambda\pi\beta(1 - \pi)\eta - \widehat{\theta}] \right\}.$$

Multiplying this inequality through by  $\lambda\eta\beta(1 - \beta + \pi\beta)\kappa$ , we have

$$(1 - \beta) [1 - \lambda + \lambda\pi(1 - \beta)\eta] + \lambda\eta\beta\pi(1 - \beta)$$

$$< \lambda\eta\beta(1 - \beta + \pi\beta) [1 - \lambda + \lambda\pi(1 - \beta)\eta] - \lambda\eta\beta^2\pi(1 - \lambda)$$

$$+ \lambda^2\eta^2\beta^2\pi(1 - \pi)(1 - \beta) - \lambda\eta\beta(1 - \beta)(1 - \lambda + \lambda\pi\eta)\theta.$$

Cancel the two terms which do not have factor  $1 - \beta$ , and divide by  $1 - \beta$ , we get

$$(1 - \lambda + \lambda\pi\eta) [1 - \lambda\eta\beta(1 - \theta)] < 0, \text{ or}$$

$$(1 - \lambda + \lambda\pi\eta) [1 - \lambda\beta(1 - \phi)] < 0.$$

This is contradiction. *QED*.