Credit Horizons

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Abstract

Why might firms borrow largely against near-term revenues? Does this mean they are unable to raise much funding against the long-term horizon? In this paper, we develop a model of credit horizons. We use our framework to examine how credit horizons interact with firm dynamics and the evolution of productivities. For an open set of parameters, we find that even though all firms start off identical, their owners may plan different paths for future productivity: some choose to improve and continue for the long haul, others choose to deteriorate and subsequently shut down. A question of particular concern to us is whether persistently low interest rates can stifle aggregate investment and growth. With this in mind, our model is of a small open economy where the world interest rate is taken to be exogenous. We show that a permanent fall in the interest rate can reduce aggregate investment and growth, and even lead to a drop in the welfare of everyone: a Pareto deterioration.

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1 Introduction

When financing long-term capital investment, entrepreneurs raise external funds either against collateral assets such as plant and buildings, or against future revenues. If the latter, lending is typically supported by near-term revenues: entrepreneurs borrow largely against, say, the first few years of their future income stream, even though investment may be of longer duration. Why? Is it that they are unable to borrow much against the long-term horizon?

In this paper, we develop a model of credit horizons. We use our framework to examine how credit horizons interact with firm dynamics – more specifically, plant dynamics – and the evolution of productivities.

A question of particular concern to us is whether persistently low real interest rates can stifle aggregate investment and growth. The question is motivated by Japan, where the economy struggles to regain robust growth despite interest rates having been close to zero for over two decades. More recently, this has become a concern for many other developed economies too.

With this in mind, we model a small open economy where the world interest rate R is exogenous. To get a flavour of our model, think of an engineer-cum-entrepreneur, Emma, raising funds to invest in plant within a building. For our purposes, it does not matter if the building is leased long-term or purchased outright: the critical thing is that there will be an ongoing flow of fixed costs that will have to be paid to maintain production in the long run. The fixed cost may be rent on the building or the opportunity cost of owning the building. There is no obstacle to Emma raising funds against the plant: this can be sold at the time of investment. What cannot be sold is Emma's engineering expertise, her human capital, which we take to be proportional to the scale of investment.

A saver, Sam, who buys the plant, together with an obligation to pay the flow of fixed costs, will subsequently need engineers' expertise to help run and maintain the enterprise. Without adequate maintenance from an engineer, the productivity (or quality) of his plant would slowly deteriorate and output would commensurately fall. We assume a form of 'roundabout' technology, inspired by the Austrian School of Böhm-Bawerk (1889): we suppose that tomorrow's plant quality is a function of both today's quality and today's engineering input.

We allow for an ex post competitive market in which plant owners hire the maintenance services of engineers. Thus Emma's share of ex post surplus is determined through competition (alternatively, Emma and Sam might engage in bilateral bargaining ex post – our findings would be broadly similar).[1] For Sam, Emma's share of surplus reflects her forward-looking marginal product: not only her immediate contribution to output but also her contribution to future quality and output.

Having bought the new plant, Sam has to decide on a maintenance plan. It turns out he has a clear-cut long-term choice. Either he curbs maintenance costs and allows quality to deteriorate slowly, to some point when he decides it is no longer worth paying the fixed costs and exits – call this his "stopping strategy." Or he pays the costs needed to maintain, or even improve, quality with a view to staying in production over the long haul – call this his "continuing strategy."

This dichotomous decision – either planning to stop within a finite horizon, or planning to continue for the long haul – reveals an intriguing feature of equilibrium. For an open set of parameters, even though all plant starts off identical in quality, their qualities diverge over time: some plant improves in quality and other plant deteriorates and eventually shuts down.[2] We think this may be a rich new vein for research into firm/plant dynamics, which should inform the study of how aggregate productivity evolves.

In the complementary part of the parameter space, all owners of new plant choose the continuing strategy and their qualities do not diverge.

A primary concern for Emma is: How much funding can she raise from savers like Sam at the time of investment? That is, her borrowing capacity is the amount Sam is willing pay per unit of new plant, which in turn depends on Sam's assessment of his share of the future surplus from that plant, net of the fixed costs. The scale of Emma's investment will be given by a critical ratio (familiar from many models of investment under financing constraints): namely, her net worth divided by the downpayment needed – unit investment cost minus borrowing capacity.

[1] Emma cannot commit ex ante to supply her maintenance services for less than the ex post market rate. This form of constraint – our sole departure from the Arrow-Debreu model – is sometimes called a "non-exclusivity constraint." See, for example, Allen (1985), Townsend (1989), Cole and Kocherlakota (2001) and Attar, Mariotti and Salanie (2011).

[2] Allowing for initial heterogeneity would purify (Aumann et al. (1983)) this mixed-strategy equilibrium so that plant owners would, more realistically, follow pure strategies. Introducing uncertainty at the individual level would greatly enrich the model, but this is for future research.

Of particular interest to us is how a lower interest rate R might impinge on Emma's investment and, ultimately, on aggregate output and growth. The key insight is that a lower R raises Emma's ex post share of surplus, insofar as her marginal product is forward-looking. This is tantamount to saying that with lower R Emma cannot, at the time of investment, credibly pledge to Sam as much of the long-term gross revenue from the project (albeit that the pledgeable revenues are discounted by Sam at the lower interest rate). But, crucially, Sam's long-run obligation to pay the fixed costs is undiminished (indeed its present value rises as R falls). Accordingly, Sam's willingness to pay for new plant can be lower when R is lower. This means that Emma's borrowing capacity is lower – overturning the usual notion that lower interest rates benefit borrowers.

Notice the driver here: the *denominator* – investment cost minus borrowing capacity per unit of investment – of the critical ratio (net worth/downpayment) rises as R falls, owing to the fall in the borrowing capacity. In the macro-finance literature, the focus has been on how the *numerator* – credit-constrained agents' net worth – might move in perverse ways following shocks to an economy. In the present model, the numerator behaves as might be expected – a fall in R raises net worth – but this can be more than offset by the rise in the denominator. Overall, aggregate investment can fall with a fall in interest rates, as can the growth rate of the economy. We show numerically that these effects may reduce the welfare of everyone in the domestic economy: a fall in R may lead to a Pareto deterioration.

2 Model

We consider a small open economy with an exogenous world real interest rate R. There is no aggregate uncertainty and, for the moment, we focus on steady state equilibrium (later, we will examine the effects of an unanticipated persistent drop in R). There is a homogeneous perishable consumption/investment good at each date $t = 0, 1, 2, ...$ We use this good as numeraire as we consider a non-monetary economy.

There is a continuum of domestic agents, each maximizing utility of consumption c_t from the present to the infinite future:

$$E_0 \left[\sum_{t=0}^{\infty} \beta^t \ln c_t \right], \tag{1}$$

where $\beta \in (0,1)$ is the utility discount factor and $\ln c$ is the natural logarithm of c. We assume that the exogenous world interest rate is nonnegative in net terms and lower than the subjective interest rate:

$$1 \leq R < \frac{1}{\beta}. \tag{2}$$

Each agent sometimes has an investment opportunity (being an entrepreneur or simply "engineer"), and sometimes not ("saver"). The transition probabilities of being an engineer conditional on being an engineer or a saver in the previous period are given by

$$\begin{aligned} \text{Prob}\,(\text{engineer at t} \mid \text{engineer at t-1}) &= \pi^E \\ \text{Prob}\,(\text{engineer at t} \mid \text{saver at t-1}) &= \pi^S. \end{aligned}$$

We assume the arrival of an investment opportunity is persistent to a limited degree so that $0 \leq \pi^S < \pi^E < 1$.

At each date t, an engineer, say E, can jointly produce plant and tools from goods: within the period, per unit of plant,

$$x \text{ goods } \} \rightarrow \left\{ \begin{array}{c} \text{plant of quality } 1 \\ \text{E-tool} \end{array} \right. . \tag{3}$$

The investment technology is constant returns to scale and scalable by any positive number i. Plant and tools are ready for use from date t+1.

Each tool is specific to the engineer ("E-tool") in that only she knows how to use it – unless she sells it to another engineer and teaches him. Because the engineer cannot sell her tools to savers, she raises funds by selling all she can sell - the plant - to savers.

The plant owner has a constant returns to scale production technology. At each date, the owner of one unit of plant of quality z can hire any number $h \geq 0$ of tools (hiring each tool along with the engineer who knows how to use it) at a competitive rental price w ("wage") to produce goods and maintain plant quality: within the period, per unit of plant,

$$\left. \begin{array}{c} \text{plant of quality } z \\ h \text{ tools} \\ f \text{ goods} \end{array} \right\} \rightarrow \left\{ \begin{array}{c} y = az \text{ goods} \\ \lambda \text{ plant of quality } z' = z^{\theta} h^{\eta} \\ \lambda h \text{ tools} \end{array} \right. . \tag{4}$$

$a > 0$ is the common productivity of all plant and z' is plant quality after maintenance. f is a fixed cost per unit of plant, and $\lambda < 1$ reflects depreciation, by which a fraction $1 - \lambda$ of plant and tools are destroyed after use.

The fixed cost can be thought of as the rental price or the opportunity cost of the building in which plant is located. The parameter θ is the share of initial plant quality and η is the share of engineers' tools in maintaining plant quality. We assume $\theta, \eta > 0$, and $\theta + \eta \leq 1$. Although we assume output is proportional to plant quality here, Appendix A shows this formulation is justified when output is a general decreasing returns to scale function of plant quality and unskilled labor and unskilled labor is hired by plant owners in a competitive market.

The plant owner always has the option to stop, so his value of a unit of plant of quality z at the end of the period is given by

$$V(z; w, R) = \frac{1}{R}\left\{0,\ \underset{h}{Max}\left[az - wh - f + \lambda V\left(z^{\theta}h^{\eta}; w, R\right)\right]\right\}. \tag{5}$$

The first term in curly bracket is the value of stopping, while the second term is the value of continuing - the sum of net cash flow (gross revenue minus wage and fixed costs) and the capital value of the remaining λ units of plant with quality $z' = z^{\theta}h^{\eta}$.

Knowing that the return from maintaining plant quality depends upon future production, the plant owner must devise a long-term plan: either stop after a finite number of periods T, or continue forever ($T = \infty$)? For each $T = 0, 1, 2, \ldots$, define recursively the owner's value of a unit of plant of current quality z stopping in T periods:

$$\begin{aligned}
S^0(z; w, R) &= 0 \\
S^1(z; w, R) &= \frac{1}{R}(az - f) \\
S^2(z; w, R) &= \frac{1}{R}\underset{h}{Max}\left[az - wh - f + \frac{\lambda}{R}(az^{\theta}h^{\eta} - f)\right] \\
&\cdots \\
S^T(z; w, R) &= \frac{1}{R}\underset{h}{Max}\left[az - wh - f + \lambda S^{T-1}\left(z^{\theta}h^{\eta}; w, R\right)\right]. \qquad (6)
\end{aligned}$$

If the plant is shut down tonight, the value $S^0(z; w, R)$ is zero. If the plant owner shuts down tomorrow night, he will not hire tools tomorrow and the value $S^1(z; w, R)$ equals the present value of tomorrow's revenue minus fixed cost. If the plant owner shuts down in two nights' time, he hires tomorrow's tools to balance the cost and benefit of maintaining plant quality for production two days later. Generally the owner's value of a unit of plant of current quality z stopping in T periods, $S^T(z; w, R)$, equals present value of sum of tomorrow's net cash flow and the value of λ units of plant of quality $z^{\theta}h^{\eta}$ stopping in $T-1$ periods.

Now, for all value of z, the plant owner chooses the optimal stopping time so that

$$V(z;w,R) \equiv \sup_{T\geq 0} S^{T}\left(z;w,R\right). \tag{7}$$

It turns out there is a clear dichotomy between stopping after a finite number of periods and continuing forever, as is shown in the following Lemma. (All proofs and details of derivations are in the Appendix.)

Lemma:

(i) If the current plant quality z is below some cutoff value, $z^{\dagger}$, it is optimal for the plant owner to stop after, say, $T_{\max}(z) < \infty$ periods.
(ii) If z is above $z^{\dagger}$, it is optimal to continue forever.
(iii) The cutoff value $z^{\dagger}$ increases with the fixed cost f. It is also a function of the wage rate and the interest rate.

In Figure 1, we plot the plant value as a function of plant quality z for a given wage and interest rate for different horizons of stopping T. The function $S^{\infty}\left(z;w,R\right)$ is the value when the plant owner chooses to maintain production forever. The upper envelope of all these functions is the value function of plant $V(z;w,R)$ with an optimal choice of stopping (including non-stopping). If the plant quality is very low as

$$z < \frac{f}{a},$$

then it is optimal for the owner to shut down the plant immediately because output does not cover fixed cost. If the plant quality z is higher than $\frac{f}{a}$ but lower than $z^{\dagger}$, then it is optimal to shut down the plant in a finite horizon, where the horizon is an increasing function of z. At $z = z^{\dagger}$, the plant owner is indifferent between continuing forever and shutting down in a finite time (in this numerical example, it is 20 periods). If plant quality is higher than $z^{\dagger}$, the plant owner will continue forever until plant dies exogenously.

In Figure 2, we plot $S^{T}\left(z;w,R\right)$ as a function of horizon T for three different levels of plant quality, $z = \frac{9}{10}z^{\dagger}$, $z^{\dagger}$, and $\frac{11}{10}z^{\dagger}$. If plant quality is relatively low at $z = \frac{9}{10}z^{\dagger}$, then the value reaches a maximum with finite horizon: in our example, around $T = 15$ so that the owner will shut down

Figure 1: plant value

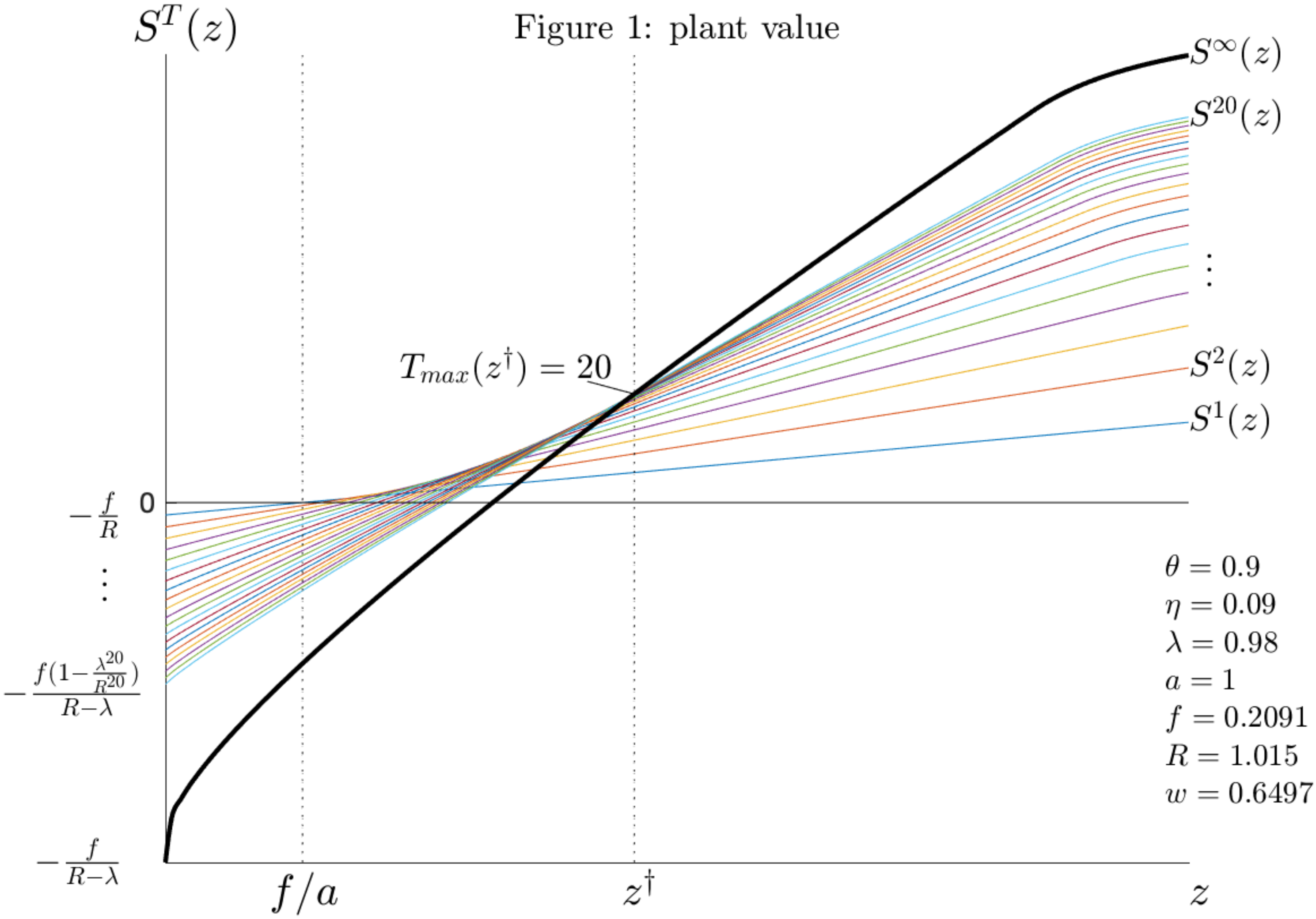
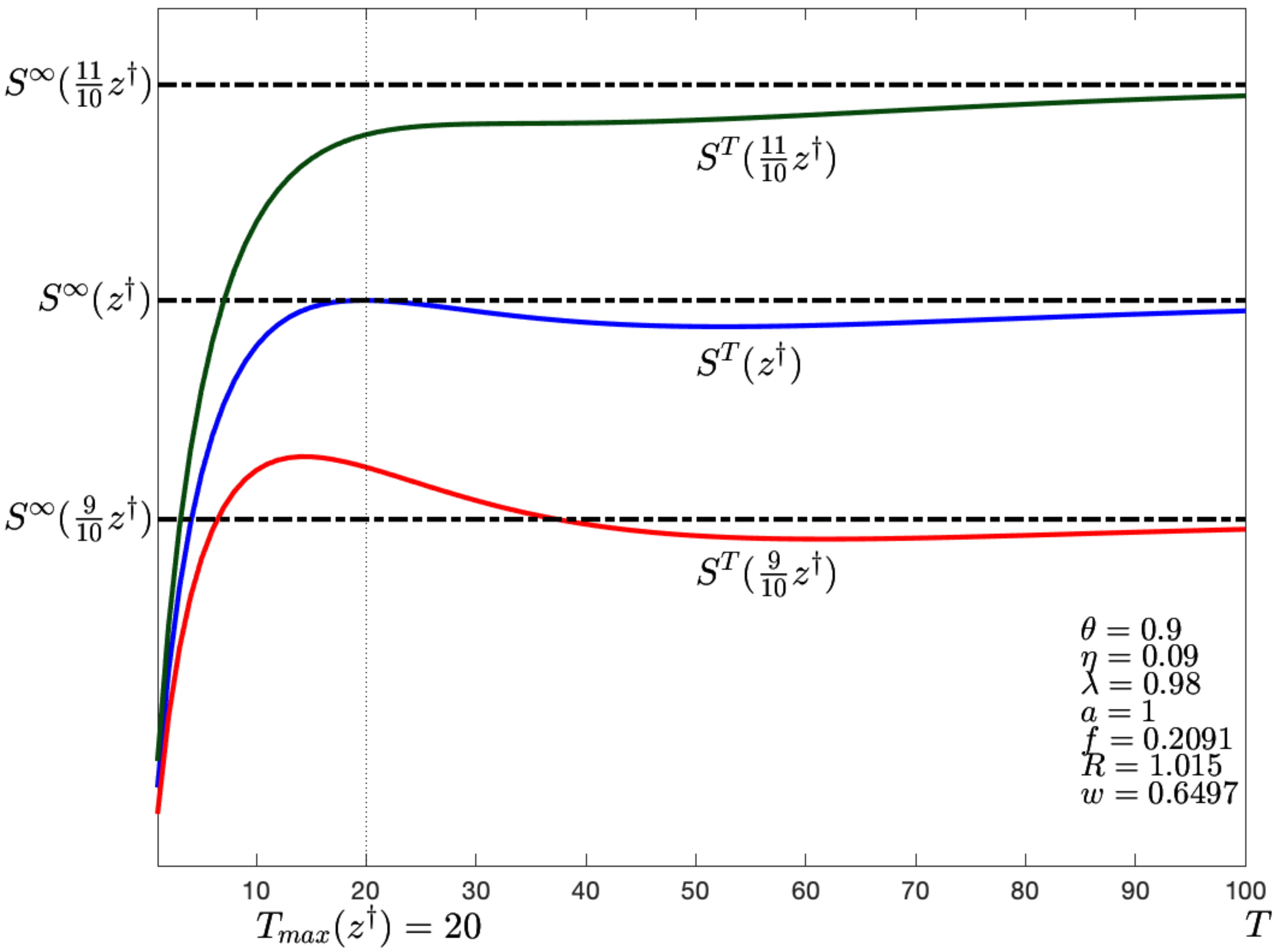


Figure 2: $S^T(z)$ as a function of horizon T



in 15 periods. If plant quality is exactly equal to $z^\dagger$, then the plant owner is indifferent between shutting down at $T = 20$ and continuing forever (asymptotically as $T \to \infty$). If plant quality is higher than $z^\dagger$, at $z = \frac{11}{10} z^\dagger$, then the owner finds that $S^\infty(z; w, R) > S^T(z; w, R)$ for any finite T so that he will continue forever.

We can use the value function to express the value as the present value of net cash flow:

$$\begin{aligned}
& V(z; w, R) \\
= \; & \frac{1}{R}(y_t - wh_t - f) + \frac{\lambda}{R^2}(y_{t+1} - wh_{t+1} - f) + \frac{\lambda^2}{R^3}(y_{t+2} - wh_{t+2} - f) \\
& + ... + \frac{\lambda^{T-2}}{R^{T-1}}(y_{t+T-1} - wh_{t+T-1} - f) + \frac{\lambda^{T-1}}{R^T}(y_{t+T} - f).
\end{aligned} \tag{8}$$

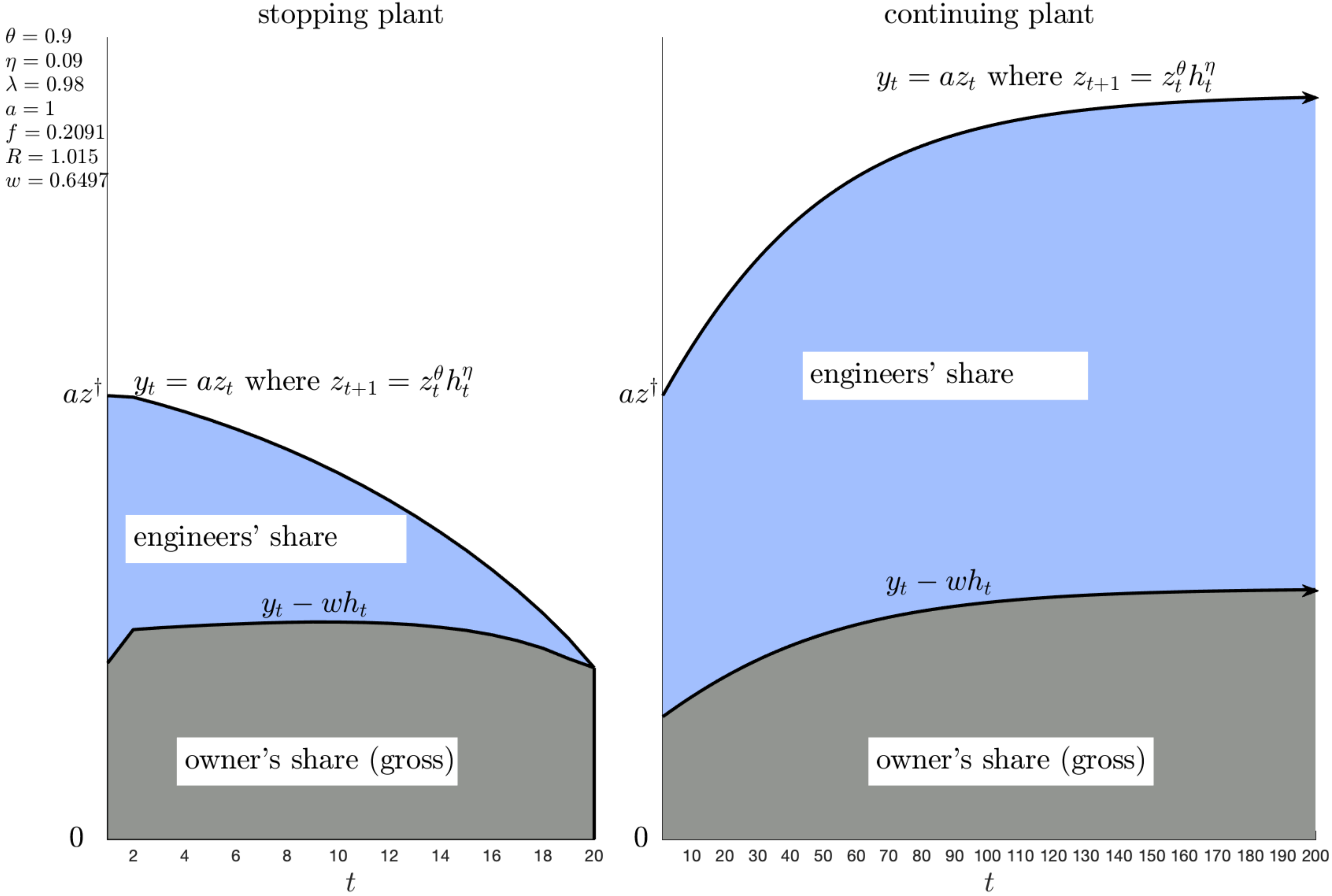
Figure 3 describes the division of gross output $\{y_t\}_{t=1,2,3,...}$ between engineers $\{wh_t\}$ and plant owner $\{y_t - wh_t\}$ for the cases of stopping in 20 periods and continuing forever when initial plant quality is $z = z^\dagger$. For stopping plant, as one gets closer to the stopping time, plant quality and gross output decrease with smaller maintenance work and wage bill $\{wh_t\}$ of engineers. For continuing plant, plant quality and gross output increase with age until they converge to a certain level. During this transition, both engineers' income $\{wh_t\}$ and plant owner's gross profit $\{y_t - wh_t\}$ increase.

Alternatively we can look into the division of gross revenues, using the plant owner's choice of hiring tools and engineers. At each date t, whether the current z_t lies above or below the cutoff $z^\dagger$, an optimal sequence $\{h_t, z_{t+1}, h_{t+1}, z_{t+2}, h_{t+2}, \ldots\}$ equates the discounted sum of marginal product to the wage (see the Appendix for the derivation):

$$\begin{aligned}
w \; = \; & \frac{\lambda}{R} a\eta \frac{z_{t+1}}{h_t} + \left(\frac{\lambda}{R}\right)^2 a\eta \frac{z_{t+1}}{h_t} \theta \frac{z_{t+2}}{z_{t+1}} + \left(\frac{\lambda}{R}\right)^3 a\eta \frac{z_{t+1}}{h_t} \theta \frac{z_{t+2}}{z_{t+1}} \theta \frac{z_{t+3}}{z_{t+2}} \\
& + ... + \left(\frac{\lambda}{R}\right)^{T-t} a\eta \frac{z_{t+1}}{h_t} \theta \frac{z_{t+2}}{z_{t+1}} \theta \frac{z_{t+3}}{z_{t+2}} \times \ldots \times \theta \frac{z_{t+T}}{z_{t+T-1}}.
\end{aligned} \tag{9}$$

The first term on the right hand side (RHS) is the marginal product of a date-t tool on output y_{t+1} through its impact on plant quality z_{t+1}. The second term is the marginal impact of the date-t tool on y_{t+2} through its impact on z_{t+1} which increases z_{t+2}. The third term is the marginal impact of the date-t tool on y_{t+3} through its impact on z_{t+1} which increases z_{t+2}

Figure 3: division of gross output, by cash flow



which in turn increases z_{t+3}. The final term is the marginal impact of the date-t tool on y_{t+T} when the plant is shut down at $t+T$.

Multiplying through by h_t, and simplifying, we get

$$wh_t = \frac{\lambda}{R}\eta y_{t+1} + \frac{\lambda^2}{R^2}\eta\theta y_{t+2} + \frac{\lambda^3}{R^3}\eta\theta^2 y_{t+3} + \ldots + \frac{\lambda^T}{R^T}\eta\theta^{T-1} y_T. \tag{10}$$

The present wage bill for engineers equals the present discounted value of a fraction η of tomorrow's output, plus a fraction $\theta\eta$ of output two period's later, plus a fraction $\theta^2\eta$ of output three period later, etc., until the plant is shut down in T periods.

An engineer raises funds by selling new plant (quality 1) at price

$$\begin{aligned} b &= V(1; w, R) \\ &= \frac{1}{R}(a-f) \ + \frac{\lambda}{R^2}[y_{t+1}(1-\eta) - f] + \frac{\lambda^2}{R^3}[y_{t+2}(1-\eta-\eta\theta) - f] \\ &\quad +\ldots + \frac{\lambda^{T-1}}{R^T}[y_{t+T}(1-\eta-\eta\theta-\ldots-\eta\theta^{T-1}) - f]. \end{aligned} \tag{11}$$

b can be thought of as the engineer's borrowing capacity per unit of investment. Figure 4 shows the division of gross output between engineers and plant owner, taking into account the plant owner's optimization. Notice that the plant owner's share of output declines through time. Intuitively, the cumulative contribution of engineers' human capital to plant quality and output increases through time, because more and more of the plant quality depends on cumulative engineers' human capital rather than on the initial plant quality. Through time, the engineers' share increases while the plant owner's share of revenue decreases. This gives us a clue to understand why an engineer borrows largely against near-term revenues.

When fixed cost is subtracted, the plant owner's net share can become negative through time. In Figure 5, the owner's net share becomes negative after period 11 before shutting down in period 20 (see the left hand side), while it becomes negative after 19 periods even if the plant continues forever (the right hand side).

This begs the question: why doesn't the plant owner shut down as soon as his net share turns negative? The reason is that, while the present wage bill equals the present value of engineers' current contribution to future revenues, *past* wage bills are sunk costs for the plant owner. See Figure 6, which is Figure 3 after subtracting fixed cost: as long as the present value of *future*

Figure 4: division of gross output, by contribution

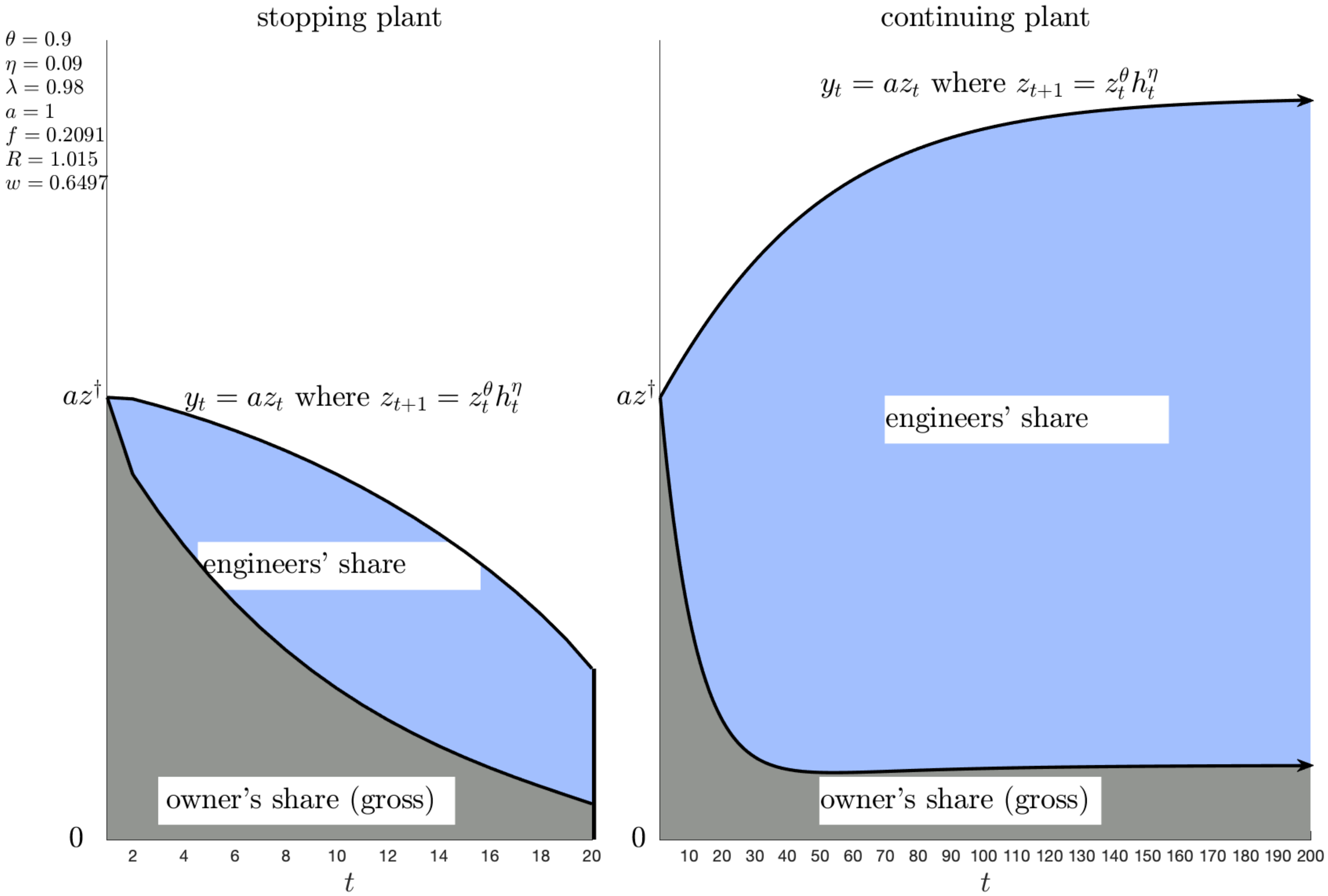
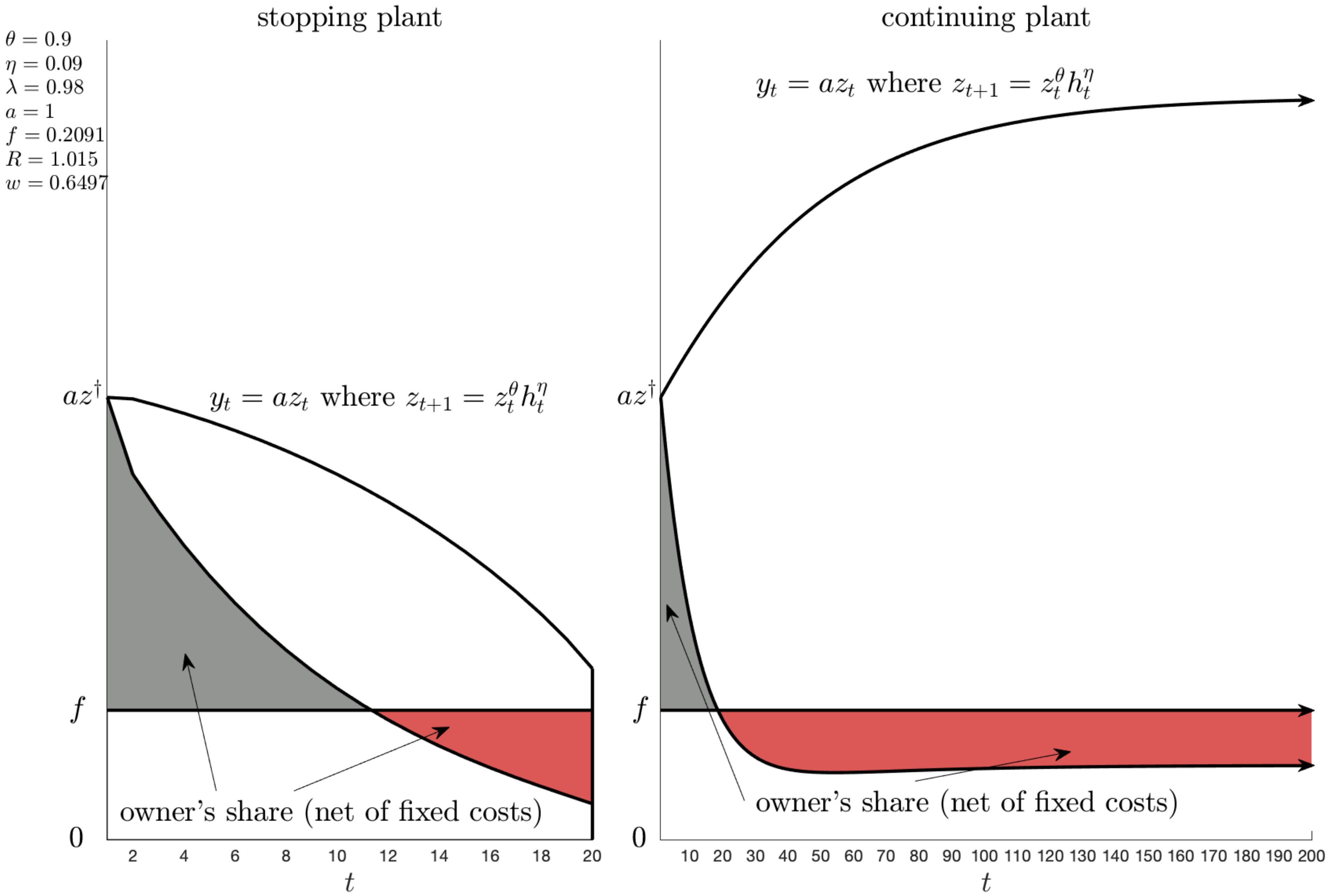


Figure 5: owner's share net of fixed costs, by contribution



net cash flow is positive, the plant owner wants to continue with maintenance and production.

The required own-funds (downpayment) per unit of investment equals

$$x - b.$$

We assume that a new saver – an engineer yesterday who switched to being a saver today – sells her tools (after use today) to an engineer, and teaches him how to use them, at a competitive price $x - b$.

The budget constraint of an agent at date t who has h_t tools and d_t financial assets is

$$c_t + (x - b)i_t + \frac{d_{t+1}}{R} = wh_t + d_t,$$

where h_t is positive if the agent was an engineer yesterday. Here, financial assets consist of returns to plant ownership as well as maturing one-period discount bonds. The discount bond is traded internationally at the interest rate R. If the agent is an engineer today, investment i_t can be positive and her tools tomorrow will be

$$h_{t+1} = \lambda h_t + i_t.$$

The budget constraint can be rewritten as

$$c_t + (x - b)h_{t+1} + \frac{d_{t+1}}{R} = [w + \lambda(x - b)]h_t + d_t = n_t,$$

where n_t is net worth - sum of flow return (wage) and capital value (replacement cost or resale value) of tools, and financial assets.

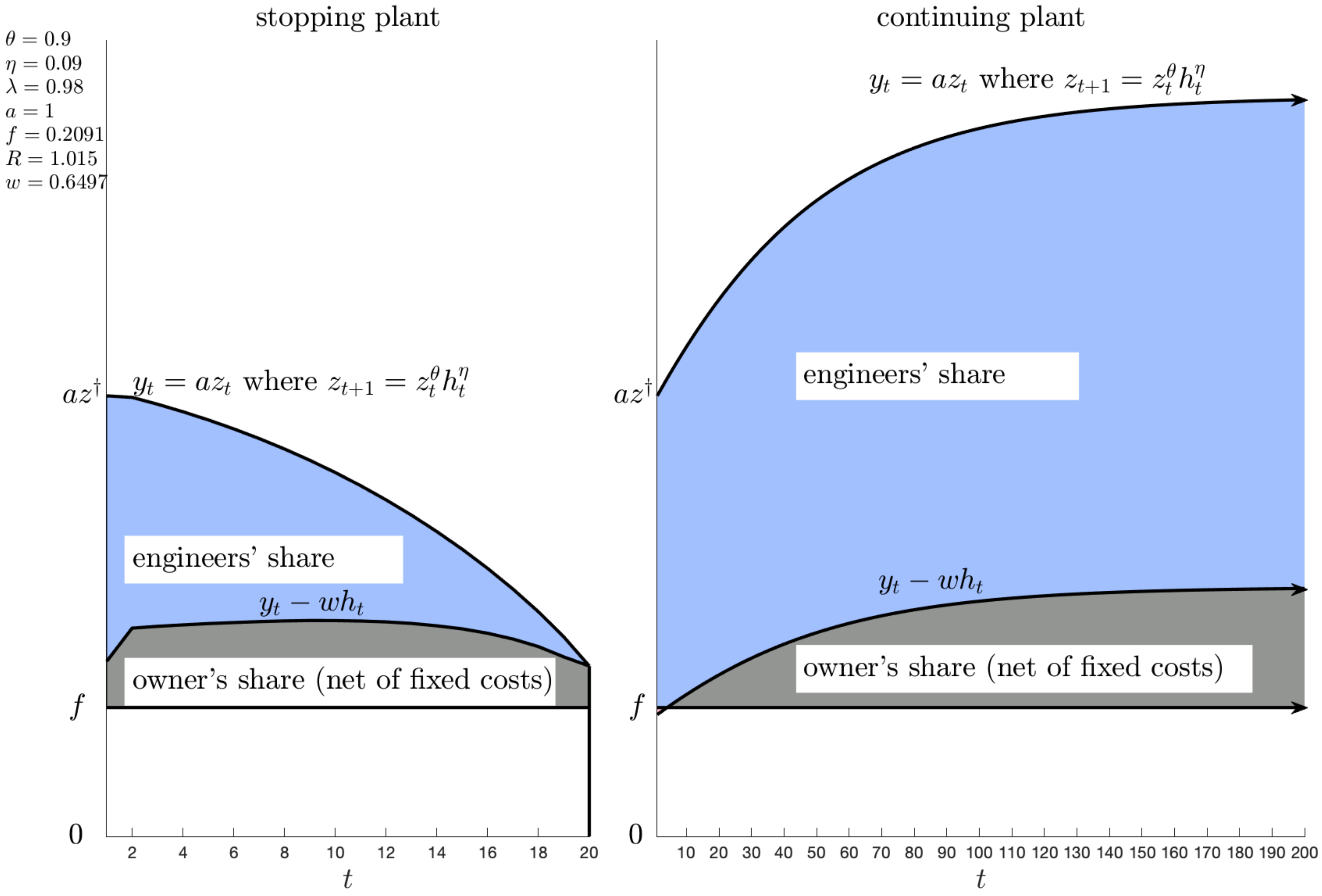
The rate of return for an engineer investing with maximal borrowing is given by

$$R^E = \frac{w + \lambda(x - b)}{x - b}, \tag{12}$$

the ratio of total returns of one tool to the downpayment of investment. (Remember she sells plant to a saver at the time of investment and so does not receive the return on plant). If the return on investment R^E exceeds the interest rate R, the engineer's consumption and investment are

$$\begin{aligned} c_t &= (1 - \beta)n_t, && \text{(13a)} \\ (x - b)h_{t+1} &= \beta n_t. && \text{(13b)} \end{aligned}$$

Figure 6: division of net output, by cashflow



A saver's consumption and asset holdings are

$$\begin{aligned} c_t &= (1-\beta)n_t & \text{(14a)} \\ \frac{d_{t+1}}{R} &= \beta n_t. & \text{(14b)} \end{aligned}$$

Notice that individual consumption only depends on present net worth and not on whether having investment opportunity today. Because marginal utility does not depend upon present investment opportunity, there is no gains from insurance against having the investment opportunity (such as the agent receives a bonus if she has an investment opportunity while paying a premium if not).

A steady state equilibrium of our small open economy is characterized by the wage w and new plant price b, together with the quantity choices of savers/plant owners (c, d, h, z, y), engineers (c, h, i), and foreigners (who have net asset holdings D^*), such that the markets for goods, tools, plant, and bonds all clear.

Aggregating across engineers and savers, we obtain aggregate tool holdings H_{t+1}, financial asset holdings D_{t+1}/R, consumption C_t, and net worth of engineers and savers $\left(N_t^E \text{ and } N_t^S\right)$:

$$\begin{aligned} (x-b)H_{t+1} &= \beta N_t^E & \text{(15a)} \\ \frac{D_{t+1}}{R} &= \beta N_t^S & \text{(15b)} \\ C_t &= (1-\beta)\left(N_t^E + N_t^S\right) & \text{(15c)} \\ N_t^E &= \pi^E \left[w + \lambda(x-b)\right] H_t + \pi^S D_t & \text{(15d)} \\ N_t^S &= \left(1-\pi^E\right) \left[w + \lambda(x-b)\right] H_t + (1-\pi^S) D_t. & \text{(15e)} \end{aligned}$$

Equation $(15a)$ shows the aggregate capital value of tools equals the aggregate net worth of engineers after subtracting their consumption. Equation $(15b)$ says the aggregate financial asset equals the aggregate net worth of savers after consumption. $(15c)$ says aggregate consumption equals a fraction $1-\beta$ of the aggregate net worth of domestic residents. Equation $(15d)$ shows the aggregate net worth of engineers is the sum of net worth of continuing and new engineers. Equation $(15e)$ shows the aggregate net worth of savers is sum of net worth of new and continuing savers.

The economy exhibits endogenous growth G: along a steady state path,

$$\frac{H_{t+1}}{H_t} = \frac{D_{t+1}}{D_t} = \frac{C_{t+1}}{C_t} = G$$

$$\begin{aligned} GN_t^E &= N_{t+1}^E = \pi^E R^E \beta N_t^E + \pi^S R\beta N_t^S \\ GN_t^S &= N_{t+1}^S = (1-\pi^E) R^E \beta N_t^E + (1-\pi^S) R\beta N_t^S. \end{aligned}$$

See Figure 7.

Substituting out $\frac{N_t^s}{N_t^E}$, we find that G solves

$$G = \pi^E R^E \beta + \pi^S R\beta \frac{(1-\pi^E) R^E \beta}{G - (1-\pi^S)\, R\beta}. \tag{16}$$

Notice that the growth rate depends on the rates of return for engineers and savers as well as on the wealth distribution between them.

In general equilibrium, the fraction of plant that is shut down after a finite number of periods depends on the fixed cost as:

Proposition 1: There exists a critical value $f^{critical}$ of the fixed cost such that

P-Region (Pure equilibrium with no stopping; low fixed cost): $f < f^{critical}$

(i) No plant owner stops: $z^{\dagger} < 1$;

(ii) Aggregate ratio of tools-to-plant stays one-to-one: $h = 1$;

(iii) All plant is maintained at initial quality: $z = z^* = 1$.

M-Region (Mixed equilibrium; high fixed cost): $f > f^{critical}$

(i) Plant owners are initially indifferent between stopping after some finite time T and continuing forever: $z^{\dagger} = 1$;

(ii) Aggregate ratio of tools-to-plant is larger than one-to-one for continuing plant: $h > 1$;

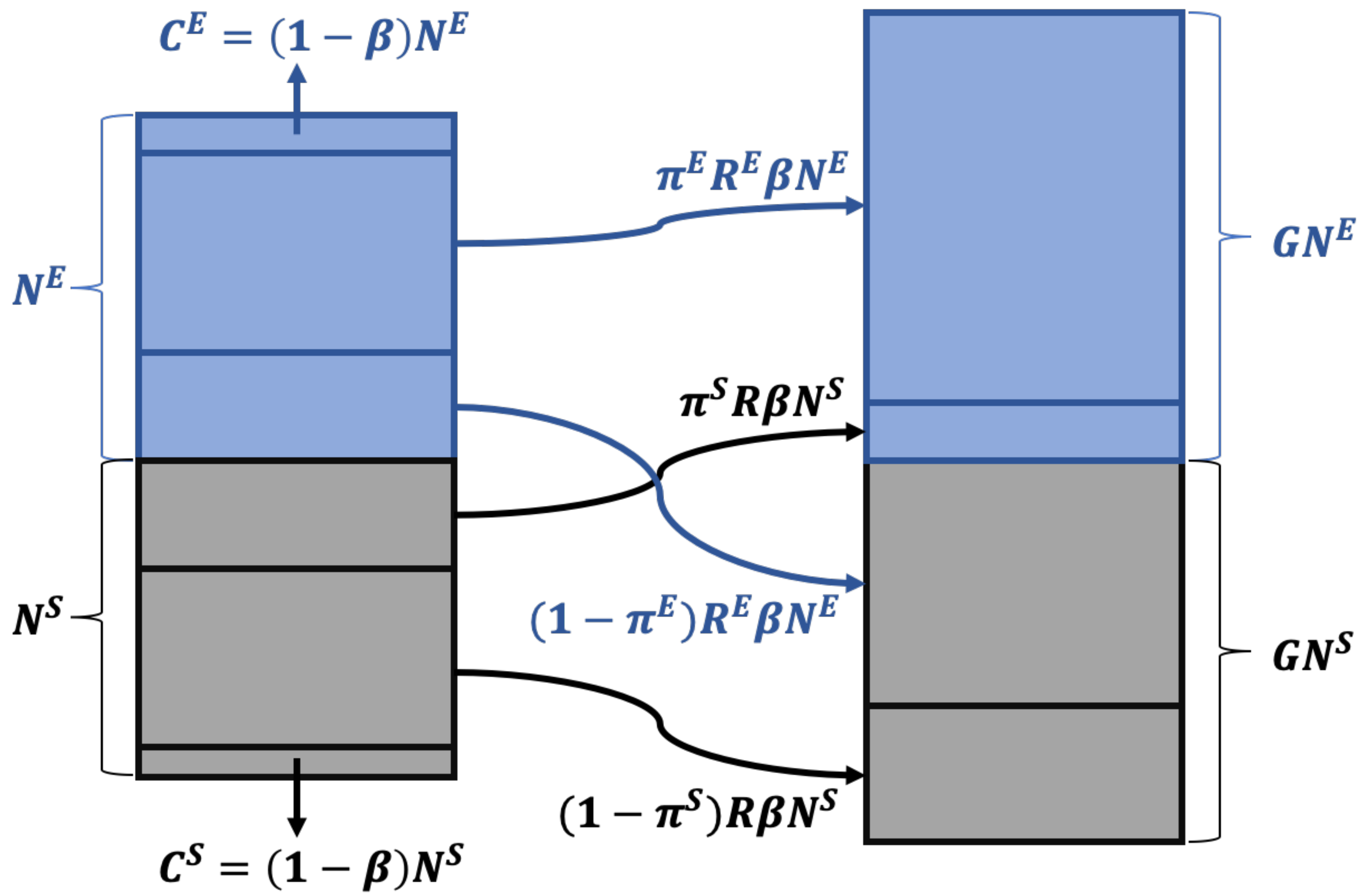
(iii) With decreasing returns to scale, $\theta + \eta < 1$, quality of continuing plant increases over time, converging to some $z^* \in (1, \infty)$;

With constant returns to scale, $\theta + \eta = 1$, continuing plant quality grows at some constant rate $g > 1$;

(iv) Stopping plant decreases in quality over time; Stop occurs just before z falls below f/a.

There is **no equilibrium with all plant stopping** in finite time.

Figure 7: the law of motion of net worth



Remember that in our model, all plant starts with quality $z = 1$. If the fixed cost is smaller than $f^{critical}$, the initial quality ($z = 1$) is higher than the threshold quality $z^{\dagger}$ for shutting down (see **Lemma**), and so no plant owner stops in the steady state equilibrium. Hence the Pure equilibrium with no stopping is the unique steady state. Moreover, investment generates an equal number of plant and tools, which have the same technological depreciation rate $1 - \lambda$. If no plant is stopped, the ratio of tools to plant stays one-to-one. Then because

$$z' = z^{\theta} h^{\eta} = 1, \text{when } z = h = 1,$$

all plant is maintained at initial quality $z = z^* = 1$ until exogenous death of plant through depreciation.

If the fixed cost is larger than the critical value $f^{critical}$, not all plant continues forever in equilibrium. The Mixed equilibrium is the unique steady state. The initial quality is exactly equal to the critical quality $z^{\dagger}$ for shutting down, so that some plant is stopped and some continues forever (modulo depreciation). Because some plant is stopped before exogenous death and the owners of stopping plant do not hire many tools, the aggregate ratio of tools-to-plant is larger than one-to-one for continuing plant, $h > 1$. With abundant supply of tools per plant, continuing plant keeps improving in quality. If the maintenance technology has decreasing returns to scale $\theta + \eta < 1$, the quality of continuing plant converges to a steady state level z^*. If the maintenance technology has constant returns to scale $\theta + \eta = 1$, the quality of continuing plant grows at some rate $g > 1$. Therefore, even though all plant is homogeneous when new, some plant improves in productivity while the other plant fails to maintain productivity and eventually exits. That is, firms evolve heterogeneously in their productivity and output even though they start off homogenous and face no idiosyncratic shocks.[3]

If all plant were to stop in finite time, the market for tools (and engineers) would be in excess supply. The quantity of active plant would be smaller than tools; moreover the demand for tools by plant owners who plan to close eventually is limited. In equilibrium, wage rate of tools and engineers would fall so that at least some plant owners would switch strategy and continue forever. Thus there is no equilibrium with all plant stopping in finite time.

[3] This is different from the standard approach taken by Jovanovic (1981) and Hopenhayn (1992) which emphasize heterogeneity and idiosyncratic shocks. Even if we do not deny the importance of heterogeneity and idiosyncratic shocks, our approach may provide a different perspective to firm dynamics. Perhaps ours is closer to Ericson and Pakes (1995) which emphasize the interaction between heterogeneity, idiosyncratic shock and investment.

Proposition 2P (P-Region):

(i) For an open set of parameter space (in particular for R and λ not too far from unity), there is an equilibrium with no stopping such that an unexpected permanent drop in the interest rate R leads to a lower steady state growth.

(ii) We show numerically that, immediately following the drop in R, all agents (engineers and savers) can be strictly worse off.

In the Appendix, we derive a sufficient (but not necessary) condition for the existence of equilibrium with no stopping:

$$f < a\frac{R(1-\theta-\eta)}{\lambda(1-\theta)}\left[1-\frac{R-\lambda}{R}\left(\frac{R-\theta\lambda}{R}\right)^{\frac{\eta}{1-\theta-\eta}}\right]. \tag{17}$$

In an equilibrium with no stopping, an unexpected permanent drop in the interest rate R leads to a lower steady state growth rate G if

$$f > a\frac{R-\lambda(\theta+\eta)}{R-\lambda\theta} - a\frac{G-\beta R\pi^E}{G-\beta\lambda\pi^E}\frac{\lambda\eta(R-\lambda)}{(R-\lambda\theta)^2}, \tag{18}$$

and $\pi^S = 0$. These inequalities are mutually consistent if R and λ are not too far from unity.[4]

Why can a permanent fall in the real interest rate stifle investment and growth? In the **P-Region** where all plant is continued forever until exogenous death, the wage in (9) becomes simply

$$w = a\frac{\lambda\eta}{R-\lambda\theta}, \tag{19}$$

because $z_t = h_t = 1$ for all t. The wage rate rises significantly with a fall in R because the engineer's marginal product has a long horizon through maintaining plant quality.

[4]If some savers may become engineers with $\pi^S > 0$, then a sufficient condition for the growth rate to fall with an unexpected permanent drop in interest rate is that

$$\lambda(1-\theta)f > (R-\lambda)^2 x + \lambda(1-\theta-\eta)a.$$

This condition guarantees that the rate of return for the engineer to invest is an increasing function of the interest rate. See Appendix B.

The engineer's borrowing capacity per unit of investment (the price of new plant) is

$$b = V(1) = \frac{a - w - f}{R - \lambda}. \tag{20}$$

In particular, under constant returns to scale, $\theta + \eta = 1$, we have

$$b = \frac{a}{R - \lambda\theta} - \frac{f}{R - \lambda}.$$

Notice that the plant owner's share decreases with the horizon by factor $\lambda\theta$, because the owner in effect has to pay to engineers an increasing fraction of future output for their maintenance work. In contrast, fixed cost decreases with horizon by factor λ. Because $\lambda > \lambda\theta$, the fixed cost has a longer horizon than the owner's share of gross output. Thus a fall in R increases the present value of fixed costs proportionately more than the present value of the plant owner's share of gross revenues. Therefore a fall in the interest rate can reduce the initial value of plant to its owner and thereby also reduce the engineer's borrowing capacity per unit of investment.

Overall, a fall in R can stifle investment and growth:

$$\text{gross investment}(H_{t+1}) \downarrow \;\; =$$

$$\text{saving rate}(\beta) \times \frac{\text{net worth of engineers}\left(N_t^E\right) \uparrow}{\text{investment cost}\,(x)\; - \text{borrowing capacity}\,(b) \downarrow}$$

Although engineers' net worth increases with a fall of interest rate, a fall in borrowing capacity may have a larger negative effect on investment and growth. Much of the macro finance literature (including Bernanke and Gertler (1989) and Kiyotaki and Moore (1997)) emphasizes a large effect on net worth in the numerator, especially when investing agents have outstanding debt. Here we emphasize the effect on borrowing capacity in the denominator.

Under these conditions, immediately following the drop in R, all agents (engineers and savers) can be strictly worse off. It is not surprising that savers may be worse off with a lower rate of return on financial assets. The reason engineers may be worse off is that their leveraged rate of return

$$R^E = \frac{w + \lambda(x - b)}{x - b}$$

can fall with lower borrowing capacity b. The Appendix derives the welfare of engineers and savers immediately after an unanticipated and permanent fall in real interest rate, taking into account the stochastic arrival of future investment opportunities.

For the mixed equilibrium in **M-Region,** we have limited analytical results. Thus we derive our findings mostly by numerical simulations.

Proposition 2M (M-Region):

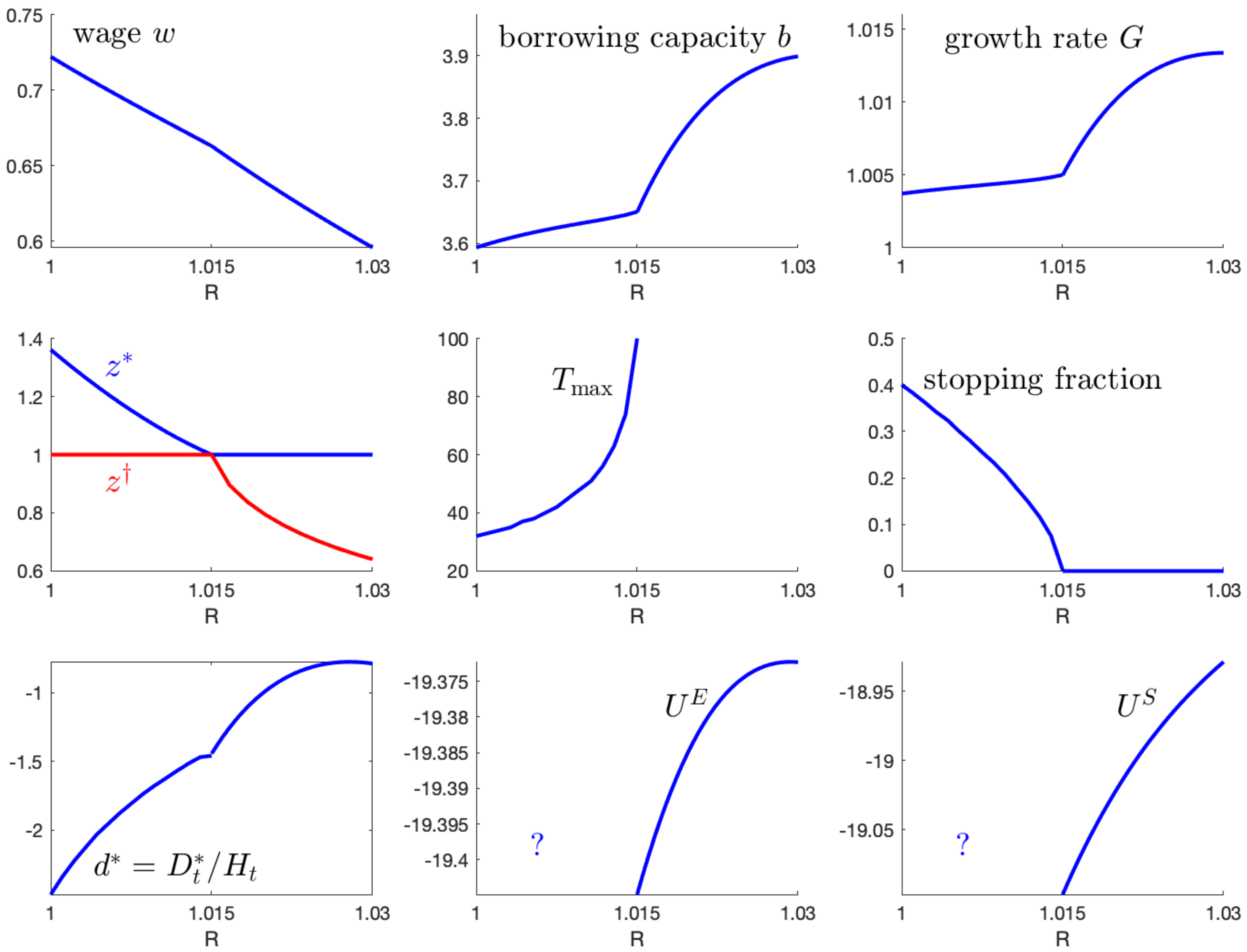
In a mixed equilibrium, we demonstrate numerically that an unexpected permanent drop in the interest rate R can lead to a lower steady state growth rate G.

For Figure 8, we numerically illustrate how nine endogenous variables depend on the real interest rate for the range between 1 and 1.03 (between 0 and 3% net real interest rate) in steady state equilibrium. We choose the parameters so that the threshold plant quality for continuing or stopping exactly equals initial quality $\left(z^{\dagger}=1\right)$ when $R=1.015$. Thus the economy is in the pure no-stopping region (P-Region) for $R\in[1.015,1.03]$ and in the mixed equilibrium region (M-Region) for $R\in[1,1.015)$.

θ	share of past productivity in maintenance	0.9
η	share of engineer in maintenance	0.09
λ	one minus depreciation rate	0.98
a	productivity	1
f	fixed cost	0.2091
x	investment cost per plant	6.127
β	utility discount factor	0.92
π^E	probability of staying to be engineer	0.7
π^S	probability of saver to become engineer	0.1

In the top-left panel of Figure 8, observe the wage rate is a decreasing function of the interest rate because the wage equals the present value of an engineer's contribution to future output through maintenance work, which has a long horizon. In the top-middle panel, the engineer's borrowing capacity increases with the interest rate because the plant owner's share has a shorter time horizon than fixed cost. Notice that this effect is smaller in the M-Region, with endogenous adjustment of the fraction of stopping plant

Figure 8: real interest and the equilibrium



(extensive margin) and of the stopping time (intensive margin). In the top-right panel, we observe that economic growth rate is an increasing function of interest rate, despite the sensitivity being weaker in M-Region.

In the middle-left panel, we show the plant quality in steady state equals $z^* = 1$ in the P-Region and is a decreasing function of R in the M-Region. The threshold plant quality for continuing and stopping $z^\dagger$ equals 1 (initial quality) in the M-Region (consistent with plant owners being indifferent between stopping and continuing) and is a decreasing function of R in the P-Region (consistent with plant owners gaining more indirectly from the lower wage rate than hurting directly from the higher interest rate). In the middle-middle panel, the number of periods before stopping ($T^{\max}$) is finite and is an increasing function of R for those who choose to stop in the M-Region. In the P-Region, no-one stops and $T^{\max} = \infty$. In the middle-right panel, the fraction of stopping plant is zero in the P-Region and is a decreasing function of R in the M-Region.

In the bottom-left panel, we see that the net financial asset holdings of foreigners is negative, i.e., domestic residents lend to foreigners in net terms. Despite the foreign interest rate being lower than the subjective interest rate $(R < 1/\beta)$, the domestic economy has a shortage of means of saving due to the financial friction and thus holds positive foreign bonds. With a lower interest rate, the financing constraint is tighter and domestic resident have a yet larger position in foreign bonds. In the bottom-middle panel, we observe that the welfare of a representative engineer (who holds the average net worth of engineers) is an increasing function of R in the P-Region, i.e., welfare is lower with lower R. In our example, when R falls from 1.03 to 1.015 unexpectedly and permanently, the welfare of a representative engineer falls by the equivalent of a 0.12% permanent fall in consumption. We do not have comparable results for the M-Region, because one cannot simply define a representative engineer. In the bottom-right panel, we see that the welfare of savers is an increasing function of R in P-Region. The effect on savers is larger: when R falls from 1.03 to 1.015 unexpectedly and permanently, their welfare falls by the equivalent of a 1.2% permanent fall in consumption.

3 Extensions

Heterogeneity across plants about initial z and/or idiosyncratic shocks to subsequent $z', z'', \ldots$

Heterogeneity across engineers concerning investment cost x
Choice of technique by engineers
Land model
Bargaining model

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5 Appendix

5.1 Appendix A:

In text, we assume output is proportional to plant quality. More generally suppose that gross output $\widehat{y}$ depends upon plant quality $\widehat{z}$ and unskilled labor $\widehat{h}$ as

$$\widehat{y} = \widehat{a}\widehat{z}^{\alpha_1}\widehat{h}^{\alpha_2}, \text{ where } \alpha_1 + \alpha_2 \leq 1.$$

Suppose there is a competitive labor market for unskilled worker at wage rate $\widehat{w}$. Then we can define the gross profit of plant owner as

$$\begin{aligned} y &= \underset{\widetilde{h}}{Max}\left(\widehat{a}\widehat{z}^{\alpha_1}\widehat{h}^{\alpha_2} - \widehat{w}\widehat{h}\right) \\ &= az \end{aligned} \tag{21}$$

where

$$\begin{aligned} z &= \widehat{z}^{\frac{\alpha_1}{1-\alpha_2}}, \\ a &= (1-\alpha_2)\left(\frac{\alpha_2}{\widehat{w}}\right)^{\frac{\alpha_2}{1-\alpha_2}}\widehat{a}. \end{aligned}$$

If supply of unskilled labor is perfectly elastic, we can treat a as exogenous - this is the case of our model. (Otherwise, we need to take into account the general equilibrium effect on a through $\widehat{w}$.)

If plant quality depends upon initial plant quality and human capital of engineer h as

$$\widehat{z}' = \widehat{z}^{\theta}h^{\widehat{\eta}}, \text{ where } \theta + \widehat{\eta} \leq 1.$$

we can rewrite this as

$$z' = z^{\theta}h^{\eta}, \text{ where } \eta = \frac{\alpha_1}{1-\alpha_2}\widehat{\eta}. \tag{22}$$

Thus we obtain the formulation in the text as $(21, 22)$.

5.2 Appendix B

5.2.1 Individual Choice

Individual agent takes wage, plant price and interest rate $\{w, b, R\}$ as given. Engineer chooses consumption, gross investment on tool and financial assets

(c, h', d') as a function of the net worth n to maximize

$$V^E(n; w, b, R) = \underset{c,h',d'}{Max} \left\{ \ln c + \beta \left[\pi^E V^E(n'; w, b, R) + (1 - \pi^E) V^S(n'; w, b, R) \right] \right\} \tag{23}$$

subject to the budget constraint

$$c + (x - b)\, h' + \frac{d'}{R} = n, \text{ and } n' = [w + \lambda\,(x - b)]\, h' + d'.$$

Defining the leveraged rate of return on investment as

$$R^E = \frac{w + \lambda\,(x - b)}{x - b},$$

the first order conditions are

$$\begin{aligned} \frac{1}{c} &\geq R^E \frac{\beta}{c'}, \text{ where } = \text{ holds if } h' > 0, \\ \frac{1}{c} &\geq R \frac{\beta}{c'}, \text{ where } = \text{ holds if } d' > 0. \end{aligned}$$

Thus if $R^E > R$, we have $d' = 0$, $(13a, 13b)$ and

$$n' = R^E \beta n. \tag{24}$$

Saver chooses consumption and financial assets (c, d') as a function of the net worth n to maximize

$$V^S(n; w, b, R) = \underset{c,h',d'}{Max} \left\{ \ln c + \beta \left[\pi^S V^E(n'; w, b, R) + (1 - \pi^S) V^S(n'; w, b, R) \right] \right\} \tag{25}$$

subject to the budget constraint

$$c + \frac{d'}{R} = n, \text{ and } n' = d'.$$

Using the first order condition

$$\frac{1}{c} = R \frac{\beta}{c'},$$

we get $(14a, 14b)$ and

$$n' = R\beta n. \tag{26}$$

From these, we conjecture that the value functions of engineer and saver are given by

$$\begin{aligned} V^E(n;w,b,R) &= \nu^E(w,b,R) + \frac{1}{1-\beta}\ln n, & (27a)\\ V^S(n;w,b,R) &= \nu^S(w,b,R) + \frac{1}{1-\beta}\ln n. & (27b)\end{aligned}$$

From $(13b, 24, 14b, 26)$, conjecture is verified if and only if

$$\begin{aligned} \nu^E(w,b,R) &= \beta\pi^E\nu^E(w,b,R) + \beta(1-\pi^E)\nu^S(w,b,R) + \frac{\beta}{1-\beta}R^E(w,b,R) + \ln(1-\beta),\\ \nu^E(w,b,R) &= \beta\pi^E\nu^E(w,b,R) + \beta(1-\pi^E)\nu^S(w,b,R) + \frac{\beta}{1-\beta}R + \ln(1-\beta),\end{aligned}$$

when there is no change of (w,b,R) in future. Then we get

$$\nu^E(w,b,R) = \beta\frac{(1-\beta+\beta\pi^S)\ln\left(R^E(w,b,R)\right) + \beta(1-\pi^E)\ln R}{(1-\beta)^2(1+\beta\pi^S-\beta\pi^E)} + \frac{\ln(1-\beta)}{1-\beta}, \tag{28}$$

$$\nu^S(w,b,R) = \beta\frac{\beta\pi^S\ln\left(R^E(w,b,R)\right) + (1-\beta\pi^E)\ln R}{(1-\beta)^2(1+\beta\pi^S-\beta\pi^E)} + \frac{\ln(1-\beta)}{1-\beta}. \tag{29}$$

Plant owner/saver's choice is given by their value function (5) in the text. The first order condition for those who choose to continue tonight is

$$\begin{aligned} w &\geq \eta z^\theta h^{\eta-1}\lambda V'(z';w,R), \text{ where } = \text{holds if } h>0, & (30)\\ V'(z;w,R) &= \frac{1}{R}[a + \theta z^{\theta-1}h^\eta\lambda V'(z';w,R)]. & (31)\end{aligned}$$

From these, if $h_t, h_{t+1}, ..., h_{t+T} > 0$ and $h_{t+T+1} = h_{t+T+2} =, ... = 0$, we have

$$\begin{aligned} w &= \frac{\lambda}{R}\left[\eta\frac{z_{t+1}}{h_t}a + \eta\frac{z_{t+1}}{h_t}\theta\frac{z_{t+2}}{z_{t+1}}\lambda V'(z';w,R)\right]\\ &= \frac{\lambda}{R}a\eta\frac{z_{t+1}}{h_t} + \left(\frac{\lambda}{R}\right)^2 a\eta\frac{z_{t+1}}{h_t}\theta\frac{z_{t+2}}{z_{t+1}} + \left(\frac{\lambda}{R}\right)^3 a\eta\frac{z_{t+1}}{h_t}\theta\frac{z_{t+2}}{z_{t+1}}\theta\frac{z_{t+3}}{z_{t+2}}\\ &\quad + ... + \left(\frac{\lambda}{R}\right)^{T-t} a\eta\frac{z_{t+1}}{h_t}\theta\frac{z_{t+2}}{z_{t+1}}\theta\frac{z_{t+3}}{z_{t+2}} \times ... \times \theta\frac{z_T}{z_{T-1}}.\end{aligned}$$

This is (9) in the text. Multiplying through by h_t, and simplifying, we get (10) in the text. Then we get

$$\begin{aligned}
& V(z; w, R) \\
= \; & \frac{1}{R}(y_t - wh_t - f) \; + \frac{\lambda}{R^2}(y_{t+1} - wh_{t+1} - f) + \frac{\lambda^2}{R^3}(y_{t+2} - wh_{t+2} - f) \\
& + ... \frac{\lambda^{T-2}}{R^{T-1}}(y_{t+T-1} - wh_{t+T-1} - f) + \frac{\lambda^{T-1}}{R^T}(y_{t+T} - f) \\
= \; & \frac{1}{R}(y_t - f) \; + \frac{\lambda}{R^2}[y_{t+1}(1-\eta) - f] + \frac{\lambda^2}{R^3}[y_{t+2}(1-\eta-\eta\theta) - f] \\
& + ... + \frac{\lambda^{T-1}}{R^T}[y_{t+T}(1-\eta-\eta\theta - ... - \eta\theta^{T-1}) - f].
\end{aligned}$$

This implies (11) in the text.

If $h_t, h_{t+1} > 0$, we can use $(30, 31)$ to derive an alternative first order condition as

$$\begin{aligned}
w &= \frac{\lambda}{R}\eta\frac{z_{t+1}}{h_t}a + \frac{\lambda}{R}w\eta\frac{z_{t+1}}{h_t}\frac{\theta\frac{z_{t+2}}{z_{t+1}}}{\eta\frac{z_{t+2}}{h_{t+1}}} \\
&= \frac{\lambda}{R}\eta\frac{z_{t+1}}{h_t}a + \frac{\lambda}{R}\theta\frac{h_{t+1}}{h_t}w \qquad (32)
\end{aligned}$$

Note that the second term on RHS equals the discounted wage rate times the marginal rate of substitution between h_t and h_{t+1} to keep z_{+2} constant. Thus equation (32) says the marginal cost of increasing h_t by one unit equals the discounted value of marginal benefit - sum of additional output through z_{t+1} and saving of wage bill keeping z_{+2} constant.

In case of constant returns to scale maintenance technology $\theta + \eta = 1$, we conjecture

$$S^\infty(z; w, r) = aA^\infty z - \frac{f}{R - \lambda},$$

$$\begin{aligned}
S^T(z; w, r) &= aA^T z - \Lambda^T f, \text{ where} \\
\Lambda^T &= f\left(\frac{1}{R} + \frac{\lambda}{R^2} + ... + \frac{\lambda^{T-1}}{R^T}\right) = f\frac{1 - \frac{\lambda^T}{R^T}}{R - \lambda}. \qquad (33)
\end{aligned}$$

For plant which continues forever, we conjecture

$$\frac{h_{t+1}}{h_t} = \frac{z_{t+1}}{z_t} = \left(\frac{h_t}{z_t}\right)^{1-\theta} = g > 1.$$

Then from (32), we get

$$w=\frac{\frac{\lambda}{R}\eta\frac{z_{t+1}}{h_t}a}{1-\frac{\lambda}{R}\theta g}=\frac{\lambda\left(1-\theta\right)a}{R-\lambda\theta g}g^{-\frac{\theta}{1-\theta}}. \tag{34}$$

Then from (5), we learn Bellman equation for continuing plant holds if and only if

$$A^{\infty}=\frac{1}{R-\lambda\theta g}. \tag{35}$$

For stopping plant in finite time, (30) implies

$$w=(1-\theta)\left(\frac{z^T}{h^T}\right)^{\theta}\lambda aA^{T-1}, \tag{36}$$

where z^T and h^T are the quality and tools of plant which closes in T periods. Then from (5), we learn Bellman equation for continuing plant holds if and only if

$$\begin{aligned} A^T &= \frac{1}{R}\left[1+\lambda\theta aA^{T-1}\left(\frac{1-\theta}{w}\lambda aA^{T-1}\right)^{\frac{1-\theta}{\theta}}\right] \\ &= \frac{1}{R}\left[1+\lambda\theta g\left(R-\lambda\theta g\right)^{\frac{1-\theta}{\theta}}\left(A^{T-1}\right)^{\frac{1}{\theta}}\right], \end{aligned} \tag{37}$$

using (34). Here A^1 is given by $A^1=\frac{1}{R}$.

When maintenance technology is decreasing returns to scale $\theta+\eta<1$, we conjecture quality of plant which continues forever will converge to the steady state quality

$$z=z^*.$$

Thus the amount of tools employed converges to

$$h=h^*=\left(z^*\right)^{\frac{1-\theta}{\eta}}.$$

We also conjecture

$$S^{\infty}(z;w,R)=az^*U^{\infty}\left(\frac{z}{z^*};R\right)-\frac{f}{R-\lambda},$$

$$S^T(z;w,R)=az^*U^T\left(\frac{z}{z^*};R\right)-\Lambda^T f.$$

From (32) for plant to continue forever in steady state, we get

$$w = \frac{\lambda \eta a}{R - \lambda \theta} (z^*)^{-\frac{1-\eta-\theta}{\eta}}. \tag{38}$$

Define $\widetilde{z} = \frac{z}{z^*}$. Using the relationship $h = \left(\frac{z'}{z^\theta}\right)^{\frac{1}{\eta}}$, we get

$$\frac{wh}{az^*} = \frac{\lambda \eta}{R - \lambda \theta} \left(\frac{\widetilde{z}'}{\widetilde{z}^\theta}\right)^{\frac{1}{\eta}}.$$

Thus the guess is verified if $U^\infty(\widetilde{z})$ and $U^T(\widetilde{z})$ solve

$$U^\infty(\widetilde{z}; R) = \frac{1}{R} \underset{\widetilde{z}'}{Max} \left[\widetilde{z} - \frac{\lambda \eta}{R - \lambda \theta} \left(\frac{\widetilde{z}'}{\widetilde{z}^\theta}\right)^{\frac{1}{\eta}} + \lambda U^\infty(\widetilde{z}'; R)\right], \tag{39}$$

$$U^T(\widetilde{z}; R) = \frac{1}{R} \underset{\widetilde{z}'}{Max} \left[\widetilde{z} - \frac{\lambda \eta}{R - \lambda \theta} \left(\frac{\widetilde{z}'}{\widetilde{z}^\theta}\right)^{\frac{1}{\eta}} + \lambda U^{T-1}(\widetilde{z}'; R)\right], \tag{40}$$

where $U^1(\widetilde{z}; R) = \frac{1}{R}\widetilde{z}$.

5.2.2 Market Clearing

In order to describe aggregate economy, let $K_t(\tau)$ be aggregate number of age-τ plant which continues forever at date t. Suppose some owners choose to stop new plant in T periods. Let $L_t^{T-\tau}(\tau)$ be aggregate number of age-τ plant which stops in $T - \tau$ periods at date t. Then we have the transition

$$\begin{aligned} K_t(\tau) &= \lambda K_{t-1}(\tau - 1) \\ L_t^{T-\tau}(\tau) &= \lambda L_{t-1}^{T-\tau+1}(\tau - 1), \text{ for } \tau = 1, 2, ..., T - 1. \end{aligned} \tag{41}$$

We also have

$$I_t = K_{t+1}(0) + L_{t+1}^T(0) \tag{42}$$

where I_t is aggregate investment at date t.

We also know

$$\begin{aligned} b &= S^\infty(1; w, R) = S^T(1; w, R) \text{ in M-Region} \\ b &= S^\infty(1; w, R) \text{ and } L_t^T(0) = 0 \text{ in P-Region.} \end{aligned} \tag{43}$$

Let $z_t^{T-\tau}(\tau)$ be the quality of age-τ plant which stops in $T-\tau$ periods at date t. Let $h_t^{T-\tau}(\tau)$ be tools employed by one unit of age-τ plant to stop in $T-\tau$ period. Then aggregate output and demand for tools (and engineers) are given by

$$Y_t = \sum_{\tau=0}^{\infty} \left[a z_t^{\infty}(\tau) - f \right] K_t(\tau) + \sum_{\tau=0}^{T-1} \left[a z_t^{T-\tau}(\tau) - f \right] L_t^{T-\tau}(\tau) \tag{44}$$

$$H_t = \sum_{\tau=0}^{\infty} h_t^{\infty}(\tau) K_t(\tau) + \sum_{\tau=0}^{T-1} h_t^{T-\tau}(\tau) L_t^{T-\tau}(\tau) \tag{45}$$

Aggregate domestic assets at the beginning of period equals sum of gross profit and value of plant from the last period minus net foreign debt as

$$\begin{aligned} D_t &= Y_t - wH_t - D_t^* \\ &\quad + \sum_{\tau=1}^{\infty} V\left(z\left(\tau\right)\right) K_t(\tau) + \sum_{\tau=1}^{T} S^{T-\tau}\left(z_t^{T-\tau}(\tau)\right) L_t^{T-\tau}(\tau). \end{aligned} \tag{46}$$

Goods market equilibrium is given by

$$C_t + xI_t + D_t^* - \frac{D_{t+1}^*}{R} = Y_t. \tag{47}$$

Output equals consumption, investment and net export (which equals net debt repayment to foreigners). One of the market clearing conditions for output, tools and financial asset is not independent by Walras Law.

5.2.3 Pure Equilibrium with No Stopping

When no plant owner stops his plant, the total number of continuing plant equals tools as

$$\sum_{\tau=0}^{\infty} K_t(\tau) = H_t,$$

and the ratio of tools to plant remains at the initial ratio

$$h_t^{\infty}(\tau) = 1, \text{ for all } \tau \text{ and } t.$$

The plant quality remains at the initial level as

$$z_t^{\infty}(\tau) = \left[z_{t-1}^{\infty}(\tau-1)\right]^{\theta} \left[h_{t-1}^{\infty}(\tau-1)\right]^{\eta} = 1, \text{ for all } \tau \text{ and } t.$$

Thus $g = 1$ with constant returns to scale maintenance technology and $z^* = 1$ with decreasing returns to scale maintenance technology, and

$$w = \frac{\lambda\eta a}{R - \lambda\theta} = w(R), \tag{48a}$$

$$b = \frac{a - w - f}{R - \lambda} = \frac{1}{R - \lambda}\left[a\frac{R - \lambda(\theta + \eta)}{R - \lambda\theta} - f\right] = b(R). \tag{48b}$$

In order to show that non-stopping is optimal strategy for the plant owner, we need to check

$$b(R) > \underset{T}{Max} S^T(1; w(R), R) = \underset{T}{Max}[aU^T(1; w(R), R) - \Lambda^T f], \tag{49}$$

for any finite T, where $U^T(1; R)$ is given by (40) with decreasing returns to scale and equals A^T with the constant returns to scale maintenance technology.

Then from (44, 46), we have

$$\begin{aligned} Y_t &= (a - f)H_t, \\ D_t &= (a - w - f)H_t + b\lambda H_t - D_t^* \end{aligned} \tag{50}$$

We also get

$$C_t = (1 - \beta)[(w + \lambda b)H_t + D_t]. \tag{51}$$

From $(15a, 47)$, we get the transition as

$$(x - b)H_{t+1} = \beta\left\{\pi^E(w + \lambda b)H_t + \pi^S D_t\right\}, \tag{52a}$$

$$\frac{D_{t+1}^*}{R} = (a - f)H_t - C_t - x(H_{t+1} - \lambda H_t) + D_t^*. \tag{52b}$$

(w, b) is a function of R and the other parameters, and (D_t, C_t) is a function of (H_t, D_t^*) and R (through w and b). Then the perfect foresight equilibrium (aside from a unanticipated permanent shock on R) is characterized recursively by $\left(H_{t+1}, D_{t+1}^*\right)$ as function of $\left(H_t, D_t^*, R\right)$.

In the steady state, we can use (16) to find steady state growth rate equilibrium where

$$R^E = \frac{w(R) + \lambda[x - b(R)]}{x - b(R)}.$$

5.2.4 Mixed Equilibrium

For mixed equilibrium, we only describe the steady state equilibrium.

Mixed equilibrium under constant returns to scale maintenance technology From $(34, 35)$, we have

$$\begin{aligned} w &= \frac{\lambda(1-\theta)a}{R-\lambda\theta g} g^{-\frac{\theta}{1-\theta}} = w(g;R) \\ b &= \frac{a}{R-\lambda\theta g} - \frac{f}{R-\lambda} = b(g;R). \end{aligned}$$

Find $\left\{A^1, A^2, A^3, ...A^T\right\}$ to solve (37) with $A^1 = \frac{1}{R}$ as a function of $(g;R)$. Find g to solve the indifference condition:

$$b(g;R) = \underset{\text{finite } T}{Max} \left[aA^T(g;R) - \Lambda^T f\right]. \tag{53}$$

Equilibrium stopping time is arg $Max \left[aA^T(g;R) - \Lambda^T f\right]$ for this equilibrium g.

Then we can find the steady state growth rate from (16) by using

$$R^E = \frac{w(g;R) + \lambda[x - b(g;R)]}{x - b(g;R)}.$$

For those continuing plant forever, because $z^\infty(0) = 1$, we get $z^\infty(\tau) = g^\tau$ and

$$h^\infty(\tau) = \left[\frac{z^\infty(\tau+1)}{(z^\infty(\tau))^\theta}\right]^{\frac{1}{1-\theta}} = g^{\frac{1}{1-\theta}+\tau}.$$

For those stopping in T period, we get from the first order condition (36) as

$$\frac{h^{T-\tau}(\tau)}{z^{T-\tau}(\tau)} = \left[\frac{(1-\theta)\lambda}{w/a} A^{T-\tau-1}\right]^{\frac{1}{\theta}} = \left(\frac{A^{T-\tau-1}}{A^\infty}\right)^{\frac{1}{\theta}} g^{\frac{1}{1-\theta}}, \tag{54}$$

for $\tau = 0, 1, 2, ..., T-2$. Because $z^T(0) = 1$, we get $\{h^{T-\tau}(\tau), z^{T-\tau-1}(\tau+1)\}$ which satisfies (54) and

$$z^{T-\tau-1}(\tau+1) = \left(\frac{A^{T-\tau-1}}{A^\infty}\right)^{\frac{1-\theta}{\theta}} g z^{T-\tau}(\tau),$$

for $\tau = 0, 1, 2, ..., T-2$.

Mixed equilibrium under decreasing returns to scale maintenance technology With decreasing returns, from (38), we get

$$w = \frac{\lambda\eta a}{R-\lambda\theta}(z^*)^{-\frac{1-\theta-\eta}{\eta}} = w(z^*;R).$$

For plant to continue forever, we have from (39) as

$$\begin{aligned} U^\infty(\widetilde{z}) &= \frac{1}{R}\underset{\widetilde{z}'}{Max}\left[\widetilde{z} - \frac{\lambda\eta}{R-\lambda\theta}\left(\frac{\widetilde{z}'}{\widetilde{z}^\theta}\right)^{\frac{1}{\eta}} + \lambda U^\infty(\widetilde{z}')\right] \\ \widetilde{z}' &= \arg\underset{\widetilde{z}'}{Max}\left[\widetilde{z} - \frac{\lambda\eta}{R-\lambda\theta}\left(\frac{\widetilde{z}'}{\widetilde{z}^\theta}\right)^{\frac{1}{\eta}} + \lambda U^\infty(\widetilde{z}')\right] \equiv \varphi^\infty(\widetilde{z}) \end{aligned}$$

Let $\widetilde{z}^\infty(\tau)$ and $\widetilde{h}^\infty(\tau)$ be quality and tools of age-τ plant which continues forever relative to the steady state. The we have

$$\begin{aligned} \widetilde{z}^\infty(\tau) &= (\varphi^\infty)^\tau\left(\widetilde{z}^\infty(0)\right) = (\varphi^\infty)^\tau\left(\frac{1}{z^*}\right) \\ \widetilde{h}^\infty(\tau) &= \left[\frac{\widetilde{z}^\infty(\tau+1)}{(\widetilde{z}^\infty(\tau))^\theta}\right]^{\frac{1}{\eta}}. \end{aligned}$$

For plant to stop in T periods, we have from (40) as

$$\begin{aligned} U^T(\widetilde{z}) &= \frac{1}{R}\underset{\widetilde{z}'}{Max}\left[\widetilde{z} - \frac{\lambda\eta}{R-\lambda\theta}\left(\frac{\widetilde{z}'}{\widetilde{z}^\theta}\right)^{\frac{1}{\eta}} + \lambda U^{T-1}(\widetilde{z}')\right] \\ \widetilde{z}' &= \arg\underset{\widetilde{z}'}{Max}\left[\widetilde{z} - \frac{\lambda\eta}{R-\lambda\theta}\left(\frac{\widetilde{z}'}{\widetilde{z}^\theta}\right)^{\frac{1}{\eta}} + \lambda U^{T-1}(\widetilde{z}')\right] \equiv \varphi^T(\widetilde{z})\ , \end{aligned}$$

where $U^1(\widetilde{z}) = \frac{1}{R}\widetilde{z}$. Let $\widetilde{z}^{T-\tau}(\tau)$ and $\widetilde{h}^{T-\tau}(\tau)$ be quality and tools of age-τ plant which stops in $T-\tau$ periods relative to the steady state. Then we have

$$\begin{aligned} \widetilde{z}^{T-\tau}(\tau) &= \varphi^T\cdot\varphi^{T-1}\cdot\ldots\cdot\varphi^{T-\tau+1}\left(\frac{1}{z^*}\right) \\ \widetilde{h}^{T-\tau}(\tau) &= \left[\frac{\widetilde{z}^{T-\tau-1}(\tau+1)}{(\widetilde{z}^{T-\tau}(\tau))^\theta}\right]^{\frac{1}{\eta}}. \end{aligned}$$

We then find z^* to satisfy the indifference condition

$$az^*U^{\infty}\left(\frac{1}{z^*}\right)-\frac{f}{R-\lambda} = \underset{\text{finite } T}{Max}\left[az^*U^T\left(\frac{1}{z^*}\right)-\Lambda^T f.\right] \tag{55a}$$

$$= b\left(z^*;R\right) \tag{55b}$$

This common value under equilibrium z^* is the engineer's borrowing capacity. Equilibrium stopping time equals arg $Max\left[az^*U^T\left(\frac{1}{z^*}\right)-\Lambda^T f,\right]$.

We can find the steady state growth rate from (16) with

$$R^E=\frac{w(z^*;R)+\lambda\left[x-b\left(z^*;R\right)\right]}{x-b\left(z^*;R\right)}=R^E\left(z^*;R\right).$$

5.2.5 Tool and goods market equilibrium in mixed equilibrium

In the steady state, we observe

$$G=\frac{H_{t+1}}{H_t}=\frac{K_{t+1}(\tau)}{K_t\left(\tau\right)}=\frac{L_{t+1}^{T-\tau}(\tau)}{L_t^{T-\tau}(\tau)}.$$

For both constant returns to scale and decreasing returns to scale maintenance technology, we have aggregate output under mixed equilibrium as (44). Using (41), we have

$$Y_t=\sum_{\tau=0}^{\infty}[az^{\infty}(\tau)-f]\frac{\lambda^{\tau}}{G^{\tau}}K_t\left(0\right)+\sum_{\tau=0}^{T-1}[az^{T-\tau}(\tau)-f]\frac{\lambda^{\tau}}{G^{\tau}}L_t^T\left(0\right).$$

Similarly aggregate demand for tools (45) becomes

$$H_t=\sum_{\tau=0}^{\infty}h^{\infty}(\tau)\frac{\lambda^{\tau}}{G^{\tau}}K_t\left(0\right)+\sum_{\tau=0}^{T-1}h^{T-\tau}(\tau)\frac{\lambda^{\tau}}{G^{\tau}}L_t^T\left(0\right). \tag{56}$$

Because $I_t=(G-\lambda)H_t=K_{t+1}\left(0\right)+L_{t+1}^T\left(0\right)$, dividing (56) by H_t, we get in the steady state as

$$1=\sum_{\tau=0}^{\infty}h^{\infty}(\tau)\frac{\lambda^{\tau}}{G^{\tau+1}}(G-\lambda)i^k+\sum_{\tau=0}^{T-1}h^{T-\tau}(\tau)\frac{\lambda^{\tau}}{G^{\tau+1}}(G-\lambda)(1-i^k), \tag{57}$$

where $i^k \equiv \frac{K_{t+1}(0)}{I_t} \in (0,1)$. We can solve for $i^k \in (0,1)$ to satisfy (57). Similarly, output per tool is

$$\frac{Y_t}{H_t} = \sum_{\tau=0}^{\infty}[az^{\infty}(\tau)-f]\frac{\lambda^{\tau}}{G^{\tau+1}}(G-\lambda)i^k+\sum_{\tau=0}^{T-1}[az^{T-\tau}(\tau)-f]\frac{\lambda^{\tau}}{G^{\tau+1}}(G-\lambda)(1-i^k) \tag{58}$$

Aggregate domestic financial asset (46) under constant returns to scale maintenance technology is given by

$$D_t = Y_t - wH_t - D_t^*$$

$$+\sum_{\tau=1}^{\infty}\left(\frac{a}{R-\lambda\theta g}-\frac{f}{R-\lambda}\right)\frac{\lambda^{\tau}}{G^{\tau}}K_t(0)+\sum_{\tau=1}^{T-1}\left(aA^{T-\tau}z^{T-\tau}(\tau)-\Lambda^T f\right)\frac{\lambda^{\tau}}{G^{\tau}}L_t^T(0),$$

or

$$\frac{D_t}{H_t} = \frac{Y_t}{H_t} - w - d_t^*$$

$$+\sum_{\tau=1}^{\infty}\left(\frac{a}{R-\lambda\theta g}-\frac{f}{R-\lambda}\right)\frac{\lambda^{\tau}}{G^{\tau+1}}(G-\lambda)i^k+\sum_{\tau=1}^{T-1}\left(aA^{T-\tau}z^{T-\tau}(\tau)-\Lambda^T f\right)\frac{\lambda^{\tau}}{G^{\tau+1}}(G-\lambda)(1-i^k),$$

where $d_t^* = D_t^*/H_t$.

Similarly domestic financial asset per tool under decreasing returns to scale is

$$\frac{D_t}{H_t} = \frac{Y_t}{H_t} - w - d_t^*$$

$$+\sum_{\tau=1}^{\infty}\left(az^*U\left(\widetilde{z}^{\infty}(\tau)\right)-\frac{f}{R-\lambda}\right)\frac{\lambda^{\tau}}{G^{\tau+1}}(G-\lambda)i^k+\sum_{\tau=1}^{T}\left(az^*U\left(\widetilde{z}^{T-\tau}(\tau)\right)-\Lambda^T f\right)\frac{\lambda^{\tau}}{G^{\tau+1}}(G-\lambda)(1-i^k).$$

We also find

$$\frac{C_t}{H_t} = (1-\beta)\left[w+\lambda(x-b)+\frac{D_t}{H_t}\right].$$

From (47), we find in the steady state as

$$\frac{Y_t}{H_t} = \frac{C_t}{H_t} + G - \lambda + d^* - \frac{G}{R}d^*$$

or

$$\left(1-\frac{G}{R}\right)d^* = \frac{Y_t}{H_t} - \frac{C_t}{H_t} - (G-\lambda). \tag{59}$$

From this, we find the ratio of net foreign debt to tools in the steady state.

5.3 Proof for Proposition 2P

We first derive a sufficient condition for the existence of pure non-stopping equilibrium in P-region:

$$V(1)=\frac{1}{R-\lambda}\left(a\frac{R-(\theta+\eta)\lambda}{R-\theta\lambda}-f\right)\geq\frac{a}{R}\left(1-\frac{\theta\lambda}{R}\right)^{\frac{\eta}{1-\theta-\eta}}\frac{1-\theta-\eta}{1-\theta}+\frac{a\eta}{(1-\theta)\,(R-\theta\lambda)}-\frac{f}{R}, \tag{60}$$

We consider a sufficient condition of (49)

$$b(R)>\ \underset{T}{Max}S^T(1;w(R),R),$$

for the case of decreasing returns to scale maintenance technology. Consider an optimal stopping strategy in the RHS as

$$\left\{z^T\,(0)>z^{T-1}\,(1)>...>z^0\,(T)\right\}=\{z_0>z_1>\ldots>z_T\}$$

such that $z_0=1$ and $z_T\geq\underline{z}=f/a$. Associated with $\{z_t\}$, there is $h_t=\left(\frac{z_{t+1}}{z_t^{\theta}}\right)^{1/\eta}$. Let $v(h|z)$ denote the flow payoff of owner of a plant of quality z who hires h units of tools.

$$v(h|z)=az-wh-f.$$

Because optimal stopping strategy $z_t>z_{t+1}$ is better than staying at z_t with $h=z_t^{\frac{1-\theta}{\eta}}$, we get

$$\begin{aligned}v(h_t|z_t)+\lambda V(z_{t+1})&\geq v\left(z_t^{\frac{1-\theta}{\eta}}|z_t\right)+\lambda V(z_t),\ \text{or}\\ V(z_t)-V(z_{t+1})&\leq\frac{1}{\lambda}\left[v(h_t|z_t)-v\left(z_t^{\frac{1-\theta}{\eta}}|z_t\right)\right].\end{aligned} \tag{61}$$

Let $\phi(z|z_t)\equiv v\left(\left(z/z_t^{\theta}\right)^{\frac{1}{\eta}}\middle|\,z_t\right)=az_t-w\left(z/z_t^{\theta}\right)^{\frac{1}{\eta}}-f$.

$$v(h_t|z_t)-v(z_t^{\frac{1-\theta}{\eta}}|z_t)=\int_{z_{t+1}}^{z_t}-\phi'(z|z_t)dz$$

where

$$-\phi'(z|z_t)=\frac{w}{\eta}\frac{z^{\frac{1}{\eta}-1}}{z_t^{\frac{\theta}{\eta}}}.$$

Notice that because

$$\frac{\partial}{\partial z_t}\left[-\phi'(z|z_t)\right] < 0,$$

we have

$$-\phi'(z|z_t) = \frac{w}{\eta}\frac{z^{\frac{1}{\eta}-1}}{z_t^{\frac{\theta}{\eta}}} \leq \frac{w}{\eta}z^{\frac{1-\theta}{\eta}-1} = -\phi'(z|z), \text{ for } z_{t+1} \leq z \leq z_t.$$

Then,

$$v(h_t|z_t) - v(z_t^{\frac{1-\theta}{\eta}}|z_t) = \int_{z_{t+1}}^{z_t} -\phi'(z|z_t)dz \leq \int_{z_{t+1}}^{z_t} -\phi'(z|z)dz$$

Combining the inequality with inequality (61), we have

$$V(z_t) - V(z_{t+1}) \leq \frac{1}{\lambda}\left[v(h_t|z_t) - v\left(z_t^{\frac{1-\theta}{\eta}}|z_t\right)\right] \leq \frac{1}{\lambda}\int_{z_{t+1}}^{z_t} \frac{w}{\eta}z^{\frac{1-\theta}{\eta}-1}dz$$

$$V(1) - V(z_T) = \sum_{t=0}^{T-1}\left[V(z_t) - V(z_{t+1})\right] \leq \frac{1}{\lambda}\int_{z_T}^{1} \frac{w}{\eta}z^{\frac{1-\theta}{\eta}-1}dz$$

where we use $z_1 = 1$ in the last inequality. Because

$$V(z_T) = \frac{1}{R}(az_T - f)$$

and

$$\frac{1}{\lambda}\int_{z_T}^{1} \frac{w}{\eta}z^{\frac{1-\theta}{\eta}-1}dz = \frac{w}{\lambda(1-\theta)}\left(1 - z_T^{\frac{1-\theta}{\eta}}\right),$$

we have

$$V(1) \leq \frac{1}{R}(az_T - f) + \frac{w}{\lambda(1-\theta)}\left(1 - z_T^{\frac{1-\theta}{\eta}}\right) \equiv RHS\left(z_T\right), \tag{62}$$

if we are not in region P, i.e., some plant owners stop their plant.

To derive a sufficient condition for Region P, we use the fact that equilibrium wage in this region satisfies

$$\frac{w}{a} = \frac{\lambda\eta}{R - \theta\lambda}.$$

Then RHS of (62) reaches the maximum when

$$
\begin{aligned}
z_T &= \left(1-\frac{\theta\lambda}{R}\right)^{\frac{\eta}{1-\theta-\eta}} \\
RHS &= \frac{a}{R}\left(1-\frac{\theta\lambda}{R}\right)^{\frac{\eta}{1-\theta-\eta}}\frac{1-\theta-\eta}{1-\theta}+\frac{a\eta}{(1-\theta)(R-\theta\lambda)}-\frac{f}{R}.
\end{aligned}
$$

A sufficient condition for the economy to be in Region P is

$$
\begin{aligned}
V(1) &= \frac{1}{R-\lambda}\left(a\frac{R-(\theta+\eta)\lambda}{R-\theta\lambda}-f\right) \\
&\geq \frac{a}{R}\left(1-\frac{\theta\lambda}{R}\right)^{\frac{\eta}{1-\theta-\eta}}\frac{1-\theta-\eta}{1-\theta}+\frac{a\eta}{(1-\theta)(R-\theta\lambda)}-\frac{f}{R}.
\end{aligned}
$$

This is equivalent to which gives an upper bound on f/a,

$$
\frac{f}{a}\leq\frac{R(1-\theta-\eta)}{\lambda(1-\theta)}\left[1-\frac{R-\lambda}{R}\left(1-\frac{\theta\lambda}{R}\right)^{\frac{\eta}{1-\theta-\eta}}\right]\equiv\overline{F}(f/a),
$$

where $\overline{F}(f/a)$ denotes an upper bound for f/a as a sufficient condition for the existence of pure equilibrium with no stopping.

Now we proceed to derive a lower bound on f/a such that the growth rate is an increasing function of real interest rate in state equilibrium. From (16), we learn

$$
\begin{aligned}
0 &= (G-\pi^E\beta R^E)[G-(1-\pi^S)\beta R]-\pi^S(1-\pi^E)\beta^2 RR^E \\
&= \left[G-\pi^E\beta\left(\lambda+\frac{w}{x-b}\right)\right][G-(1-\pi^S)\beta R]-\pi^S(1-\pi^E)\beta^2 R\left(\lambda+\frac{w}{x-b}\right) \\
&\equiv \Psi\left(G;R,\frac{w}{x-b}\right). \qquad (63)
\end{aligned}
$$

Because we assume $\beta R<1$, we restrict our attention the case

$$
G>(1-\pi^S)\beta R.
$$

Then we learn

$$
G\geq\pi^E\beta\left(\lambda+\frac{w}{x-b}\right).
$$

Then we learn

$$\frac{\partial}{\partial G}\Psi\left(G;R,\frac{w}{x-b}\right)>0,$$

in the neighborhood of the equilibrium G. We can easily check

$$\frac{\partial}{\partial R}\Psi\left(G;R,\frac{w}{x-b}\right)<0$$

$$\frac{\partial}{\partial\left(\frac{w}{x-b}\right)}\Psi\left(G;R,\frac{w}{x-b}\right)<0.$$

Thus a sufficient condition for

$$\frac{dG}{dR}=-\frac{\frac{\partial}{\partial G}\Psi\left(G;R,\frac{w}{x-b}\right)}{\frac{\partial}{\partial R}\Psi\left(G;R,\frac{w}{x-b}\right)+\frac{\partial}{\partial\left(\frac{w}{x-b}\right)}\Psi\left(G;R,\frac{w}{x-b}\right)\frac{d}{dR}\left(\frac{w}{x-b}\right)}>0$$

is

$$\begin{aligned}0 &< \frac{d}{dR}\left(\frac{w}{x-b}\right)\\ &= \frac{w}{(x-b)^2(R-\lambda)^2(R-\lambda\theta)}\left[\lambda\left(1-\theta\right)f-(R-\lambda)^2x-\lambda\left(1-\theta-\eta\right)a\right],\end{aligned}$$

or

$$\lambda\left(1-\theta\right)f>(R-\lambda)^2x+\lambda\left(1-\theta-\eta\right)a. \tag{64}$$

If $\pi^S=0$, then from (16), we have $G=\pi^E\beta\left(\lambda+\frac{w}{x-b}\right)$, or

$$\begin{aligned}x=F(R,G)&=\frac{a-f-w}{R-\lambda}+\frac{\beta\pi^E}{G-\beta\lambda\pi^E}w\\ &=\frac{a-f}{R-\lambda}-\frac{G-\beta R\pi^E}{(R-\lambda)\left(G-\beta\lambda\pi^E\right)}w\end{aligned}$$

Because $F_G<0$, $dG/dR>0$ if and only if $F_R>0$. Because

$$(R-\lambda)F_R=-\frac{a-f-w}{R-\lambda}+\frac{G-\beta R\pi^E}{G-\beta\lambda\pi^E}\frac{a\eta\lambda}{(R-\theta\lambda)^2},$$

$dG/dR>0$ iff

$$f/a>\frac{R-(\theta+\eta)\lambda}{R-\theta\lambda}-\frac{G-\beta R\pi^E}{G-\beta\lambda\pi^E}\frac{\eta\lambda\left(R-\lambda\right)}{(R-\theta\lambda)^2}\equiv\underline{F}(f/a),$$

when $\pi^S = 0$. For the growth enhancing effect of interest rate in Region P, we need

$$\overline{F}(f/a) - \underline{F}(f/a) > 0$$

or

$$\begin{aligned}\frac{\overline{F}(f/a) - \underline{F}(f/a)}{R-\lambda} &= \frac{R(1-\theta-\eta) - \lambda(1-\theta)(\theta+\eta)}{\lambda(1-\theta)(R-\theta\lambda)} \\ &\quad - \frac{1-\theta-\eta}{\lambda(1-\theta)}\left(1-\frac{\theta\lambda}{R}\right)^{\frac{\eta}{1-\theta-\eta}} + \frac{G-\beta R\pi^E}{G-\beta\lambda\pi^E}\frac{\eta\lambda}{(R-\theta\lambda)^2} > 0\end{aligned}$$

Suppose both R and λ are close to 1,

$$\begin{aligned}\frac{\overline{F}(f/a) - \underline{F}(f/a)}{R-\lambda} &\approx \frac{1-\theta-\eta-(1-\theta)(\theta+\eta)}{(1-\theta)^2} - \frac{1-\theta-\eta}{(1-\theta)}(1-\theta)^{\frac{\eta}{1-\theta-\eta}} + \frac{\eta}{(1-\theta)^2} \\ &= \frac{1-\theta-\eta}{1-\theta}\left[1-(1-\theta)^{\frac{\eta}{1-\theta-\eta}}\right] \\ &> 0\end{aligned}$$

This proves that for any f/a, there exists an open set of interest rate and depreciation rate both of which are close to 1 where we have the property that growth rate is an increasing function of interest rate in Region P.

To examine the effect of unanticipated fall in real interest rate on welfare in pure non-stopping region, we use $(27a, 27b, 28, 29)$. Continue to assume $\pi^S = 0$. Then we have

$$\begin{aligned}\frac{dV^E}{dR} &= \frac{1}{1-\beta}\frac{d}{dR}(\ln n^E) \\ &\quad + \frac{\beta}{(1-\beta)(1-\beta\pi^E)}\frac{d}{dR}\left[\ln\left(\frac{w+\lambda(x-b)}{x-b}\right)\right] \\ &\quad + \frac{\beta^2(1-\pi^E)}{(1-\beta)^2(1-\beta\pi^E)}\frac{d}{dR}\ln R \qquad (65)\end{aligned}$$

From $(48a, 48b)$, we have

$$\begin{aligned}\frac{dw}{dR} &= -\frac{w}{R-\lambda\theta}, \\ \frac{db}{dR} &= \frac{1}{R-\lambda}\left(\frac{w}{R-\lambda\theta} - b\right).\end{aligned}$$

Then we get

$$
\begin{aligned}
\frac{d}{dR}\ln\left[w+\lambda(x-b)\right] &= \frac{1}{w+\lambda(x-b)}\frac{1}{(R-\lambda)^2}\left(a-f-\frac{R^2-\lambda^2\theta}{(R-\lambda\theta)^2}\eta a\right), \\
\frac{d}{dR}\ln\left(\lambda+\frac{w}{x-b}\right) &= \frac{w}{[w+\lambda(x-b)](x-b)(R-\lambda)^2(R-\lambda\theta)^2} \\
&\quad\cdot\left[\lambda\eta a-\lambda(1-\theta)(a-f)-(R-\lambda)^2 x\right]
\end{aligned}
$$

When $\pi^S=0$, then $n^E=\left[w+\lambda(x-b)\right]h$. Then from (65), we have

$$
(1-\beta)(1-\beta\pi^E)(R-\lambda)^2(R-\lambda\theta)^2\left[w+\lambda(x-b)\right]\frac{dV^E}{dR}/\lambda
$$

$$
\begin{aligned}
= &\ (1-\beta\pi^E)\left[(R-\lambda\theta)^2(a-f)-(R^2-\lambda^2\theta)\eta a\right] \\
&+\frac{\beta a\eta}{x-b}\left[\lambda\eta a-\lambda(1-\theta)(a-f)-(R-\lambda)^2 x\right] \\
&+\frac{\beta^2(1-\pi^E)}{1-\beta}\frac{(R-\lambda)(R-\lambda\theta)}{R}\{(R-\lambda\theta)\left[(R-\lambda)x-(a-f)\right]+R\eta a\}.
\end{aligned}
$$

5.4 Calibration strategy

We choose the following parameter values, θ, η, λ, β, π^E and π^S. We normalize the productivity of plant quality a to be 1.

We solve for f such that the economy is at boundary between Region P and Region M at $R=1.015$. We design an algorithm to solve for the infimum of the set of f, which plant owner stops in finite number of periods.

Suppose that the plant owner stops in T period at a particular value of f. Then, $S^t(1;f,w,R)$ as a function of t reaches its peak at T. Define a sequence of f_t such that at $f=f_t$, for $z^*=1$:

$$
S^{t+1}(1;f_t,w,R)=S^t(1;f_t,w,R).
$$

Intuitively, f_t tracks the movement in the peak as we vary f. If $f=f_t$, the peak is either t or $t+1$. As t goes to infinity, the peak shifts to infinity. Because

$$
S^{t+1}(1;a,w,r)=U^{t+1}\left(1;R\right)-\left(\frac{1}{R}+\frac{\lambda}{R^2}+...+\frac{\lambda^t}{R^{t+1}}\right)f
$$

and

$$S^t(1; a, w, r) = U^t\,(1; R) - \left(\frac{1}{R} + \frac{\lambda}{R^2} + ... + \frac{\lambda^{t-1}}{R^t}\right) f,$$

we have

$$f_t = \frac{R^{t+1}}{\lambda^t}\left[U^{t+1}\,(1; R) - U^t\,(1; R)\right].$$

The calibrated value of f is equal to $\inf_{t=1,2,...} f_t$, which we approximate by $\min_{t=1,2,...,T} f_t$ with T large enough. For any value of f strictly above $\inf_{t=1,2,...} a_t$, there must exist a finite optimal stopping time. For any value of f strictly below $\inf_{t=1,2,...} f_t$, there cannot exist a finite stopping time.

After we calibrate the value of f, we solve for x to target a growth rate of 0.5% at gross interest rate $R = 1.015$.

$$x = \frac{a - f - w}{R - \lambda} + \frac{\beta\Pi}{G - \beta\lambda\Pi} w$$

where $w = \frac{\lambda\eta}{R - \theta\lambda} a$ and

$$\Pi = \pi^E + \pi^S \frac{\beta R(1 - \pi^E)}{G - \beta R(1 - \pi^S)}.$$

.