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LIQUIDITY AND ASSET PRICES*

BY NOBUHIRO KIYOTAKI AND JOHN MOORE[1]

London School of Economics, U.K.;
Edinburgh University and London School of Economics, U.K.

We broadly define liquid assets, or monetary assets, as any asset that can be readily sold in the market and can be held by a number of people in succession before maturity. We ask in what environment is the circulation of liquid assets essential for the smooth running of the economy. By developing a canonical model of a monetary economy (i.e., where the circulation of liquid assets is essential), we are able to examine the interaction between liquidity, asset prices, and aggregate economic activity.

1. INTRODUCTION

The quantity of liquid assets (broad money) and the value of fixed assets (such as capital and real estate) fluctuate considerably over the business cycle. Standard asset-pricing models with a representative agent do not pay much attention to monetary matters, nor are they very successful in explaining large procyclical movements in asset prices (at least not with standard utility functions). Beside the volatility of asset prices, there are well-known puzzles in the asset-pricing literature, such as the low risk-free rate puzzle and the equity premium puzzle—puzzles that presume that the underlying economy is nonmonetary.[2] But these puzzles can be related to a traditional question in monetary economics: Why do people hold money, even though the rate of return is low and often dominated by the return on other assets? This article develops a canonical model of a monetary economy, in order to examine the interaction between the circulation of liquid assets, resource allocation, and asset prices.

We broadly define liquid assets, or monetary assets, as any asset that can be readily sold in the market and can be held by a number of people in succession before maturity. When an asset circulates among many people as a means of short-term saving (liquidity), it also serves as a medium of exchange (money): People hold it not for its maturity value, but for its exchange value. Thus we will use

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[2] See Mehra and Prescott (1985), Weil (1989), Campbell et al. (1997), and Campbell (1999).

liquid asset and monetary asset interchangeably. We ask in what environment is the circulation of monetary assets essential for the smooth running of the economy. Here, we contrast broad monetary assets with other assets solely in terms of their liquidity; how quickly they can be sold in the market. That is, in this article, all assets are real; a broad liquid asset is not denominated in cash, and we ignore the issue of fiat money and inflation.[3]

In order to analyze the role of liquid assets for resource allocation, we consider an economy in which output is produced from two types of asset, capital and land. Capital stock can be accumulated through productive investment, whereas the supply of land is fixed. We depart from a standard model of a stochastic production economy with a representative agent (a real business cycle model) in two respects. First, we assume that only a fraction of agents have a productive investment opportunity to accumulate capital stock at each point in time, even though agents are equally likely to find investment opportunities in the future. We also assume that there is no insurance contingent on the arrival of an investment opportunity. Thus, the economy must transfer purchasing power through financial markets from those who do not have a productive investment opportunity to those who do.[4] The second departure is that, at the time of productive investment, people can sell only a fraction of their capital stock (or equivalently, a claim to the future returns from capital stock). Thus capital stock is an asset with limited liquidity. Investing people may, therefore, face binding liquidity constraints. One interpretation of our model is that the productive investment opportunities disappear so quickly that investing agents do not have enough time to raise funds against their entire capital holding, nor to process an insurance claim, in order to finance new investment. In contrast, land is a liquid asset, and people can raise funds against their entire land holdings at the time of investment.[5]

[3] In a companion paper (Kiyotaki and Moore, 2001), we consider an economy in which the only liquid asset is cash, and we address the determination of the nominal price level. In some popular monetary frameworks, such as the cash-in-advance model or the dynamic sticky-price model, the circulation of monetary assets is not indispensable for efficient resource allocation. Other monetary frameworks, overlapping-generations models and random-matching models, do explain why the circulation of money improves efficiency. However, these models are not easy to apply to an economy with well-developed financial markets. Perhaps the closest ancestors of this article are Townsend (1987) and Townsend and Wallace (1987). Although our model does not start from as fundamental assumptions as theirs, our framework is closer to standard business cycle models.

[4] A large part of the asset-pricing literature with credit constraints uses an endowment economy, in which the focus is on risk sharing for households who face idiosyncratic utility shocks or income shocks (Cochrane, 2001). Here, we consider a production economy in which the role of financial markets is to transfer resources to those agents who have a productive investment opportunity. Holmstrom and Tirole (2001) develop a liquidity-based asset-pricing model of a three-period production economy with financial intermediaries in which the arrival of an investment opportunity is contractible. Our analysis largely abstracts from financial intermediation and contingent contracting of this kind in order to concentrate on the dynamic general equilibrium effects. Our framework is perhaps more comparable to a standard asset-pricing model, given that, in our economy, agents are identical ex ante, risk averse, and infinitely lived.

[5] In reality, land (or a claim to the future returns on land) is often less liquid than capital. The term "land" in this article may be taken to represent the productive assets of old and well-established sectors of the economy. Such sectors consist of publicly traded firms; their stock market is well organized and their productive assets are relatively constant. In contrast, the term "capital" might represent the productive assets of new and dynamic sectors of the economy, comprising less-established businesses.

We show that the circulation of the liquid asset is essential for resource allocation—i.e., the economy is "monetary"—if each agent rarely has a productive investment opportunity, if investing agents can sell only a small fraction of capital, and if the income share of land is small relative to capital. In the monetary economy, people with investment opportunities are liquidity constrained in the equilibrium. Also, there is a liquidity premium: a gap in the expected rates of returns between the illiquid asset (capital) and the liquid asset (land). The expected rate of return on the liquid asset is lower than the time preference rate. These phenomena are closely related. If people anticipate a binding liquidity constraint at the time of investment, they will hold the liquid asset in their portfolios even if its expected rate of return is dominated by that of the illiquid asset, and even if it is lower than their time preference rate, because the liquid asset is more valuable for financing the down payment for investment than the illiquid asset. That is, the liquidity constraint for investing agents, the liquidity premium, and the low return on the liquid asset are all equilibrium features of the monetary economy.

In a standard asset price model, an asset price is the expected present value of dividends, where dividends are either exogenous (as in an endowment economy model) or determined in equilibrium without feedback from the asset price itself (as in a real business cycle model). In contrast, in a monetary economy, there is a two-way interaction between asset prices and aggregate quantities. As in the standard framework, the higher future expected dividends are, the higher are asset prices. But, at the same time, the higher are asset prices, the more liquidity there is, which helps transfer resources from those who save to those who invest, thus encouraging aggregate investment and future production. This two-way interaction helps us explain large fluctuations in asset prices and aggregate output.

In the later part of the article, we extend our basic model. We introduce labor as a factor of production and workers as a new group of agents. Workers do not have productive investment opportunities, nor can they borrow against their future wage income. We show that, when the economy is monetary, the workers do not save, while entrepreneurs save using both the liquid asset and the illiquid asset. The difference in savings behavior between workers and entrepreneurs stems from the difference in their respective future investment opportunities: workers, who never invest, do not save because of the low rates of return on assets relative to the time preference rate, whereas entrepreneurs, anticipating binding liquidity constraints, save using assets despite the low rates of return in order to finance the down payment for their future investment. It is not because they have different preferences.

Although our framework does not have nominal assets or nominal price stickiness and the modeling strategy is close to real business-cycle theory, the behavior of our model has some Keynesian features: The circulation of liquid assets plays an essential role in transferring resources from those who save to those who invest (as emphasized in Keynes, 1937); there is an intimate interaction between the goods and asset markets (as in the formulations by Hicks, 1937; Tobin, 1969); and the main savers are not workers but entrepreneurs, including entrepreneurial households (as in Kaldor, 1955–56; Pasinetti, 1962). The main difference from traditional

Keynesian approaches is that we systematically deduce these characteristics of a monetary economy from a dynamic general equilibrium model with liquidity constraints.

2. BASIC MODEL

Consider a discrete-time economy with one homogeneous output. There is a continuum of infinitely lived agents with population size of unity. The utility of an agent at date 0 is described by

$$V_0 = E_0\left[\sum_{t=0}^{\infty} \beta^t \ln c_t\right] \tag{1}$$

where c_t is his or her consumption at date t, $\beta \in (0, 1)$ is the constant discount factor, ln x is the natural logarithm of x, and $E_0[x]$ is the expected value of x conditional on information at date 0.

The individual agent produces output from two types of homogeneous asset, capital stock and land, according to a constant returns to scale production function:

$$y_t = a_t k_t'^{\alpha} \ell_t^{1-\alpha}, \qquad 0 < \alpha < 1 \tag{2}$$

where y_t is output, k_t' is capital stock, and ℓ_t is land used for production. The variable $a_t > 0$ is aggregate productivity, which is common to all individuals. We assume that the aggregate productivity follows a stationary Markov process that fluctuates exogenously in the neighborhood of some constant level. The individual agents own capital and land, and can freely rent capital and land for production in a perfectly competitive rental market. The capital stock depreciates at a constant rate $1 - \lambda \in (0, 1)$. The total supply of land is fixed at L.

Each agent meets an opportunity to invest in capital with probability Π in each period. The arrival of the investment opportunity is independent across periods and across agents. When an agent has a productive investment opportunity, he can convert goods into capital one for one instantaneously.

In order to finance investment, the agent can sell his land and capital holdings, in exchange for goods. But, crucially, the transfer of ownership of capital is not instantaneous; delivery is overnight. This delay opens the door to a potential moral hazard problem. After receiving goods from an agreed sale of capital (in the evening before the overnight transfer of ownership), the agent can abscond and start a new life the next day with a fresh identity and clear record. We assume that he cannot take all his capital with him, though; he can steal at most a fraction $1-\theta \in (0,1)$. To ensure that sales agreements are honored, they must be incentive compatible. Hence an investing agent can credibly sell at most a fraction θ of the capital he holds before investment. Equally, he can sell only a fraction θ of the new capital he produces from the investment. The exogenous parameter θ can be thought of as a measure of the liquidity of capital.[6] Note that if θ were to equal unity, then, because the agent could not abscond with any capital, there would be no moral hazard problem, and in effect capital would be perfectly liquid. The transfer of ownership of land is also overnight, but we assume that agents cannot abscond with any land, so that land (or a claim to the future rental income on land) is a liquid asset, which serves as money in our model.

One way to think of the parameter θ is that it reflects how quickly an agent must exploit an investment opportunity once he meets one. The opportunity disappears the next day (although with probability Π he will meet another), and, therefore, he does not have much time to sell capital. In principle he could eventually disinvest his entire capital holding by selling off a fraction θ every period, but this would be too late.

We assume that the arrival of an investment opportunity is not contractible, so that an agent cannot arrange contingent insurance. Again, we are thinking of an environment in which, by the time an insurance company has verified that the agent does indeed have an investment opportunity, the opportunity is gone.[7]

Let p_t be price of land, let q_t be the price of capital installed, and let r_t^K and r_t^L be the rental prices of capital and land—all in terms of goods, the numeraire. The flow-of-funds constraint of an agent at date t is

$$
\begin{aligned}
(3) \qquad & c_t + i_t + q_t(k_{t+1} - i_t - \lambda k_t) + p_t(m_{t+1} - m_t) \\
& \quad = r_t^K k_t + r_t^L m_t + \left(y_t - r_t^K k_t' - r_t^L \ell_t\right)
\end{aligned}
$$

where i_t is investment, m_t and k_t are land and capital owned before investment, and ℓ_t and k_t' are land and capital used for production. The left-hand side is expenditure on consumption and investment plus the net purchase of capital and land. The right-hand side is rental income from capital and land plus the profit from production of goods. Because the agent can rent capital and land for goods production, his usage need not match his capital and land ownership. The constraints governing his ownership of capital and land are

$$(4) \qquad k_{t+1} \geq (1 - \theta)(\lambda k_t + i_t)$$

$$(5) \qquad m_{t+1} \geq 0$$

[6] There are other justifications for why capital, or claims secured against capital, may not be perfectly liquid. For example, it may take time to inspect the capital or to verify the authenticity of a claim, in which case θ would be a proxy for the speed of verification. Or there may be different qualities of capital, and buyers may be less informed than sellers so that there is adverse selection in the second-hand market.

[7] Self-reporting insurance arrangements as in Diamond and Dybvig (1983) are not incentive compatible here, because we assume that an insurance company cannot monitor individual transactions in financial markets (Jacklin, 1987; Cole and Kocherlakota, 2001). We will not consider the possibility of insurance payments being made after the investment opportunity has gone, because this kind of insurance is not very useful for improving resource allocation; we expect our main findings are robust to allowing for delayed insurance payments.

Inequality (4) says that, since the agent cannot sell more than a fraction θ of his existing (depreciated) stock λk_t or of his new capital i_t, he has to retain at least a fraction $1 - \theta$. Inequality (5) says that he cannot short sell land. Taken together, these inequalities constitute the agent's liquidity constraints.[8]

We will see that the aggregate state of the economy, s_t, is summarized by the aggregate capital stock, K_t, and the aggregate productivity, a_t; $s_t = (K_t, a_t)$. Competitive equilibrium is described as price functions $p_t = p(s_t)$, $q_t = q(s_t)$, $r_t^K = r^K(s_t)$, and $r_t^L = r^L(s_t)$, and quantities of consumption, investment, output, and ownership and usage of capital and land, $(c_t, i_t, y_t, k_t, m_t, k_t', \ell_t)$, such that (i) individual agents choose rules of consumption, investment, asset portfolio, and production to maximize their utility subject to the flow-of-funds constraints (3) and the liquidity constraints (4) and (5), taking the price functions as given; (ii) the sum of the individual capital holdings equals the aggregate capital stock K_t; (iii) the sum of the individual land holdings equals the total land supply L; (iv) aggregate consumption and investment equals aggregate output; and (v) the rental markets for capital and land clear.

The decision by an individual agent of how many goods to produce is straightforward. He chooses output and usage of capital and land to maximize his profit, $y_t - r_t^K k_t' - r_t^L \ell_t$, subject to the production function. Profit is maximized when the marginal products of capital and land equal their rental prices. Maximized profit is zero, so as to be consistent with competitive equilibrium with positive production. That is,

$$r_t^K = \alpha y_t / k_t' \quad \text{and} \quad r_t^L = (1-\alpha) y_t / \ell_t \tag{6}$$

Thus, when the agent does not have an opportunity to invest in capital stock, the flow of funds constraint (3) can be written as

$$c_t^n + q_t k_{t+1}^n + p_t m_{t+1}^n = \left(r_t^K + \lambda q_t\right) k_t + \left(r_t^L + p_t\right) m_t \equiv b_t^n \tag{7}$$

where the superscript n represents an agent without an investment opportunity and b_t is his total wealth.

When the agent has an opportunity to invest at date t, he has two ways of acquiring capital in Equation (3): one through investment at his production cost of unity and the other through the market at the price q_t. If $q_t < 1$, the agent will not invest. The agent is indifferent if $q_t = 1$. Thus, his flow-of-funds constraint is the same as (7) if the agent has an investment opportunity, but $q_t \leq 1$. On the other hand, if $q_t > 1$, the agent will invest and sell as much capital as he can, subject to the constraint (4). Such an agent is liquidity constrained because his investment is limited by his available funds. The investment choice is similar to Tobin's q theory

[8] Strictly speaking, to avoid the possibility that the agent might abscond, leaving behind his land plus a fraction θ of his capital, we need to place a lower bound on a combination of k_{t+1} and m_{t+1}: A linear combination of (4) and (5) must hold. But, whenever an agent is credit constrained in equilibrium, (4) and (5) both bind, so nothing is lost by imposing them as separate inequality constraints.

(Tobin, 1969). Tobin's q may exceed unity here, because at each date only a small fraction (Π) of agents have the opportunity to invest in capital stock and they are liquidity constrained. When $q_t > 1$, the flow-of-funds constraint (3) of an investing agent, given that (4) binds, can be written as

$$c_t^i + \frac{1-\theta q_t}{1-\theta} k_{t+1}^i + p_t m_{t+1}^i = \left(r_t^K + \lambda\right) k_t + \left(r_t^L + p_t\right) m_t \equiv b_t^i \tag{8}$$

where the superscript i represents an agent with an investment opportunity that is strictly profitable. Think of $[(1-\theta q_t)/(1-\theta)]$ as the cost of investment with maximum leverage: Because the agent sells the maximum fraction θ of each unit of new capital, he needs to finance only the down payment $(1-\theta q_t)$ from his own funds in order to retain the residual fraction $1-\theta$. Capital in total wealth on the right-hand side is valued at the replacement cost λk_t rather than at the market value $q_t \lambda k_t$, given that the agent is unable to sell as much capital as he would wish in the market. Thus, when the agent has an investment opportunity at date t, the rate of return on capital from the last period is $[(r_t^K + \lambda)/q_{t-1}]$ rather than $[(r_t^K + \lambda q_t)/q_{t-1}]$.

In what follows, we concentrate on the case where Tobin's q exceeds unity so that the liquidity constraint is binding for the investing agents; we will later derive the condition that guarantees $q_t > 1$.

When an agent does not have an investment opportunity at date t, he chooses between consumption and saving in order to maximize his expected utility. The marginal utility of consumption must be equal to the expected marginal benefit of acquiring capital and land:

$$\frac{1}{c_t^n} = \frac{1}{q_t} \beta E_t \left(\Pi \frac{r_{t+1}^K + \lambda}{c_{t+1}^{ni}} + (1-\Pi) \frac{r_{t+1}^K + \lambda q_{t+1}}{c_{t+1}^{nn}} \right) \tag{9a}$$

$$\frac{1}{c_t^n} = \frac{1}{p_t} \beta E_t \left(\Pi \frac{r_{t+1}^L + p_{t+1}}{c_{t+1}^{ni}} + (1-\Pi) \frac{r_{t+1}^L + p_{t+1}}{c_{t+1}^{nn}} \right) \tag{9b}$$

where c_{t+1}^{ni} (resp. c_{t+1}^{nn}) is the date $t+1$ consumption of the agent, who does not have an investment opportunity at date t, if he turns out to have an investment opportunity at date $t+1$ (resp. if he still doesn't have an investment opportunity at date $t+1$). Because utility is the logarithm of consumption, marginal utility is the inverse of consumption on the left-hand sides of the above equations. On the right-hand side of Equation (9a), the agent can acquire $(1/q_t)$ units of capital by giving up one unit of consumption at date t. With probability Π, the agent has an investment opportunity at date $t+1$, and the return on capital becomes $r_{t+1}^K + \lambda$ with a marginal utility of $(1/c_{t+1}^{ni})$. With probability $1-\Pi$, the agent does not have an investment opportunity at date $t+1$, and the return on capital is $r_{t+1}^K + \lambda q_{t+1}$ with a marginal utility of $(1/c_{t+1}^{nn})$. On the right-hand side of

Equation (9b), the return on land is always equal to $r^L_{t+1} + p_{t+1}$, given that land is liquid.[9]

When an agent has an investment opportunity at date t, he chooses between consumption and investment with maximum leverage:

$$\frac{1}{c^i_t} = \frac{1-\theta}{1-\theta q_t}\beta E_t\left(\Pi\frac{r^K_{t+1}+\lambda}{c^{ii}_{t+1}} + (1-\Pi)\frac{r^K_{t+1}+\lambda q_{t+1}}{c^{in}_{t+1}}\right) \tag{10}$$

where c^{ii}_{t+1} (resp. c^{in}_{t+1}) is the date $t+1$ consumption of the agent, who has an investment opportunity at date t, if he turns out also to have an investment opportunity at date $t+1$ (resp. if he doesn't have an investment opportunity at date $t+1$). In Equation (10), the agent can acquire $(1-\theta)/(1-\theta q_t)$ units of capital by giving up one unit of consumption with maximum leverage. The investing agent will not hold any land, if the marginal cost of acquiring land exceeds the marginal benefit:

$$\frac{1}{c^i_t} > \frac{1}{p_t}\beta E_t\left(\Pi\frac{r^L_{t+1}+p_{t+1}}{c^{ii}_{t+1}} + (1-\Pi)\frac{r^L_{t+1}+p_{t+1}}{c^{in}_{t+1}}\right) \tag{11}$$

Later, we will verify that inequality (11) holds.

From Equations (7)–(11), we can show that

$$c^n_t = (1-\beta)b^n_t \tag{12a}$$

$$c^i_t = (1-\beta)b^i_t \tag{12b}$$

$$m^i_{t+1} = 0 \tag{13a}$$

$$(1-\theta q_t)i_t = \left(r^K_t + \theta\lambda q_t\right)k_t + \left(r^L_t + p_t\right)m_t - c^i_t \tag{13b}$$

$$(1-\Pi)E_t\left(\frac{\frac{r^K_{t+1}+\lambda q_{t+1}}{q_t} - \frac{r^L_{t+1}+p_{t+1}}{p_t}}{\left(r^K_{t+1}+\lambda q_{t+1}\right)k^n_{t+1} + \left(r^L_{t+1}+p_{t+1}\right)m^n_{t+1}}\right) = \Pi E_t\left(\frac{\frac{r^L_{t+1}+p_{t+1}}{p_t} - \frac{r^K_{t+1}+\lambda}{q_t}}{\left(r^K_{t+1}+\lambda\right)k^n_{t+1} + \left(r^L_{t+1}+p_{t+1}\right)m^n_{t+1}}\right) \tag{14}$$

[9] On a day immediately after investment, an agent holds only capital stock and no land. Then, even if he no longer has an investment opportunity, (4) may be binding, in which case Equation (9a) becomes an inequality. In examining the aggregate economy, we will ignore this because the proportion of such agents is small when Π is not large. For most of our comparative static and dynamic analysis, we use the continuous-time approximation, where we can show that the effects of such agents on the aggregate economy is infinitesimal.

In Equations (12a) and (12b), consumption is proportional to the total wealth.[10] In Equations (13a) and (13b), the investing agent holds no land and uses all his funds after consumption to finance downpayment for investment. Equation (14) describes the portfolio decision of the agent who does not have an investment opportunity at date t, which is derived from (9). The numerator in the bracket on the left-hand side represents how much the rate of return on capital exceeds that of land if there is no investment opportunity at date $t+1$ (which happens with probability $1-\Pi$). Because marginal utility equals the inverse of consumption (which is proportional to total wealth from (12a)), the left-hand side is the expected advantage of capital over land in terms of utility (times a constant $1-\beta$), if there is no investment opportunity in the next period. The right-hand side is the expected advantage of land over capital when the agent meets an investment opportunity in the next period (which happens with probability Π).

Since the arrival of the productive investment opportunity is independent across agents, we can aggregate the individuals' investment functions (13b):

(15)
$$(1-\theta q_t)I_t = \Pi\big(\big(r_t^K+\theta\lambda q_t\big)K_t + \big(r_t^L+p_t\big)L - (1-\beta)\big[\big(r_t^K+\lambda\big)K_t + \big(r_t^L+p_t\big)L\big]\big)$$

where I_t is aggregate investment. Equation (15) shows that when investing agents are liquidity constrained, aggregate investment is an increasing function of capital price and land price. This two-way interaction between aggregate investment and asset prices is an important propagation channel for the effects of shocks on aggregate output.[11] The aggregate capital stock evolves over time as

$$K_{t+1} = I_t + \lambda K_t \tag{16}$$

Market-clearing condition for capital holdings implies

$$K_{t+1} = (1-\theta)(I_t + \Pi\lambda K_t) + K_{t+1}^n \tag{17}$$

where K_{t+1}^n is the aggregate date $t+1$ capital holdings of the agents who did not have an investment opportunity at date t. The first term on the right-hand side is the aggregate capital holdings of the agents who do have an investment opportunity and who face the binding liquidity constraint (4), which is a $1-\theta$ fraction of new capital (I_t) and capital held from the previous period ($\Pi\lambda K_t$).

Through the competitive rental markets for capital and land, both the marginal products are equalized across producers. Thus aggregate output Y_t is a function of aggregate capital stock:

$$Y_t = a_t K_t^{\alpha} L^{1-\alpha} \tag{18}$$

[10] The logarithmic utility in (1) is isomorphic to a Cobb–Douglas utility function, where the expenditure share of present consumption is constant and equal to $1/(1+\beta+\beta^2+\cdots) = 1-\beta$.

[11] There is a similar two-way interaction in our earlier article, (Kiyotaki and Moore, 1997), but there the mechanism relied on the fact that debt could not be indexed to asset prices. There is no such ad hoc restriction here.

In these rental markets, prices equal marginal products:

(19a) $$r_t^K = \alpha Y_t / K_t$$

(19b) $$r_t^L = (1-\alpha) Y_t / L$$

Aggregating the individuals' consumption and investment, we have the goods-market-clearing condition:

(20) $$Y_t = I_t + (1-\beta)\left(\left[r_t^K + \Pi\lambda + (1-\Pi)\lambda q_t\right]K_t + \left(r_t^L + p_t\right)L\right)$$

All the land is held by the agents who do not have an investment opportunity. Their portfolio behavior (14) is now

(21) $$(1-\Pi)E_t\left(\frac{\frac{r_{t+1}^K + \lambda q_{t+1}}{q_t} - \frac{r_{t+1}^L + p_{t+1}}{p_t}}{\left(r_{t+1}^K + \lambda q_{t+1}\right)K_{t+1}^n + \left(r_{t+1}^L + p_{t+1}\right)L}\right)$$
$$= \Pi E_t\left(\frac{\frac{r_{t+1}^L + p_{t+1}}{p_t} - \frac{r_{t+1}^K + \lambda}{q_t}}{\left(r_{t+1}^K + \lambda\right)K_{t+1}^n + \left(r_{t+1}^L + p_{t+1}\right)L}\right)$$

A competitive equilibrium of our economy in the aggregate is described recursively as prices (p_t, q_t, r_t^L, r_t^K) and aggregate quantities $(I_t, K_{t+1}, K_{t+1}^n, Y_t)$, all as functions of the aggregate state (K_t, a_t), which satisfy (15)–(21), together with an exogenous evolution of aggregate productivity a_t.

It is worth remarking that our model is almost as simple as standard asset-pricing models of a production economy, real business cycle models, and IS-LM models. The main difference from standard asset-pricing models (such as Merton, 1975; Brock, 1982) is the portfolio behavior, which takes into account the illiquidity of capital in financing investment, Equation (21). Our model differs from standard real business cycle models (such as Kydland and Prescott, 1982) because we have a two-way interaction between asset prices and aggregate quantities. In a typical real business cycle model, quantities are determined first, before deriving implied prices. In this aspect, our model is closer to traditional IS-LM models (in particular to Tobin, 1969, where q is the key variable linking goods market equilibrium and asset market equilibrium). Our framework, however, differs substantively (beyond modeling strategy) because, while IS-LM compares cash and interest-bearing assets in determining the nominal interest rate, we draw a contrast between broad liquid assets and illiquid assets in determining the liquidity premium.[12]

Before analyzing dynamics in the next section, let us examine the steady-state equilibrium for constant aggregate productivity, $a_t \equiv a$. In steady state, capital stock is constant so that $I = (1-\lambda)K$, and

[12] In this respect, we are close to Keynes (1937), who defines "money" broadly as liquid assets, including Treasury bills.

(22a) $$(1-\theta q)(1-\lambda)K = \Pi((r^K+\theta\lambda q)K + (r^L+p)L - (1-\beta)[(r^K+\lambda)K + (r^L+p)L])$$

(22b) $$aK^\alpha L^{1-\alpha} = (1-\lambda)K + (1-\beta) \times ([r^K + \Pi\lambda + (1-\Pi)\lambda q]K + (r^L+p)L)$$

(22c) $$\frac{r^K+\lambda q}{q} - \frac{r^L+p}{p} = \Pi\lambda\frac{q-1}{q}\left\{1 + \left(\frac{r^L+p}{p} - \frac{r^K+\lambda}{q}\right) \times \left(\frac{qK^n}{(r^K+\lambda)K^n + (r^L+p)L}\right)\right\}$$

(22d) $$r^L = (1-\alpha)aK^\alpha L^{-\alpha}$$

(22e) $$r^K = \alpha a K^{\alpha-1}L^{1-\alpha}$$

(22f) $$K^n = [1-(1-\theta)(1-\lambda+\lambda\Pi)]K$$

From these conditions, we can show the following proposition:

PROPOSITION 1. *Suppose that the parameters of the economy satisfy:*

(Assumption 1) $$\theta < \theta^* \equiv 1 - \frac{\Pi}{1-\lambda+\Pi\lambda}\left(1 + \frac{1-\alpha}{\alpha}\frac{1-\beta\lambda}{1-\beta}\right)$$

Then, in the steady-state equilibrium

(i) *Tobin's q exceeds* 1, *so that investing agents are liquidity constrained;*
(ii) $\frac{r^L}{p} < \frac{r^K}{q} - (1-\lambda) < \frac{1-\beta}{\beta}$;
(iii) $K < K^*$, *where* K^* *is the capital stock in the first-best steady state, which solves* $\alpha a(K^*)^{\alpha-1}L^{1-\alpha} = (1/\beta) - \lambda$;
(iv) *investing agents do not hold land at the end of the period,* $m^i_{t+1} = 0$.

PROOF. See Part A of the Appendix.

Assumption 1 requires that the fraction of capital that agents can sell in order to finance their investment is smaller than some critical level θ^*. The value of θ^* is a decreasing function both of the arrival rate of investment opportunities, Π, and of the ratio of the land value to capital in a first-best allocation, $(\frac{1-\alpha}{1-\beta})/(\frac{\alpha}{1-\beta\lambda})$. Thus, Proposition 1(i) says that investing agents are liquidity constrained if and only if capital is sufficiently illiquid in relation to the arrival rate of investment opportunities and the ratio of land value to capital. Proposition 1(ii) says that the rate of return on the liquid asset, land, is dominated by the rate of return on the illiquid asset, capital, which in turn is smaller than the time preference rate. The gap between the rates of returns on illiquid and liquid assets may be called the liquidity premium, which arises if and only if investing agents are liquidity constrained. From (22c), the magnitude of the liquidity premium is roughly equal to $\Pi(q-1)$, which can be substantial.[13] Proposition 1(iii) says that there is underinvestment relative to the first-best allocation if the investing agents' liquidity constraints are binding: Not enough resources are transferred from savers to investors. All of these features—liquidity constraints, liquidity premium, low liquid asset return, underinvestment—are features of a "monetary economy," in which the circulation of a liquid asset is essential for the smooth running of the economy.

The reverse of Proposition 1 is also true. Suppose that $\theta \geq \theta^*$. Then the steady-state equilibrium is the first-best allocation. Tobin's q equals 1 and the investing agents are not liquidity constrained. The rates of returns on capital and land are both equal to the time preference rate. (Incidentally, notice that if $\theta > (1-\Pi)\times(1-\lambda)/(1-\lambda+\Pi\lambda)$ the economy achieves first-best even if the liquid asset, land, is unimportant for production, $\alpha \cong 1$. In this case, the economy ceases to be monetary in the sense that the circulation of a liquid asset is no longer necessary for an efficient allocation.)

3. DYNAMICS

We now examine the dynamics of the economy with stochastic fluctuations in aggregate productivity a_t. Let us assume that a_t follows a two-point Markov process:

$$(23)\qquad a_t = \text{either } a(1+\Delta_a), \quad \text{or} \quad a(1-\Delta_a), \text{ where } \Delta_a \in (0,1) \text{ is small}$$

and the arrival of the productivity changes follows a Poisson process with arrival rate $\bar{\eta}$.

There are two ways to investigate the dynamics. One is to calibrate the recursive equilibrium system, $(p_t, q_t, r_t^L, r_t^K, I_t, Y_t, K_{t+1}, K_{t+1}^n)$, as functions of the aggregate state (K_t, a_t), that satisfies Equations (15)–(21).

The other way is to use analytical methods, in a continuous-time approximation. Let the length of a period be Δ instead of 1, and define the continuous-time parameters ρ, δ, π and η that satisfy

$$(24)\qquad \beta = e^{-\rho\Delta}, \quad \lambda = e^{-\delta\Delta}, \quad \Pi = 1 - e^{-\pi\Delta}, \quad \bar{\eta} = 1 - e^{-\eta\Delta}$$

The parameter ρ is time preference rate, δ is depreciation rate, π is the Poisson arrival rate of an investment opportunity for each agent, and η is the arrival rate of an aggregate productivity shock. We assume that Assumption 1 holds with strict inequality in the continuous time limit, so that investing agents are always liquidity constrained in the neighborhood of the steady-state equilibrium:

$$(\text{Assumption } 1')\qquad \theta < \tilde{\theta}^* \equiv 1 - \frac{\pi}{\pi+\delta}\left(1 + \frac{1-\alpha}{\alpha}\frac{\rho+\delta}{\rho}\right)$$

[13] For example, if Tobin's q is 5% above 1 and each agent has an investment opportunity once every 2 years on average, then the liquidity premium is 2.5% at an annual rate.

We also assume that the arrival rate of an investment opportunity is greater than the depreciation rate:

(Assumption 2) $$\pi > \delta$$

Assumption 2 is a mild condition. For example, if the depreciation rate is 10% a year, the arrival rate of investment opportunity for each agent is more frequent than once every 10 years.

Define I_t and Y_t as the investment and output rates. Let r_t^K and r_t^L be the rental prices of capital and land per unit of time. Take the limit of the aggregate equilibrium as Δ goes to 0. Then we have

$$(1-\theta q_t)I_t = \pi(\theta q_t K_t + p_t L) \tag{25}$$

$$\dot{K}_t = I_t - \delta K_t \tag{26}$$

$$K_t^n = K_t \tag{27}$$

$$Y_t = I_t + \rho(q_t K_t + p_t L) \tag{28}$$

$$\frac{r_t^K}{q_t} - \delta - \frac{r_t^L}{p_t} + \lim_{dt\to 0} E_t\left(\frac{1}{dt}\left(\frac{q_{t+dt}}{q_t} - \frac{p_{t+dt}}{p_t}\right)\frac{q_t K_t + p_t L}{q_{t+dt}K_t + p_{t+dt}L}\right) = \pi\left(1-\frac{1}{q_t}\right)\frac{q_t K_t + p_t L}{K_t + p_t L} \tag{29}$$

Equations (18) and (19) are unchanged except that a_t is aggregate productivity per unit of time. In Equation (25), the investing agents use maximum sales of capital and land to finance the downpayment of investment. Because the fraction of agents who invest in an infinitesimal period $[t, t+dt]$ is equal to $\pi\, dt$ (which is infinitesimal), effectively all the capital stock is owned by the agents who do not have an investment opportunity in Equation (27). The left-hand side of (29) is the liquidity premium: the difference between the expected rates of return on capital and land, taking into account risk aversion. The right-hand side is the expected "capital loss" on illiquid capital if an investment opportunity arrives. When the agent has an investment opportunity (with arrival rate π), the implied value of capital falls from q_t to 1, and the holder suffers from the capital loss of $1 - (1/q_t)$. (Liquid land does not suffer such a capital loss.) This is adjusted by the marginal rate of substitution between consumption with an investment opportunity and consumption without an opportunity. As a check, in Part C of the Appendix we lay out a model of a continuous time economy in order to derive the above equations directly, instead of taking a limit of the discrete time economy.

Because it is easier to analyze the model in intensive form, let us define $i_t = I_t/K_t$ (i_t is now the investment rate, not investment by an individual agent),

$v_t = P_t/K_t$ (ratio of land value to capital), and $x_t = Y_t/K_t$ (ratio of output to capital). Equations (25), (26), (18), (28), and (29), respectively, become

$$(1-\theta q_t)i_t = \pi(\theta q_t + v_t) \tag{30}$$

$$\dot{K}_t/K_t = i_t - \delta \tag{31}$$

$$x_t = a_t K_t^{\alpha-1}L^{1-\alpha} = i_t + \rho(q_t + v_t) \tag{32}$$

$$\left[\frac{\alpha}{q_t} - \frac{1-\alpha}{v_t}\right]x_t - i_t + \lim_{dt\to 0} E_t\left(\frac{1}{dt}\left(\frac{q_{t+dt}}{q_t} - \frac{v_{t+dt}}{v_t}\right)\frac{q_t+v_t}{q_{t+dt}+v_{t+dt}}\right) = \pi\frac{q_t-1}{q_t}\frac{q_t+v_t}{1+v_t} \tag{33}$$

We now examine a linearized dynamic system in the neighborhood of the steady state. Using $\hat{x}_t$ to denote the proportional deviation of variable x_t from the steady-state value x, i.e., $\hat{x}_t \equiv (x_t - x)/x$, we can postulate the endogenous variables as functions of the state variables:

$$\hat{i}_t = \phi\hat{K}_t + \psi\hat{a}_t \tag{34a}$$

$$\hat{v}_t = \phi_v\hat{K}_t + \psi_v\hat{a}_t \tag{34b}$$

$$\hat{q}_t = \phi_q\hat{K}_t + \psi_q\hat{a}_t \tag{34c}$$

From (23), (24), (31), and (32),

$$d\hat{x}_t = d\hat{a}_t - (1-\alpha)\delta\,\hat{i}_t\,dt \tag{35}$$

Equation (35) says that the proportional change of the output/capital ratio in an infinitesimal period $[t, t+dt]$ is equal to the proportional change in productivity (if the productivity shock occurs) plus the effect on capital accumulation through investment.

In Part B of the Appendix, we show that there is a unique negative ϕ that satisfies saddle-point stability of the dynamic system. (If ϕ were positive, the growth rate of capital stock would be an increasing function of capital stock, which would make the economy unstable.) Part B of the Appendix also shows

$$\psi > 0 \tag{36}$$

$$\psi_v < 0 \tag{37}$$

Inequality (36) implies that the investment rate (I_t/K_t) is an increasing function of aggregate productivity. Inequality (37) implies that the ratio of land value to capital stock is a decreasing function of capital stock, even though the total land values may increase with capital stock.

The stochastic process of asset prices and aggregate quantities are described by the recursive rules (34) together with the evolution of aggregate productivity (23) and (24). It is surprisingly difficult to derive the signs of the other values, ϕ_v, ϕ_q, and ψ_q, in (34) generally. But, for reasonable parameters, Figure 1 summarizes the typical fluctuations of aggregate quantities and prices.

When the productivity rises to a higher level at date t, such a change is considered good news, even if people anticipate occasional changes of the productivity. The prices of capital and land rise discontinuously. Investment increases vigorously, because the investing entrepreneurs can raise more funds against land and capital. Output and consumption also increase but, proportionately, not as much as investment. After date t, capital stock starts accumulating with the increased investment. With capital accumulation, output, consumption, and land price

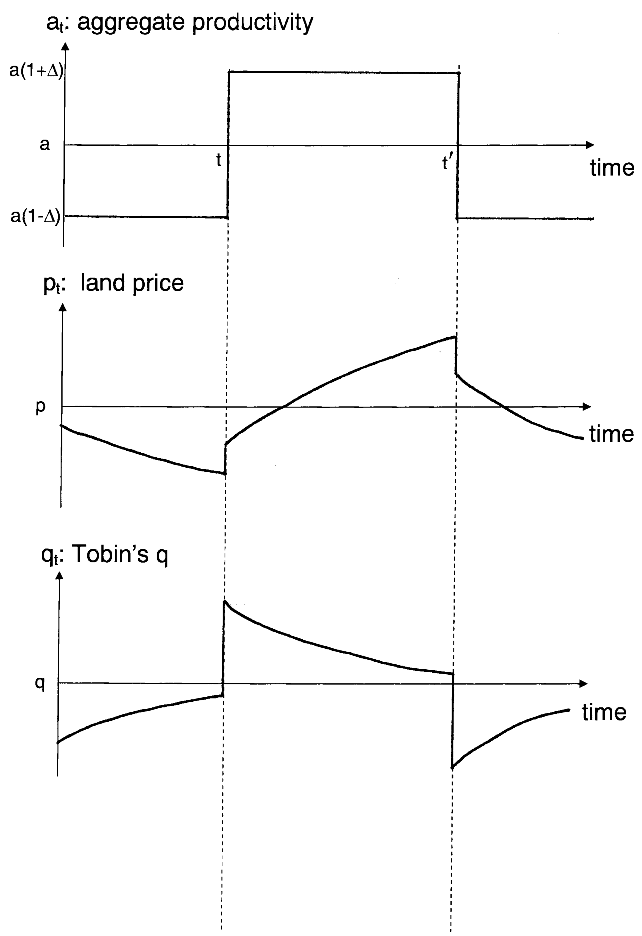

FIGURE 1

RESPONSE TO PRODUCTIVITY SHOCKS (FIGURE CONTINUES ON NEXT PAGE)

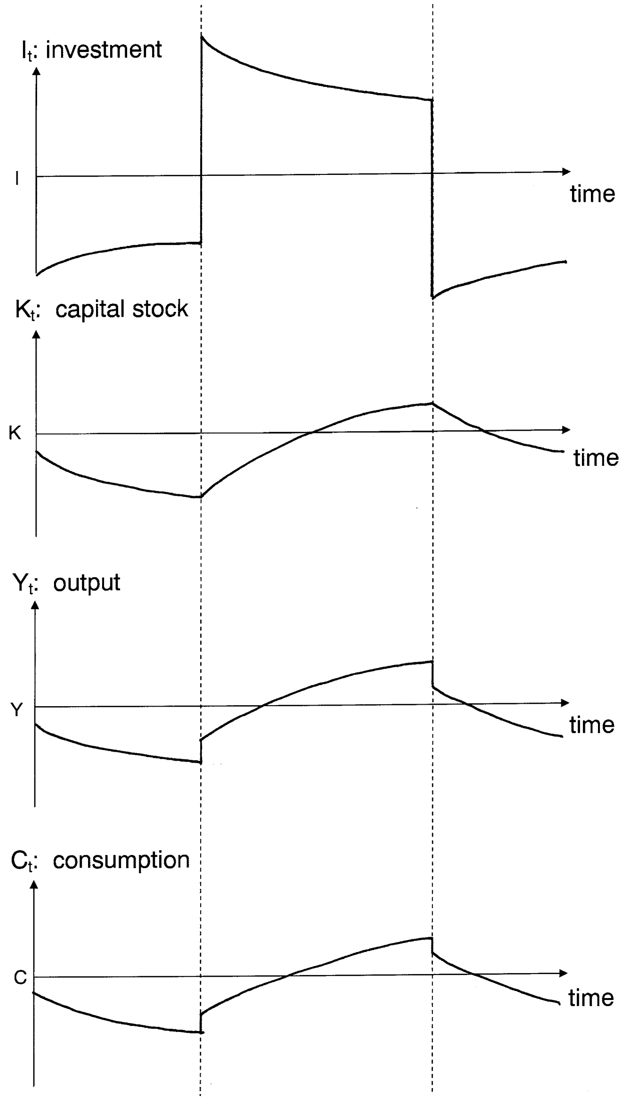

FIGURE 1

(CONTINUED)

further increase. On the other hand, Tobin's q starts falling to a normal level as the liquidity constraint loosens with the capital accumulation—that is, until a negative productivity shock arrives and the above dynamics reverse.

For applications to asset pricing, it is perhaps more natural to suppose that aggregate productivity follows a geometric Brownian motion, instead of a two-point Markov process:

$$da_t = \sigma a_t \, dz_t \tag{38}$$

where z_t is a Wiener process and $\sigma > 0$ is the standard deviation of the innovation in aggregate productivity. Such an economy can be seen as the economy of Merton (1975), but with limited liquidity of capital stock. With this stochastic process of productivity, we no longer can presume that the investing agents are always liquidity constrained. Depending on the state of the economy, the investing agents may invest up to the maximum, may invest without a liquidity constraint, or may not invest at all. In Part C of the Appendix, we derive some preliminary results for such an economy.

4. WORKERS

We now introduce workers into the basic model from Section 2. Suppose that output is produced from homogeneous capital, land, and labor, according to the aggregate production function

$$Y_t' = a_t' K_t^{\alpha'} L_t^{\gamma'} N_t^{1-\alpha'-\gamma'}, \qquad \alpha', \gamma', 1-\alpha'-\gamma' > 0 \tag{39}$$

where Y_t' is aggregate output, K_t is capital, L_t is land, and N_t is labor. Suppose also that there is a continuum of a new group of people, called workers, with a population size of unity. The workers supply labor, but they never have investment opportunities. The utility of a worker at date 0 is described by

$$E_0\left(\sum_{t=0}^{\infty} \beta^t u\left(c_t - \frac{\omega}{1+\nu} N_t^{1+\nu}\right)\right), \qquad \nu, \omega > 0 \tag{40}$$

where c_t is consumption, N_t is labor supplied, and $u(\cdot)$ is an instantaneous utility function satisfying $u' > 0$, $u'' < 0$, $u'(0) = \infty$ and $u'(\infty) = 0$. We assume that the workers cannot borrow against their future wage income. Let the agents described in the basic model, those who do meet investment opportunities, be called entrepreneurs.

There is a competitive labor market in which the producing entrepreneurs hire labor. The real wage rate, w_t, equals both the marginal product of labor and the marginal disutility of labor

$$(1-\alpha'-\gamma') a_t' K_t^{\alpha'} L_t^{\gamma'} N_t^{-(\alpha'+\gamma')} = w_t = \omega N_t^{\nu} \tag{41}$$

Define the gross profit of entrepreneurs, Y_t, as the aggregate output minus the aggregate wage for the workers. Then from (39), we have

$$Y_t \equiv Y_t' - w_t N_t = a_t \left(K_t^{\alpha} L^{1-\alpha}\right)^{\frac{(\alpha'+\gamma')(1+\nu)}{\alpha'+\gamma'+\nu}} \tag{42}$$

where

$$\alpha \equiv \frac{\alpha'}{\alpha' + \gamma'} \quad \text{and} \quad a_t \equiv (\alpha' + \gamma') \left(\frac{1 - \alpha' - \gamma'}{\omega} \right)^{\frac{1-\alpha'-\gamma'}{\alpha'+\gamma'+\nu}} a_t'^{\frac{1+\nu}{\alpha'+\gamma'+\nu}}$$

Note that because there is no income effect in the labor supply of workers, we have been able to write down the gross profit function as a function of capital and land. And since the marginal disutility of labor is an increasing function of labor ($\nu > 0$), the gross profit function has decreasing returns to scale in capital and land, i.e., $(\alpha' + \gamma')(1 + \nu)/(\alpha' + \gamma' + \nu) < 1$. Think of α as the income share of capital in gross profit, and $1 - \alpha$ as the share of land. Note that from the entrepreneur's perspective, this economy is the same as that in the basic model without workers.

As in Section 3, let us suppose that the aggregate productivity shock a_t follows the two-point Markov process (23), where the shock Δ_a is small, so that we are in a neighborhood of steady state. The interesting case is where liquidity constraints bind—we later derive the condition that guarantees this—with the implication that the equilibrium rates of return on land and capital are both lower than the time preference rate.

Because the returns on assets are low, the workers will not save at all, i.e., they always consume their entire wage income. (This is provided the arrival of an aggregate productivity switch is not so frequent that there is a large incentive to smooth consumption.) Notice that workers save nothing not because they are myopic or irrational, but rather because they do not expect any future investment opportunities and the rates of return on land and capital are lower than their time preference rate. Remember that the workers' time preference rate is the same as the entrepreneurs'. The feature that entrepreneurs save whereas workers do not is common in the Keynesian literature, e.g., Kaldor (1955–56) and Pasinetti (1962). The difference between our model and this traditional literature is that we derive differences in saving patterns as the equilibrium outcome of a liquidity-contained economy, rather than making ad hoc assumptions about differences in propensities to save.

In the goods market, aggregate output is the sum of investment and consumption by entrepreneurs and workers. Because the consumption of workers equals their wage income, goods-market clearing is equivalent to the condition that the gross profit (aggregate output minus wage income) equals investment and entrepreneurs' consumption:

$$Y_t = I_t + (1 - \beta)\left(\left[r_t^K + \pi\lambda + (1 - \pi)\lambda q_t\right]K_t + (r_t^L + p_t)L\right) \tag{43}$$

This equation is same as the goods-market-clearing condition from the basic model, Equation (20), except that Y_t represents gross profit rather than aggregate output. Moreover, given that workers neither save nor hold assets, the entrepreneurs hold all the assets, so the asset-market-clearing condition is unchanged from Section 2. That is, competitive equilibrium is described recursively by (p_t, q_t, r_t^L, r_t^K, I_t, Y_t, K_{t+1}) as functions of aggregate states (K_t, a_t) that satisfy Equations (15)–(17), (19), (21), (42), and (43). Proposition 1 still holds: Assumption 1 continues to be the condition for the liquidity constraint to be binding in the neighborhood of the steady state.

Arguably, we may have gone too far in predicting that workers do not save at all. In reality, workers do hold some liquid assets (money), albeit they may not hold illiquid assets. How can this be explained? One way might be to suppose that the aggregate productivity shock Δ_a were larger, or that instead of a Poisson arrival there were continuous shocks (as in geometric Brownian motion), in which cases, as in a buffer stock model, workers would save in order to smooth their consumption, despite the low returns on assets (see Bewley, 1977, 1980, 1983; Carroll, 1992; Deaton, 1992). However, in the usual buffer stock model there is only one means of saving. By itself, this would not necessarily explain why workers choose to save holding mainly liquid assets.

Alternatively, we can extend our model to give workers (small) investment opportunities of their own. Consider the continuous-time version of our model from Section 3. Suppose that each worker suffers a "health" shock according to a Poisson process with arrival rate ε. Following a shock, a worker has to spend ζ goods instantaneously in order to maintain his human capital. There is then a time interval T during which he is immune from shocks, where the equilibrium wage greatly exceeds ζ/T. Thereafter, times revert to normal, i.e., the worker faces the possibility of a further shock, again with arrival rate ε. Shocks cannot be contracted on, so workers can only insure themselves through saving. We restrict attention to the case where ε and ζ are sufficiently small that the economy continues to be monetary, under Assumption 1′.

Then we can show that, (i) in normal times, a worker holds liquid assets worth ζ, which he sells to meet a shock when it arrives; (ii) following the shock, during the period of immunity, he steadily accumulates a liquid asset holding that is worth ζ by the end of the interval T; and (iii) the worker never holds the illiquid asset. Intuitively, the worker need not save more than ζ units of the liquid asset, given that its rate of return (and that of the illiquid asset) is below his time preference rate. And he does not save in the illiquid asset, even though it yields a higher expected return than the liquid asset, because

$$\frac{1}{\theta}\left\{\rho+\delta-E_t\left[\frac{r_t^K\,dt+dq_t}{q_t\,dt}\right]\right\} > \rho - E_t\left[\frac{r_t^L\,dt+dp_t}{p_t\,dt}\right] \tag{44}$$

Inequality (44) says that the opportunity cost of holding $1/\theta$ units of capital is greater than the opportunity cost of holding one unit of land (remember that the agent can sell only a fraction θ of capital in order to meet the cost ζ of the shock). In Part A of the Appendix, we prove (44) holds in the neighborhood of steady state, if the economy is monetary.

The contrast between the entrepreneurs' savings behavior and the workers' savings behavior throws up another feature of a monetary economy: An agent's portfolio choice depends upon his future investment opportunities. If people were not anticipating liquidity constraints, their savings portfolios would not be sensitive to their future investment needs. But, in a monetary economy, liquidity constraints are expected to bind, and the nature of future investment affects the extent to which people keep their savings liquid. On the one hand, entrepreneurs save substantial amounts of both liquid and illiquid assets, because their future investment opportunities are large (constant returns to scale) and arrive steadily (Poisson). On the other hand, workers save only a small amount of liquid assets, because their investment needs are small (fixed size) and interspersed (a minimum interval T between each).[14]

5. INDIVIDUAL PERSISTENCE IN INVESTMENT OPPORTUNITIES

So far, we have assumed that the arrival rate of an investment opportunity is the same for every entrepreneur in every period. In effect, each date the identity of investing entrepreneurs and saving entrepreneurs is completely reshuffled. As we have seen, this greatly simplifies aggregation: Aggregate quantities and prices are functions of just two state variables, aggregate capital stock and aggregate productivity. At the aggregate level, then, the distribution of wealth (between those who invest and those who save) does not matter. In this section, we sketch out an extension of the basic model from Section 2 in which distribution matters—although aggregation is still kept relatively simple.

Let us introduce individual persistence in investment opportunities. That is, if an agent has an investment opportunity in period t, he is more likely to have one in period $t+1$ as well. Now the model has an interesting new twist: There is an interaction between investment and the wealth distribution between savers and investors.[15]

More specifically, we assume that this period's arrival rate of an investment opportunity for those who invested last period is Π^i, whereas the rate for those who didn't invest is Π^n, and

$$\text{(Assumption 3)} \qquad \Pi^i > \Pi^n > 0$$

Now the equilibrium depends on not only aggregate capital K_t and aggregate productivity a_t, but also on the total capital stock held by those who invested last period, K_t^i, versus the stock held by those who didn't invest, $K_t^n = K_t - K_t^i$. The equilibrium Equations (15)–(21) are modified to

$$\begin{aligned}(1-\theta q_t)I_t &= \Pi^i\big((r_t^K + \theta\lambda q_t)K_t^i - (1-\beta)(r_t^K + \lambda)K_t^i\big) \\ &\quad + \Pi^n\big((r_t^K + \theta\lambda q_t)K_t^n + (r_t^L + p_t)L - (1-\beta) \\ &\qquad \cdot \big[(r_t^K + \lambda)K_t^n + (r_t^L + p_t)L\big]\big) \\ &= \big[r_t^K + \theta\lambda q_t - (1-\beta)\lambda\big]\big((\Pi^i - \Pi^n)K_t^i + \Pi^n K_t\big) + \beta(r_t^L + p_t)\Pi^n L\end{aligned} \tag{45}$$

[14] If the workers were expecting large investments in the future, say buying a house or paying for their children's college education, then they would save substantial amounts. In national income accounts, when households build houses it counts as investment, entrepreneurial activity. We might think of households with large investments as entrepreneurs instead of workers.

[15] This has been emphasized in, for example, Bernanke and Gertler (1989), Kiyotaki and Moore (1997), and Bernanke et al. (1999).

(46) $$Y_t = I_t + (1-\beta)\big(\big[r_t^K + \Pi^n\lambda + (1-\Pi^n)\lambda q_t\big]K_t + \big(r_t^L + p_t\big)L - (\Pi^i - \Pi^n)\lambda(q_t - 1)K_t^i\big)$$

(47) $$K_{t+1}^i = (1-\theta)\big(I_t + \Pi^n\lambda K_t + (\Pi^i - \Pi^n)\lambda K_t^i\big)$$

(48) $$(1-\Pi^n)E_t\left(\frac{\frac{r_{t+1}^K + \lambda q_{t+1}}{q_t} - \frac{r_{t+1}^L + p_{t+1}}{p_t}}{\big(r_{t+1}^K + \lambda q_{t+1}\big)\big(K_{t+1} - K_{t+1}^i\big) + \big(r_{t+1}^L + p_{t+1}\big)L}\right) = \Pi^n E_t\left(\frac{\frac{r_{t+1}^L + p_{t+1}}{p_t} - \frac{r_{t+1}^K + \lambda}{q_t}}{\big(r_{t+1}^K + \lambda\big)\big(K_{t+1} - K_{t+1}^i\big) + \big(r_{t+1}^L + p_{t+1}\big)L}\right)$$

together with (16), (18), and (19). Competitive equilibrium is described by $(p_t, q_t, r_t^K, r_t^L, I_t, K_{t+1}, K_{t+1}^i, Y_t)$ as functions of the aggregate state (K_t, K_t^i, a_t). From (45) and (46), we notice that investment and saving are functions of the capital stock held by the previously investing agents, K_t^i. The economy tends to exhibit more persistence because we have K_t^i as an additional state variable through which the effects of this period persist to the next period.

APPENDIX

A. *Proof of Propositions.*

PROOF OF PROPOSITION 1. Define $x \equiv Y/K \equiv a(L/K)^{1-\alpha}$, $v \equiv pL/K$, and $\xi \equiv K^n/K = \lambda(1-\Pi) + \theta(1-\lambda+\lambda\Pi) \in (0,1)$. From (22), we get

(A.1a) $$(1-\theta q)(1-\lambda) = \Pi[\beta x + \theta\lambda q - (1-\beta)\lambda + \beta v]$$

(A.1b) $$\beta x = 1 - \lambda + \lambda\Pi(1-\beta) + (1-\beta)(1-\Pi)\lambda q + (1-\beta)v$$

(A.1c) $$\left(\frac{\alpha}{q} - \frac{1-\alpha}{v}\right)x - (1-\lambda) = \Pi\lambda\frac{q-1}{q}\frac{(1-\alpha)x + v}{v} \times \frac{\xi q + v}{\xi(\alpha x + \lambda) + (1-\alpha)x + v}$$

Then we have

(A.2a) $$x = x(q;\theta) = \frac{1}{\Pi\beta}((1-\beta+\Pi\beta)[1-\lambda+\lambda\Pi(1-\beta)] + (1-\beta)[\lambda\beta\Pi(1-\Pi) - (1-\lambda+\lambda\Pi)\theta]q)$$

(A.2b) $$v = v(q;\theta) = \frac{1}{\Pi}((1-\Pi)[1-\lambda+\lambda\Pi(1-\beta)] - [\lambda\Pi(1-\beta)(1-\Pi) + (1-\lambda+\lambda\Pi)\theta]q)$$

(A.2c)
$$F(q;\theta) \equiv 1-\lambda+\frac{(1-\alpha)x}{v}-\frac{\alpha x}{q} + \Pi\lambda\frac{q-1}{q}\left(1+\xi q\frac{\frac{(1-\alpha)x}{v}+1-\frac{\alpha x+\lambda}{q}}{(\alpha\xi+1-\alpha)x+\lambda\xi+v}\right) = 0$$

Part 1(i): From (22c) in the text, we have that the second term in the large bracket of the right-hand side (RHS) of (A.2c) is positive. From (A.2a) and (A.2b), it follows that $\delta v(q;\theta)/\partial q < 0$, $\partial[x\ (q;\theta)/q]/\partial q < 0$, $\partial[x(q;\theta)/v\ (q;\theta)]/\partial q > 0$, and

$$\frac{\partial}{\partial q}[(\alpha\xi+1-\alpha)x(q;\theta)+\lambda\xi+v(q;\theta)] < 0$$

Therefore, F is an increasing function of q. And hence there is a unique q, which is larger than 1 if and only if $F(1;\theta) < 0$. Because F is an increasing function of θ, $F(1;\theta) < 0$ if and only if Assumption 1 holds. ■

Part 1(ii): From the optimal portfolio condition (22c), the return on land is dominated by that on capital, $(r^L+p)/p < (r^K+\lambda q)/q$, if and only if q exceeds unity, or Assumption 1 holds. Also, the left-hand side (LHS) of (A.1c) can be written as

$$\text{LHS of (A.1c)} = \left(\alpha\frac{x}{q}+\lambda-\frac{1}{\beta}\right)\left(1+\frac{q}{v}\right)+\frac{1}{\beta}-1-\frac{1}{v}\left(x+q\left(\lambda-\frac{1}{\beta}\right)\right)$$
$$= \left(\alpha\frac{x}{q}+\lambda-\frac{1}{\beta}\right)\left(1+\frac{q}{v}\right)+(q-1)\frac{1-\lambda+\lambda\Pi(1-\beta)}{\beta v},$$

(from (A.1b))

Thus, because the RHS of (A.1c) is an increasing function of q and $\xi < 1$, a sufficient condition for $(r^K+\lambda q)/q < 1/\beta$ is

$$\Pi\lambda\frac{v+(1-\alpha)x}{q}\frac{q+v}{x+\lambda+v} < \frac{1-\lambda+\lambda\Pi(1-\beta)}{\beta} \quad \text{or}$$

(A.3)
$$\frac{\lambda\beta\Pi[1+(v/q)]}{1-\lambda+\lambda\Pi(1-\beta)} < \frac{x+\lambda+v}{v+(1-\alpha)x}$$

From (A.2a) and (A.2b), we know that the RHS of (A.3) is an increasing function of q and that the LHS of (A.3) is a decreasing function of q. Hence a sufficient condition for (A.3) is

$$\left.\frac{\lambda\beta\Pi[1+(v/q)]}{1-\lambda+\lambda\Pi(1-\beta)}\right|_{q=1} < \left.\frac{x+\lambda+v}{v+(1-\alpha)x}\right|_{q=1} \quad \text{or}$$

$$\frac{\lambda\beta(1-\lambda+\lambda\Pi)(1-\theta)}{1-\lambda+\lambda\Pi-\lambda\Pi\beta} < \frac{(1-\lambda+\lambda\Pi)(1-\theta)}{(1-\lambda+\lambda\Pi)(1-\theta)-\lambda\Pi\beta-\Pi\beta\alpha x}$$

But, this holds for any $x > 0$, which implies $(r^K + \lambda q)/q < 1/\beta$. ■

Part 1(iii): From (A.1c):

$$\text{LHS of (A.1c)} = \left(\alpha x+\lambda-\frac{1}{\beta}\right)\left(1+\frac{1}{v}\right)-\frac{x}{v}+\left(\frac{1}{\beta}-\lambda\right)\frac{1}{v}+\frac{1}{\beta}-1-\alpha x\left(1-\frac{1}{q}\right)$$

$$= \left(\alpha x+\lambda-\frac{1}{\beta}\right)\left(1+\frac{1}{v}\right)-(q-1)\frac{(1-\beta)(1-\Pi)\lambda}{\beta v}-\alpha x\frac{q-1}{q},$$

(from (A.1b))

But, RHS of (A.1c) > 0. Together with $q > 1$, we have $r^K + \lambda > 1/\beta$. ■

Part 1(iv): We know $m^i_{t+1} = 0$ iff (11) is true. From (8), (12), and (13b)

$$c^{ii}_{t+1} = (1-\beta)\left(r^K_{t+1}+\lambda\right)k^i_{t+1} = (1-\beta)\left(r^K_{t+1}+\lambda\right)\frac{1-\theta}{1-\theta q_t}\beta b^i_t$$

$$c^{in}_{t+1} = (1-\beta)\left(r^K_{t+1}+\lambda q_{t+1}\right)k^i_{t+1} = (1-\beta)\left(r^K_{t+1}+\lambda q_{t+1}\right)\frac{1-\theta}{1-\theta q_t}\beta b^i_t$$

Thus, together with (12b), the inequality (11) in the steady-state is equivalent to

$$1 > \left(1+\frac{r^L}{p}\right)\frac{1-\theta q}{1-\theta}\left(\frac{\Pi}{r^K+\lambda}+\frac{1-\Pi}{r^K+\lambda q}\right) \quad \text{or}$$

$$\left(1+\frac{r^L}{p}\right)\frac{1-\theta q}{(1-\theta)q}\left(1+\frac{\Pi\lambda(q-1)}{r^K+\lambda}\right) < \frac{r^K+\lambda q}{q}$$

$$= \left(1+\frac{r^L}{p}\right)\left(1+\Pi\lambda\frac{q-1}{q}\frac{qK^n+pL}{(r^K+\lambda)K^n+(r^L+p)L}\right), \quad \text{(from (22c))}$$

This is equivalent to

$$1 > \Pi\lambda\frac{1-\theta q}{r^K+\lambda}-\Pi\lambda(1-\theta)\frac{qK^n+pL}{(r^K+\lambda)K^n+(r^L+p)L}$$

But, this is true because the first term in RHS is smaller than 1. ■

PROOF OF INEQUALITY (44). The inequality (44) holds in the neighborhood of the steady-state equilibrium if and only if

$$\theta\left(\rho-(1-\alpha)\frac{x}{v}\right)<\rho+\delta-\alpha\frac{x}{q} \tag{44'}$$

From (30)–(33), the steady-state conditions for the continuous-time model are $i=\delta$ and

$$(1-\theta q)\delta=\pi(\theta q+v) \tag{A.4a}$$

$$x=\delta+\rho(q+v) \tag{A.4b}$$

$$\frac{\alpha}{q}x-\delta-\frac{1-\alpha}{v}x=\pi\frac{q-1}{q}\frac{q+v}{1+v} \tag{A.4c}$$

The LHS of (A.4c) can be written as $(\frac{\alpha}{q}x-\delta-\rho)\frac{q+v}{v}+\delta\frac{q-1}{v}$. Thus from (A.4c) and (A.4a), we obtain

$$\rho+\delta-\frac{\alpha}{q}x=\frac{q-1}{q+v}\left(\delta-\pi v\frac{q+v}{q(1+v)}\right)=\frac{q-1}{q(1+v)}\left(\delta(q-1)\frac{v}{q+v}+(\pi+\delta)\theta q\right)$$

Then, again from (A.4c), we also obtain

$$\rho-\frac{1-\alpha}{v}x=\rho+\delta-\frac{\alpha}{q}x+\pi\frac{q-1}{q}\frac{q+v}{1+v}$$

Thus, condition (44′) is equivalent to

$$\begin{aligned}0&<(1-\theta)\frac{q-1}{q(1+v)}\left(\delta(q-1)\frac{v}{q+v}+(\pi+\delta)\theta q\right)-\theta\pi\frac{q-1}{q}\frac{q+v}{1+v}\\&=\frac{q-1}{q(1+v)}\left((1-\theta)\delta(q-1)\frac{v}{q+v}+\theta\delta(q-1)\right),\quad\text{(using (A.4a))}\end{aligned}$$

But, the last expression is positive by $q>1$. ■

B. Derivation of Local Dynamics. From (A.4), the steady-state of the continuous-time version of the model is

$$x=\frac{1}{\pi}((\pi+\rho)\delta+[\pi-\theta(\pi+\delta)]\rho q) \tag{A.5a}$$

$$v=\frac{1}{\pi}(\delta-\theta(\pi+\delta)q) \tag{A.5b}$$

$$\text{(B.1c)}\quad \left(\frac{\alpha}{\pi q} - \frac{1-\alpha}{\delta - \theta(\pi+\delta)q}\right)((\pi+\rho)\delta + [\pi - \theta(\pi+\delta)]\rho q) - \delta - \pi\frac{q-1}{q}\left(1 + \frac{\pi(q-1)}{(\pi+\delta)(1-\theta q)}\right) \equiv \tilde{F}(q,\theta) = 0$$

From (B.1c), we learn that $\tilde{F}(q,\theta)$ is a decreasing function of q and θ, and that $\tilde{F}(q,\theta) = 0$ has a solution $q > 1$ if and only if Assumption 1′ satisfies, i.e.,

$$\theta < \tilde{\theta}^* \equiv \frac{1}{\pi+\delta}\left(\delta - \pi\frac{1-\alpha}{\alpha}\frac{\rho+\delta}{\rho}\right)$$

Assumption 1′ is the continuous-time version of Assumption 1 in the text, which we will assume in the following argument.

From (34),

$$\lim_{dt\to 0} E_t\left(\frac{1}{dt}\left(\frac{q_{t+dt}}{q_t} - \frac{v_{t+dt}}{v_t}\right)\frac{q_t+v_t}{q_{t+dt}+v_{t+dt}}\right) = (\phi_q - \phi_v)\delta\hat{i}_t + (\psi_q - \psi_v)\eta(-2\hat{a}_t)$$

The first term on the RHS is the difference in the rate of capital gains between capital and land due to capital accumulation. The second term is the difference in the expected rates of capital gains associated with the productivity change, noting that the arrival rate of the change is η and that the proportional size of the change $(d\hat{a}_t)$ is equal to $-2\hat{a}_t$ by (23) and (24). Thus linearizing (30), (32), and (33) around the steady state, we have

$$\text{(B.2)}\quad (1-\theta q)\delta\hat{i}_t = \pi v\hat{v}_t + \theta(\pi+\delta)q\hat{q}_t$$

$$\text{(B.3)}\quad x\hat{x}_t = \delta\hat{i}_t + \rho v\hat{v}_t + \rho q\hat{q}_t$$

$$\text{(B.4)}\quad 0 = G_x\hat{x}_t + G_q\hat{q}_t + G_v\hat{v}_t - (1+\phi_v-\phi_q)\delta\hat{i}_t - 2\eta(\psi_q-\psi_v)\hat{a}_t$$

where

$$G_x \equiv \left(\frac{\alpha}{q} - \frac{1-\alpha}{v}\right)x,\; G_q \equiv -\left(\frac{\alpha}{q}x + \frac{1}{q}\frac{v+q^2}{v+1}\right), \quad \text{and}$$

$$G_v \equiv \left[\frac{1-\alpha}{v}x + \pi\frac{v}{q}\left[\frac{q-1}{v+1}\right]^2\right]$$

Substituting (34) and (35) into (B.2), (B.3), and (B.4), these equations must hold for any $\hat{K}_t$ and $\hat{a}_t$. Thus we have six unknowns variables $(\phi, \phi_q, \phi_v, \psi, \psi_q, \psi_v)$, which satisfy six equations: equating the coefficients of $\hat{K}_t$ and $\hat{a}_t$ in each of (B.2), (B.3), and (B.4).

Solving (B.2), (B.3) for $\hat{v}_t$ and $\hat{q}_t$ with respect to $\hat{i}_t$ and $\hat{x}_t$, we have

(B.5)
$$\begin{pmatrix}\hat{v}_t\\ \hat{q}_t\end{pmatrix} = \frac{1}{d}\begin{pmatrix}[\rho(1-\theta q)+\theta(\pi+\delta)]\delta q & -\theta(\pi+\delta)qx\\ -[\rho(1-\theta q)+\pi]\delta v & \pi v x\end{pmatrix}\begin{pmatrix}\hat{i}_t\\ \hat{x}_t\end{pmatrix}$$
$$\equiv \begin{pmatrix}D_{vi} & D_{vx}\\ D_{qi} & D_{qx}\end{pmatrix}\begin{pmatrix}\hat{i}_t\\ \hat{x}\end{pmatrix}, \quad \text{where } d \equiv [\pi - \theta(\pi+\delta)]\rho q v$$

From (34), (35), and (B.5), we learn

(B.6)
$$\begin{pmatrix}\phi_v(\phi) & \psi_v(\psi)\\ \phi_q(\phi) & \psi_q(\psi)\end{pmatrix} = \begin{pmatrix}D_{vi} & D_{vx}\\ D_{qi} & D_{qx}\end{pmatrix}\begin{pmatrix}\phi & \psi\\ -(1-\alpha) & 1\end{pmatrix}$$

Substituting (B.5) and (B.6) into (B.4), we have two equations for the coefficients of $\hat{K}_t$ and $\hat{a}_t$

(B.7)
$$0 = -(1-\alpha)G_x - [1+\phi_v(\phi)-\phi_q(\phi)]\delta\phi + G_q\phi_q(\phi) + G_v\phi_v(\phi)$$
$$\equiv \Phi(\phi) \equiv \Phi_2\phi^2 + \Phi_1\phi + \Phi_0$$

(B.8)
$$0 = G_x - [1+\phi_v(\phi)-\phi_q(\phi)]\delta\psi - 2\eta[\psi_q(\psi)-\psi_v(\psi)]$$
$$+ G_q\psi_q(\psi) + G_v\psi_v(\psi)$$
$$\equiv \Psi(\psi,\phi) \equiv \Psi_1(\phi)\psi + \Psi_0$$

From (B.7), we learn that $\Phi(\phi)=0$ is a quadratic equation in ϕ with

(B.9a)
$$\Phi_2 = -\delta(D_{vi}-D_{qi}) = \frac{\delta^2}{d}[\delta\theta q + x(1-\theta q)]$$

(B.9b)
$$\Phi_0 = -(1-\alpha)[G_x + G_q D_{qx} + G_v D_{vx}]$$

Under Assumptions 2, $d \equiv [\pi - \theta(\pi+\delta)]\rho q v > 0$ and thus $\Phi_2 < 0$. Also, because we have $D_{qx} > 1$ and $G_x + G_q < 0$ and $G_v D_{vx} < 0$, we have $\Phi_0 > 0$. Therefore, there is a unique ϕ that satisfies the saddle-point stability condition $\phi < 0$. Also, from (B.6) and (B.8), we see that $\Psi(\psi,\phi)=0$ is a linear equation in ψ with

(B.10a)
$$\Psi_0 = G_x + G_q D_{qx} + G_v D_{vx} - 2\eta(D_{qx} - D_{vx})$$

and

(B.10b)
$$\Psi_1 = -\delta[1+\phi_v(\phi)-\phi_q(\phi)] - 2\eta(D_{qi}-D_{vi}) + G_q D_{qi} + G_v D_{vi}$$

Using the same argument that we used for Φ_0, we learn $\Psi_0 < 0$. Also using (B.7), we can rewrite (B.10b) as

$$\Psi_1 = -\frac{\Phi_0}{\phi} - 2\eta(D_{qi} - D_{vi})$$

Because $\phi < 0$ and $D_{qi} - D_{vi} < 0$, we obtain $\Psi_1 > 0$. Therefore, $\psi > 0$, from (B.8). Again from (B.5), (B.6), (B.7), and (B.8), we can derive the property (37) in the text.

C. Continuous-Time Formulation of the Agent's Behavior. Here we describe the individual agent's behavior using a continuous-time formulation. The agent's utility is given by

$$V_0 = E_0\left[\int_0^{\infty} \ln c_t e^{-\rho t}\, dt\right] \tag{C.1}$$

where c_t is the consumption rate at date t, ρ is a constant time preference rate. The individual agent can produce according to production function (2), where y_t is output per unit of time. The aggregate productivity a_t follows the continuous-time two-point Markov process (23) and (24). The depreciation rate of capital is δ.

Each agent meets an opportunity to invest in capital according to a Poisson process with arrival rate π. At the time of the investment, the agent can sell up to a fraction θ of the capital held before the investment plus a fraction θ of the newly invested capital. Also the agent cannot short sell land. Thus the flow-of-funds constraint of the investing agent is

$$(1 - \theta q_t)i_t \le p_t m_t + \theta q_t k_t \tag{C.2}$$

where i_t is investment, m_t is land, and k_t is capital owned before the investment. Let b_t be the total assets of the individual:

$$b_t = p_t m_t + q_t k_t \tag{C.3}$$

If the agent has a productive investment opportunity at date t, he can choose whether or not to invest the maximum amount subject to the flow-of-funds constraint. After maximal investment, the agent has sold all his land, and holds a fraction $1 - \theta$ of the capital he held previously plus a fraction $1 - \theta$ of his newly invested capital. Thus, from (C.2), his total assets after the investment are

$$\begin{aligned} b_{t+dt} &= \text{Max}(q_t(1-\theta)(k_t + i_t), b_t) \\ &= \text{Max}\left(\frac{(1-\theta)q_t}{1-\theta q_t}(p_t m_t + k_t), b_t\right) \end{aligned} \tag{C.4}$$

Let r_t^L be the rental price of land and r_t^K be the profit per unit of capital, each per unit of time. Then the accumulation of total assets of the agent between dates t and $t + dt$ is

$$db_t = \left(r_t^L\,dt + dp_t\right)m_t + \left(r_t^K\,dt + dq_t\right)k_t - c_t\,dt + \mathrm{Max}\left(\frac{(1-\theta)q_t}{1-\theta q_t}(p_t m_t + k_t) - b_t,\quad 0\right)dM_t \tag{C.5}$$

where $dM_t = 1$ if the agent has an investment opportunity at date t and $dM_t = 0$ otherwise.

In the following, we concentrate on the case in which Tobin's q exceeds unity so that the credit constraint is binding for the investing agents. Let $V(b, s)$ be the value function of an individual with total assets b when the aggregate state is $s = (K, a)$. Then the Bellman equation is

$$\rho V(b,s) = \underset{c,m,k}{\mathrm{Max}}\Bigg\{ \ln c + V_b(b,s)[(r^L + \dot{p})m + (r^K + \dot{q})k - c]_{da=0} + \pi\left[V\left(\frac{(1-\theta)q}{1-\theta q}(pm+k), s\right) - V(b,s)\right] + \eta[V(p(s')m + q(s')k, s') - V(b,s)]_{da\neq 0}\Bigg\} \tag{C.6}$$

subject to (C.3) and (C.5), where s is the aggregate state at date t and s' is the state at date $t + dt$. The first line in the RHS is the utility of consumption and the value of saving when there is no switch in aggregate productivity. With the same productivity, there is no discontinuous jump in the asset prices and $\dot{x}$ denotes $\frac{dx}{dt}$. The second line is the expected gain in the value with the arrival of a productive investment opportunity, corresponding to Equation (C.4). The last term is the capital gains associated with a switch of the aggregate productivity. The first-order conditions for consumption and portfolio choice are

$$\frac{1}{c} = V_b(b,s) \tag{C.7}$$

(C.8)

$$V_b(b,s)\left[\frac{r^L + \dot{p}}{p} - \frac{r^K + \dot{q}}{q}\right]_{da_t=0} + \eta V_b(p(s')m + q(s')k, s')\left[\frac{p(s')}{p} - \frac{q(s')}{q}\right]_{da_t\neq 0} + \pi V_b\left(\left(\frac{(1-\theta)q}{1-\theta q}\right)(pm+k), s\right)\frac{(1-\theta)q}{1-\theta q}\left(1 - \frac{1}{q}\right) = 0$$

Equation (C.7) says that the marginal utility of consumption must be equal to the marginal value of assets. Equation (C.8) says that the expected rates of return on liquid land and illiquid capital should be equal in terms of utility for an optimal portfolio. In particular, the second line is the expected return from financing the downpayment of investment, in which the rate of return on illiquid capital is $(1/q_t)$ times as much as the rate of return on liquid land. Hence we have

$$V(b_t, s_t) = \tilde{V}(s_t) + \frac{1}{\rho}\ln b_t \tag{C.9}$$

$$c_t = \rho b_t \tag{C.10}$$

$$\left[\frac{r_t^L}{p_t} - \frac{r_t^K}{q_t}\right] + \pi\frac{q_t - 1}{q_t}\frac{p_t m_t + q_t k_t}{p_t m_t + k_t} + \left[\frac{\dot{p}_t}{p_t} - \frac{\dot{q}_t}{q_t}\right]_{da_t=0} + \eta\left[\left(\frac{p_{t+dt}}{p_t} - \frac{q_{t+dt}}{q_t}\right)\frac{P_t m_t + q_t k_t}{p_{t+dt} m_t + q_{t+dt} k_t}\right]_{da_t\neq 0} = 0 \tag{C.11}$$

Because the instantaneous utility function is the logarithm of the consumption rate, the value function is a log-linear function of total assets in Equation (C.3).

In aggregate, since between dates t and $t + dt$ exactly a fraction $\pi\, dt$ of agents invest with binding flow-of-funds constraint (C.2), we get (25). From (C.10), goods market equilibrium is given by (28). And from (C.11) with (27), asset market equilibrium is given by (29). This confirms that the formulation in the text, as a limit of the discrete time model, is consistent with the continuous-time formulation.

Next in this Appendix, we lay out the basic model with an aggregate productivity shock that follows the geometric Brownian motion (38). We first guess that the aggregate output/capital ratio $x_t = a_t K_t^{\alpha-1} L^{1-\alpha}$ and the aggregate productivity a_t summarize the aggregate state of nature. We also guess that the land price/capital ratio, capital price, and investment rate are functions of x_t only:

$$v_t = v(x_t), \quad q_t = q(x_t), \quad \text{and} \quad i_t = i(x_t) \tag{C.12}$$

Next we derive the equilibrium conditions that these functions must satisfy, and then we verify that our initial guess was correct.

Using Ito's lemma with (31) and (32), we have

$$\frac{dx_t}{x_t} = (\alpha - 1)[i(x_t) - \delta]\, dt + \sigma\, dz_t \tag{C.13}$$

Write down the rates of returns on land and capital as

$$\frac{r^L dt + dp}{p} = \frac{(1-\alpha)x\, dt + dv(x)}{v(x)} + [i(x) - \delta]\, dt \equiv r^p(x)\, dt + \omega^p(x)\, dz \tag{C.14a}$$

$$\frac{(r^K - \delta q)\, dt + dq}{q} = \frac{\alpha x\, dt + dq(x)}{q(x)} - \delta dt \equiv r^q(x)\, dt + \omega^q(x)\, dz \tag{C.14b}$$

Let b_t be the total assets as in (C.3) and let h_t be the share of liquid assets in the portfolio, $h_t \equiv p_t m_t / b_t$. Then the budget constraint of the agent (C.5) is

$$db_t = \left(\left[h_t r_t^p + (1-h_t) r_t^q\right] b_t - c_t\right) dt + \left[h_t \omega_t^p + (1-h_t)\omega_t^q\right] b_t \, dz_t + \mathrm{Max}\left(\frac{1-\theta}{1-\theta q_t}(1-h_t+h_t q_t) b_t - b_t, 0\right) dM_t \tag{C.15}$$

The Bellman equation of an agent is

$$\begin{aligned}\rho V(b_t, x_t) = \underset{c_t, h_t}{\mathrm{Max}} \Bigg\{ & \ln c_t + V_b(b_t, x_t)\left(\left[h_t r_t^p + (1-h_t) r_t^q\right] b_t - c_t\right) \\ & + \frac{1}{2} V_{bb}(b_t, x_t)\left[h_t \omega_t^p + (1-h_t)\omega_t^q\right]^2 b_t^2 \\ & + \pi \,\mathrm{Max}\left[V\left(\frac{1-\theta}{1-\theta q_t}(1-h_t+h_t q_t) b_t, x_t\right) - V(b_t, x_t), 0\right]\Bigg\}\end{aligned} \tag{C.16}$$

The optimal investment rule is the same as Tobin's q theory of investment, as in the text. From the first-order conditions, we get the optimal consumption rule and optimal portfolio rule:

$$c_t = \rho b_t \tag{C.17}$$

$$r_t^p - r_t^q - \left(\omega_t^p - \omega_t^q\right)\left[h_t \omega_t^p + (1-h_t)\omega_t^q\right] + \pi \frac{\mathrm{Max}(q_t - 1, 0)}{1 - h_t + h_t q_t} = 0 \tag{C.18}$$

These are very similar to consumption and portfolio rules (C.10) and (C.11), except for the effect of risk due to the difference in the stochastic process of aggregate productivity. From (C.18), we see that the optimal share of liquid assets is

$$\begin{aligned} & h_t = h\left(r_t^p - r_t^q, q_t, \pi, \omega_t^p, \omega_t^q\right), \quad \text{where} \\ & h_{(r^p - r^q)} > 0, \quad h_q \geq 0, \quad h_\pi \geq 0, \quad h_{\omega^p} < 0, \quad \text{and} \quad h_{\omega^q} > 0 \end{aligned} \tag{C.19}$$

The difference from standard portfolio theory is that the liquid asset ratio is a weakly increasing function of Tobin's q and the arrival rate of a productive investment opportunity.

Now we can combine the individuals' behavior with the market-clearing conditions to define equilibrium as price functions $v(x)$, $q(x)$ and investment rate $i(x)$ that satisfy

(C.20) $$[1-\theta q(x)]i(x) \leq \pi[v(x)+\theta q(x)], \quad \text{where} \quad \begin{cases} = \text{holds}, & \text{if } q(x) > 1 \\ \leq \text{holds}, & \text{if } q(x) = 1 \\ i(y) = 0, & \text{if } q(x) < 1 \end{cases}$$

(C.21) $$x = i(x) + \rho[v(x) + q(x)]$$

(C.22) $$[r^p(x) - r^q(x)] - [\omega^p(x) - \omega^q(x)]\frac{v(x)\omega^p(x) + q(x)\omega^q(x)}{v(x) + q(x)} + \pi\frac{\text{Max}[q(x)-1, 0]}{q(x)}\frac{v(x)+q(x)}{v(x)+1} = 0$$

where the returns characteristics $r^p(x)$, $r^q(x)$, $\omega^p(x)$, and $\omega^q(x)$ are defined in (C.14). The stochastic process of a_t and x_t follow (38) and (C.13). Equation (C.20) describes the behavior of aggregate investment. Equation (C.21) describes the goods market equilibrium, and (C.22) describes the asset market equilibrium. In (C.22), the first term is the difference of the expected rates of return on the liquid asset and illiquid capital, the second is the effect of risk aversion, and the last is the expected advantage of the liquid asset over illiquid capital for financing productive investment. This last term distinguishes our model from a standard capital asset-pricing model. All the above equilibrium conditions are functions of the output/capital ratio x_t only, and aggregate productivity together with the output/capital ratio are sufficient statistics for the aggregate state of the economy. Our initial conjecture was correct.

This system is not much more complicated than a real business cycle model, or Merton (1975). Thus, in principle we can simulate the above system to examine its dynamics. Alternatively, if $v(x)$ and $q(x)$ were three-times differentiable, we would know from Ito's Lemma that

$$r^p(x) = \frac{(1-\alpha)x}{v(x)} + \left(1 + \frac{xv'(x)}{v(x)}(\alpha - 1)\right)[i(x) - \delta] + \frac{1}{2}\sigma^2\frac{x^2v''(x)}{v(x)}$$

$$r^q(x) = \frac{\alpha x}{q(x)} - \delta + \frac{xq'(x)}{q(x)}(\alpha - 1)[i(x) - \delta] + \frac{1}{2}\sigma^2\frac{x^2q''(x)}{q(x)}$$

$$\omega^p(x) = \sigma\frac{xv'(x)}{v(x)}$$

$$\omega^q(x) = \sigma\frac{xq'(x)}{q(x)}$$

However, such a property may not hold here, because the pattern of investment differs qualitatively depending on the value of $q(x)$—unless the liquidity constraint is always biding. But, here it is not guaranteed that the liquidity constraint does always bind, given that aggregate productivity follows a Brownian motion. Obviously, more study is needed before we fully understand this economy.

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