Controller Switching Based on Output Prediction Errors

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***Abstract*—We consider a switched nonlinear feedback control strategy for controlling a plant with unknown parameters so that the output asymptotically tracks a reference signal. The controller is selected on-line from a given set of controllers according to a switching rule based on output prediction errors. For control problems requiring asymptotic tracking of a reference input we provide sufficient conditions under which the switched closed-loop control system is exponentially stable and asymptotically achieves good control even if the switching does not stop. Our results are illustrated with three examples.**

***Index Terms*— Adaptive control, hybrid control, supervisory control, switching.**

I. Introduction

We examine the problem of controlling a fixed linear continuous-time plant Σ with unknown parameters so that the plant output asymptotically tracks, with some desired accuracy, a bounded reference input. The control strategy that we analyze is based on switching among a family of fixed controllers at certain decision times based on a switching logic that attempts to select a good predictor for the plant.

There are several elements to the controller architecture. First, we have available a family of model-controller pairs (Σ_p, Γ_p), $p \in \mathcal{P}$. The index set $\mathcal{P}$ may be finite, countable, or a compact subset of some metric space. Controller Γ_p stabilizes model Σ_p and yields desired asymptotic tracking performance for a class of admissible reference signals. These requirements will be made precise in Section II. For each model we run a corresponding predictor O_p driven by the inputs and outputs of the plant, and the resultant prediction errors are used to form a real-valued performance measure for predictor p. Then at certain decision times, a supervisor uses the performance measures to select a controller from the family $\{\Gamma_p, p \in \mathcal{P}\}$ to be connected in feedback with the plant. The resultant nonlinear switched control system must ensure boundedness of the process states and satisfy an asymptotic tracking performance criterion; see Fig. 1.

We consider the simplest case in which the transfer function of the unknown plant exactly matches that of one of the known deterministic models $\{\Sigma_p,\ p \in \mathcal{P}\}$. This may be regarded as a case of purely parametric uncertainty. Although unrealistic,

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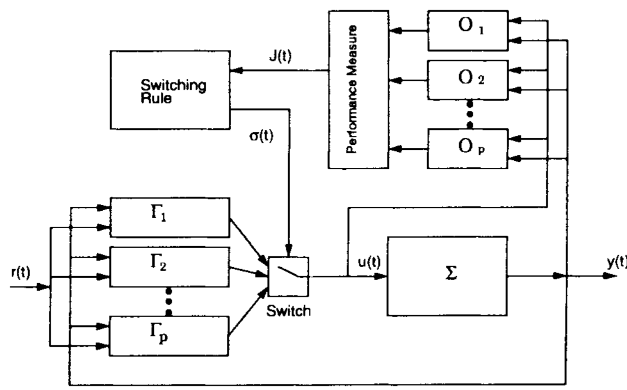

Fig. 1. A prediction error-based switched control system.

this situation is of theoretical interest since it provides a lower bound on what can be expected in practice.

The above architecture for on-line controller switching has been proposed and examined in several special cases in [12]–[14], [7], [11], [18], and [2]. In [12] and [13] the problem of tracking a constant set point for single-input/single-output (SISO) linear time-invariant (LTI) systems is studied. At a sequence of decision times, the performance of the predictors is compared, and the controller corresponding to the best predictor at that time is selected. The sequence of selections is not required to converge and in general will not do so. Nevertheless, the system variables remain bounded and the output of the SISO system converges to the constant set-point. In [11] and [18] switching is used to select a controller structure matched to the similarity invariants of the plant, and in [2], [15], and [16] it is used to improve the transient performance of stable adaptive control schemes. In [7] convergent decision rules are studied. It is shown that for multi-input/multi-output (MIMO) LTI systems there exists a convergent selection rule under which the supervised control system is stable and satisfies a performance criterion with respect to a class of admissible inputs. Several other controller switching strategies have been examined in the literature. Generally, these involve strategies that use a predefined search sequence, e.g., [5], [9], and [10]. Roughly, these operate by switching into feedback with the plant a predefined sequence of controllers in which each controller appears infinitely often.

The main contributions of this paper are as follows.

- We examine an asymptotic tracking problem in a general setting that extends previous work on set-point control problems.

- We show that the output of the prediction-based supervised control system is the sum of the output of a time-varying system in which at every moment Σ_p is controlled by the concurrent controller Γ_p and a prediction error term. The stability of the above-mentioned time-varying system can be assured using standard results from the literature on slowly time-varying systems [4], [3], [6], [17].
- Our first result, Theorem 4.3, provides sufficient conditions on the models and controllers such that even if switching between the candidate controllers does not stop, selecting a good predictor will imply good tracking control. These conditions are illustrated with three examples. As a special case, we provide a more direct proof of a result given in [13].
- Finally, in our second result, Theorem 5.1, we show that, under a mild additional assumption, the key condition in our set of sufficient conditions is always satisfied if the switched control system is required to asymptotically exactly track the reference signal.

II. Formulation

We can select the input $u(t)$ and observe the output $y(t)$ of an unknown SISO system Σ, hereafter called the "plant," with McMillan degree at most n and a stabilizable and detectable $\bar{n}$-dimensional state space realization

$$\Sigma\colon\quad \begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t), \qquad x(t_0) = x_0\\ y(t) &= Cx(t).\end{aligned} \tag{1}$$

The objective is to select the input $u(t)$ so that the output $y(t)$ asymptotically adequately tracks a reference signal $r(t)$ generated as the output of a finite-dimensional autonomous linear system of the form

$$\Xi\colon\quad \begin{aligned}\dot{\phi}(t) &= f\phi(t), \qquad \phi(0) = \phi_0\\ r(t) &= g\phi(t)\end{aligned} \tag{2}$$

where for each initial condition ϕ_0 the state trajectory of Ξ is bounded. The reference signal is thus bounded and completely specified by the selection of an initial condition for Ξ.

We are given (or have constructed) a family of LTI systems $O_p, p \in \mathcal{P}$, with a common state realization of the form

$$O_p\colon\quad \begin{aligned}\dot{w}(t) &= Mw(t) + Nu(t) + Ky(t), \qquad w(t_0) = w_0\\ \hat{y}_p(t) &= C_p w(t)\\ e_p(t) &= C_p w(t) - y(t)\end{aligned} \tag{3}$$

where $w(t) \in \mathbf{R}^{k_1}$, $u(t) \in \mathbf{R}$, $y(t) \in \mathbf{R}$; the dimensions of the matrices M, N, K, and C_p are appropriate. This system will be used as a predictor of the plant output. The predictor outputs are: $\hat{y}_p$, the prediction of $y(t)$, and e_p, the corresponding prediction error, $p \in \mathcal{P}$.

If we set $y(t) = \hat{y}_p(t)$ in O_p, then we obtain a realization of an LTI system Σ_p, $p \in \mathcal{P}$

$$\Sigma_p\colon\quad \begin{aligned}\dot{x}_p(t) &= A_p x_p(t) + Nu(t), \qquad x_p(t_0) = x_0\\ y_p(t) &= C_p x_p(t)\end{aligned} \tag{4}$$

where $A_p \triangleq M + KC_p$. We call Σ_p the pth system model.

Similarly, we are given (or have constructed) a family of LTI systems Δ_p, $p \in \mathcal{P}$, with a common state realization of the form

$$\Delta_p\colon\quad \begin{aligned}\dot{z}(t) &= Fz(t) + Gy(t) + Lu(t) + Rr(t), \qquad z(t_0) = z_0\\ u_p(t) &= H_p z(t) + S_p y(t) + T_p r(t)\end{aligned} \tag{5}$$

where $z(t) \in \mathbf{R}^{k_2}$ and the dimensions of F, G, L, R, H_p, S_p, and T_p are appropriate. Under the feedback connection $u(t) = u_p(t)$ in Δ_p we obtain a realization of an LTI system Γ_p which we will call the pth controller.

The assumption of a common state realization for the predictors and controllers is standard in adaptive control; see for example [13] and [14]. One such realization can be constructed as follows. Set

$$M = \begin{pmatrix} A_I & 0\\ 0 & A_I\end{pmatrix}, \qquad N = \begin{pmatrix} 0\\ b_I\end{pmatrix}, \qquad K = \begin{pmatrix} b_I\\ 0\end{pmatrix}$$

where (A_I, b_I) is a parameter-independent n-dimensional controllable pair, and A_I is stable. For simplicity suppose that (A_I, b_I) is in controllable canonical form. By choice of C_p we can ensure that $\{C_p, A_p, N\}$ is detectable and stabilizable and is a realization of any given strictly proper transfer function $h_p(s) = b_p(s)/a_p(s)$ with McMillan degree at most n. Detectability follows immediately from the stability of A_I and the fact that $A_p = M + KC_p$. To verify stabilizability and that we can select C_p to achieve $h_p(s)$, compute the transfer function of $\{C_p, A_p, N\}$

$$\begin{aligned}C_p(sI - A_p)^{-1}N &= \frac{C_p^2(sI - A_I)^{-1}b_I}{1 - C_p^1(sI - A_I)^{-1}b_I}\\ &= \frac{\beta_p(s)/\omega(s)}{(\omega(s) - \alpha_p(s))/\omega(s)} = \frac{b_p(s)}{a_p(s)}\end{aligned}$$

where $C_p \triangleq (C_p^1, C_p^2)$, $\omega(s) \triangleq \det(sI - A_I)$, and $\deg(\beta_p) \le n-1$, $\deg(\alpha_p) = n$. Since $\deg(a_p) \le n$, it is possible to select C_p such that $\beta_p(s) = \delta(s) \cdot b_p(s)$ and $\omega(s) - \alpha_p(s) = \delta(s) \cdot a_p(s)$ where $\delta(s)$ is a stable polynomial. Thus $\{C_p, A_p, N\}$ is stabilizable with transfer function $h_p(s)$.

Similarly, a common state realization of the controllers can be obtained by setting

$$F = \begin{pmatrix} A_I & 0 & 0\\ 0 & A_I & 0\\ 0 & 0 & A_I\end{pmatrix}, \qquad K = \begin{pmatrix} b_I\\ 0\\ 0\end{pmatrix}$$

$$L = \begin{pmatrix} 0\\ b_I\\ 0\end{pmatrix}, \qquad R = \begin{pmatrix} 0\\ 0\\ b_I\end{pmatrix}.$$

Then selecting $u = u_p$ yields

$$\begin{aligned}U_p(s) &= \frac{H_p(sI - A_I)^{-1}b_I}{1 - S_p(sI - A_I)^{-1}b_I}Y(s)\\ &\quad + \frac{T_p(sI - A_I)^{-1}b_I}{1 - S_p(sI - A_I)^{-1}b_I}R(s).\end{aligned} \tag{6}$$

Thus by suitable choice of H_p, S_p, and T_p, Γ_p is a stabilizable and detectable realization of any given strictly proper transfer function $g_p(s) = [b_p^1(s)/c(s), b_p^1(s)/c(s)]$ with McMillan degree at most n.

The equations of the system (Σ_p, Γ_p) consisting of Σ_p in feedback connection with Γ_p are

$$\begin{aligned} \dot{x}_{pp}(t) &= A_{pp}x_{pp}(t) + \begin{pmatrix} R + LT_p \\ NT_p \end{pmatrix} r(t) \\ y_{pp}(t) &= (0, C_p)x_{pp}(t) \end{aligned} \tag{7}$$

where

$$A_{pp} \triangleq \begin{pmatrix} F + LH_p & GC_p + LS_pC_p \\ NH_p & M + KC_p + NS_pC_p \end{pmatrix}. \tag{8}$$

The models and controllers, parameterized as indicated above, will be assumed to satisfy the following constraints.

Assumption A1) The matrices M and F are Hurwitz.

Assumption A2) The matrix A_{pp} is Hurwitz with stability margin γ.

Assumption A3) The controlled system (Σ_p, Γ_p) yields acceptable asymptotic tracking performance over the admissible class of reference signals.

Assumption A4) $\mathcal{P}$ is a compact metric space with metric $d(\cdot, \cdot)$.

Assumption A5) The functions f_1: $p \mapsto C_p$, f_2: $p \mapsto H_p$, f_3: $p \mapsto S_p$, and f_4: $p \mapsto T_p$ are continuous with respect to the metric on $\mathcal{P}$ and any matrix norm.

Together, A3) and A2) impose the constraint that the pth controller should stabilize the pth model with a stability margin γ that is independent of p and satisfy the asymptotic tracking criterion. It will not be necessary at this point to give a specific criterion for acceptable asymptotic tracking. However, as examples of suitable criteria we mention: $\lim_{t\to\infty} |y(t) - r(t)| = 0$ (asymptotic exact tracking) and $\limsup_{t\to\infty} |y(t) - r(t)| \leq \epsilon$ (asymptotic ϵ-tracking). Other criteria are also clearly possible. Assumption A4) can be satisfied by the discrete metric if $\mathcal{P}$ is finite. If $\mathcal{P}$ is a closed and bounded subset of $\mathbf{R}^k$ for some positive integer k, then we might take $d(\cdot, \cdot)$ to be the metric induced by a suitable norm on $\mathbf{R}^k$. If $\mathcal{P}$ is finite and $d(\cdot, \cdot)$ is the discrete metric, then Assumption A5) is trivially satisfied. However, when $\mathcal{P}$ is a compact subset of $\mathbf{R}^k$, it is a nontrivial assumption—it requires that we have designed the controllers so that they satisfy A2) and A3) *and* vary continuously with respect to $p \in \mathcal{P}$. To show that such parameterizations exist it is sufficient to assume that there exists a design procedure for determining a controller transfer function $g(s)$ from a stabilizable and detectable plant realization such that A2) and A3) are satisfied and such that the parameters in $g(s)$ vary continuously as the parameters in the plant realization vary over some open set containing their nominal values. In this case, we can take p to be the vector of entries in C_p and f_1: $p \mapsto C_p$ is then obviously continuous. Suppose that p_o is the parameter of a nominal plant; then, Σ_{p_o} is a stabilizable and detectable realization of the corresponding plant transfer function. Then let $\mathcal{P}_o$ be the closure of any open ball containing p_o such that $\{\Sigma_p$: $p \in \mathcal{P}_o\}$ is contained in an open set about Σ_{p_o} on which the controller design procedure is continuous. Then let the maps f_2: $p \mapsto H_p$, f_3: $p \mapsto S_p$, and f_4: $p \mapsto T_p$ select the appropriate values of H_p, S_p, and T_p from the designed controller transfer function using the method indicated in (6). By choice of $\mathcal{P}_o$ these functions are continuous. More generally, we can take $\mathcal{P} = \bigcup_{k=1}^{N} \mathcal{P}_k$ where the $\mathcal{P}_k$ are constructed like $\mathcal{P}_o$.

We now describe how the given controllers are connected to the plant. At each term τ_k of a strictly monotone increasing sequence of switching times (that may depend on the initial condition of the plant, predictor, and controller), the controller connected to the plant will be "switched" among the family parameterized by $p \in \mathcal{P}$. Given $\tau_D > 0$ we require that the sequence $\{\tau_k, k \geq 0\}$ satisfies $\tau_{k+1} - \tau_k \geq \tau_D$, for each $k \geq 0$. Such a sequence is said to be τ_D- *admissible*. It will not be important exactly how the switching times are selected, only that they satisfy this form of constraint.

The selection of which controller to be connected into feedback with the plant at switching time τ_k is based on performance indexes for each of the predictors. The performance index $J(t, p)$ of predictor p at time t is a function of the prediction error signal $e_p(s)$, $s \in [0, t]$. For example, for fixed $\lambda > 0$ we might set

$$J(t, p) = \int_0^t e^{-\lambda(t-s)} |e_p(s)|^2 \, ds. \tag{9}$$

Then at each switching time τ_k, an index $q_k \in \mathcal{P}$ is selected based on the values $\{J(t, p), t \in [0, \tau_k], p \in \mathcal{P}\}$ according to specified decision rule. The controller Γ_{q_k}, driven by the reference input r, is then connected in feedback with the plant over the time interval $[\tau_k, \tau_{k+1})$. The simplest instance of a decision rule is a fixed function $g: \mathbf{R}^{\mathcal{P}} \to \mathcal{P}$ with $q_k = g(J(\tau_k, p), p \in \mathcal{P})$. For example, the rule used in [13] is (roughly)

$$q_k = \operatorname{argmin}_{p\in\mathcal{P}} \{J(\tau_k, p)\}. \tag{10}$$

More complex rules can easily be envisioned; see, e.g., [7] and [11]. However, all that will be important for our investigation is that the rule has certain basic properties.

Let $\sigma(t)$ denote the piecewise-constant signal taking values in $\mathcal{P}$ that specifies the controller in use at time t, and let $\bar{\sigma}$ denote its set of limit points in $\mathcal{P}$. By Assumption A4), $\bar{\sigma}$ is nonempty. Let $\mathcal{P}^* \subseteq \mathcal{P}$ denote the set of predictors for which the prediction error decays to zero along the system trajectory, i.e., $\mathcal{P}^* = \{p \in \mathcal{P}$: $e_p \to 0\}$. Clearly $\bar{\sigma}$ and $\mathcal{P}^*$ depend on the initial conditions.

We restrict our attention to switching rules that satisfy one or more of the following.

Assumption R0) For all initial states of the plant and the predictor, $e_{\sigma(t)} \to 0$ as $t \to \infty$.

Assumption R1) There exist constants $C, \alpha > 0$ such that for all plant initial states $x(0)$ and predictor initial states $w(0)$, $|e_{\sigma(t)}(t)| < Ce^{-\alpha t}(\|x(0)\| + \|w(0)\|)$.

Assumption R2) $\bar{\sigma} \subseteq \mathcal{P}^*$.

Assumptions R0)–R2) require that the performance measures and decision rule result in the selection of a "good" predictor. Clearly, Assumption R1) implies R0). Assumption R2) requires that if $q \in \bar{\sigma}$, then $e_q(t) \to 0$ as $t \to \infty$, i.e., the only predictors that are selected "in the limit" are those that do "good prediction" along the state trajectory. Since $\bar{\sigma}$ is nonempty, under Assumption R2) so is $\mathcal{P}^*$, and under Assumptions A4) and A5), R2) implies R0).

These assumptions implicitly define classes of switching rules and performance measures. The question of whether these classes are nontrivial (e.g., nonempty) is closely connected to the stability of the closed-loop controlled system, and we defer its discussion until after the section on stability.

In summary, the closed-loop switched system is described by the following set of equations:

$$\tilde{x}(t) = \tilde{A}_{\sigma(t)}\tilde{x}(t) + \tilde{R}_{\sigma(t)}r(t) \quad \tilde{x}(0) = \tilde{x}_0 \tag{11}$$

$$y(t) = \tilde{C}\tilde{x}(t) \tag{12}$$

$$e_{\sigma(t)}(t) = \tilde{E}_{\sigma(t)}\tilde{x}(t) \tag{13}$$

$$\dot{\phi}(t) = f\phi(t), \qquad \phi(0) = \phi_0 \tag{14}$$

$$r(t) = g\phi(t) \tag{15}$$

$$J(t,p) = h_t(e_p(s)|_{s=0}^t) \tag{16}$$

$$\sigma(t) = g_k(J(s,p)|_{s=0}^{\tau_k}, p \in \mathcal{P}), \quad \text{if } t \in [\tau_k, \tau_{k+1}). \tag{17}$$

In the above

$$\begin{aligned}\tilde{x}(t) &= \begin{pmatrix} x(t) \\ z(t) \\ w(t) \end{pmatrix} \\ \tilde{A}_{\sigma(t)} &= \begin{pmatrix} A + BS_{\sigma(t)}C & BH_{\sigma(t)} & 0 \\ GC + LS_{\sigma(t)}C & F + LH_{\sigma(t)} & 0 \\ KC + NS_{\sigma(t)}C & NH_{\sigma(t)} & M \end{pmatrix} \\ \tilde{R}_{\sigma(t)} &= \begin{pmatrix} BT_{\sigma(t)} \\ R + LT_{\sigma(t)} \\ NT_{\sigma(t)} \end{pmatrix} \\ \tilde{C} &= (C, \quad 0, \quad 0) \\ \tilde{E}_{\sigma(t)} &= (-C, \quad 0, \quad C_{\sigma(t)})\end{aligned} \tag{18}$$

and $h_t, t \geq 0$ are functions mapping continuous real-valued functions on the interval $[0,t]$ into the real line, and g_k is a sequence of functions mapping $\mathbf{R}^{\mathcal{P}\times[0,\tau_k]}$ into $\mathcal{P}$.

In the sequel it will be necessary to consider the joint trajectories of several state-space systems. If $x_1(t) \in \mathbf{R}^{k_1}$ and $x_2(t) \in \mathbf{R}^{k_2}$ are vector valued signals, then the notation $(x_1(t), x_2(t))$ will denote the vector in $\mathbf{R}^{k_1+k_2}$ formed by concatenating the vectors x_1 and x_2.

III. Stability Analysis

To show that the nonlinear system (11)–(17) is exponentially stable, we analyze the system for each initial state and its corresponding switching signal. Following [7] we first show that along any trajectory of the closed-loop system the plant output is the sum of two separate signals: one is the response of a switched linear system to the reference input r, and the other is the zero state response of the same system to a prediction error related disturbance. This leads to the following result.

Proposition 3.1: Let $\tilde{x}(t) = (x(t), z(t), w(t))$ be the state trajectory of the closed-loop system (11)–(17) with initial condition $\tilde{x}_0 = (x_0, w_0, z_0)$, and let σ be the associated switching signal. Then along this trajectory

$$\begin{aligned}\begin{pmatrix} z(t) \\ w(t) \end{pmatrix} &= x_s^\sigma(t) + x_s^{\sigma,e}(t) \\ y(t) &= y_s^\sigma(t) + \varepsilon(t), \qquad t \geq 0\end{aligned} \tag{19}$$

where x_s^σ and y_s^σ are the state and output, respectively, of the switched linear system $(\Sigma_{\sigma(t)}, \Gamma_{\sigma(t)})$ with input r and initial condition (z_0, x_0), and $x_s^{\sigma,e}$ and ε are the state and output of this system to a disturbance $e_{\sigma(t)}(t)$.

Proof: Since the switching signal is fixed, write $\tilde{A}(t) = \tilde{A}_{\sigma(t)}$, $\tilde{R}(t) = \tilde{R}_{\sigma(t)}$, and $\tilde{E}(t) = \tilde{E}_{\sigma(t)}$. Then applying algebraic manipulations to (11) yields

$$\begin{aligned}\tilde{x}(t) &= (\tilde{A}(t) + \tilde{K}(t)\tilde{E}(t))\tilde{x}(t) - \tilde{K}(t)\tilde{E}(t)\tilde{x}(t) + \tilde{R}(t)r(t) \\ &= (\tilde{A}(t) + \tilde{K}(t)\tilde{E}(t))\tilde{x}(t) - \tilde{K}(t)e_{\sigma(t)}(t) + \tilde{R}(t)r(t).\end{aligned} \tag{20}$$

Let $\tilde{K}(t) \triangleq (\bar{K} + BS_{\sigma(t)}, G + LS_{\sigma(t)}, K + NS_{\sigma(t)})$. Then

$$\begin{aligned}\tilde{A}_2(t) &\triangleq \tilde{A}(t) + \tilde{K}(t)\tilde{E}(t) \\ &= \begin{pmatrix} A - \bar{K}C & BH_{\sigma(t)} & \bar{K}C_{\sigma(t)} + BS_{\sigma(t)}C_{\sigma(t)} \\ 0 & F + LH_{\sigma(t)} & GC_{\sigma(t)} + LS_{\sigma(t)}C_{\sigma(t)} \\ 0 & NH_{\sigma(t)} & M + KC_{\sigma(t)} + NS_{\sigma(t)}C_{\sigma(t)} \end{pmatrix}.\end{aligned}$$

Write $\tilde{C}_2(t) = \tilde{E}(t) + \tilde{C}(t)$, and let $\tilde{x}_1(t)$, $\varepsilon(t)$ be the solutions to the equations

$$\begin{aligned}\tilde{x}_1(t) &= \tilde{A}_2(t)\tilde{x}_1(t) - \tilde{K}(t)e_{\sigma(t)}(t), \qquad \tilde{x}_1(0) = 0 \\ \varepsilon(t) &= \tilde{C}_2(t)\tilde{x}_1(t) - e_{\sigma(t)}(t)\end{aligned} \tag{21}$$

and let $\tilde{x}_2(t)$, $y_s^\sigma(t)$ be the solutions of

$$\begin{aligned}\tilde{x}_2(t) &= \tilde{A}_2(t)\tilde{x}_2(t) + \tilde{R}(t)r(t), \qquad \tilde{x}_2(0) = \tilde{x}(0) \\ y_s^\sigma(t) &= \tilde{C}_2(t)\tilde{x}_2(t).\end{aligned} \tag{22}$$

It is clear that $\tilde{x}(t) = \tilde{x}_1(t) + \tilde{x}_2(t)$ and $y(t) = y_s^\sigma(t) + \varepsilon(t)$.

Writing (21) and (22) out in detail, we see that ε and y_s^σ are the solutions of

$$\begin{aligned}\dot{x}_s^{\sigma,e}(t) &= A_{\sigma(t)\sigma(t)}x_s^{\sigma,e}(t) - \begin{pmatrix} G + LS_{\sigma(t)} \\ K + NS_{\sigma(t)} \end{pmatrix} e_{\sigma(t)}(t) \\ x_s^{\sigma,e}(0) &= (0,0) \\ \varepsilon(t) &= \big(0, C_{\sigma(t)}\big)x_s^{\sigma,e}(t) - e_{\sigma(t)}(t)\end{aligned} \tag{23}$$

$$\begin{aligned}\dot{x}_s^\sigma(t) &= A_{\sigma(t)\sigma(t)}x_s^\sigma(t) + \begin{pmatrix} R + LT_{\sigma(t)} \\ NT_{\sigma(t)} \end{pmatrix} r(t) \\ x_s^s(0) &= (z_0, w_0) \\ y_s^\sigma(t) &= (0, C_{\sigma(t)})x_s^\sigma(t)\end{aligned} \tag{24}$$

where $(z(t), w(t)) = x_s^\sigma(t) + x_s^{\sigma,e}(t)$ and

$$A_{\sigma(t)\sigma(t)} = \begin{pmatrix} F + LH_{\sigma(t)} & GC_{\sigma(t)} + LS_{\sigma(t)}C_{\sigma(t)} \\ NH_{\sigma(t)} & M + KC_{\sigma(t)} + NS_{\sigma(t)}C_{\sigma(t)} \end{pmatrix}$$

is the dynamical matrix of the closed-loop system $(\Sigma_{\sigma(t)}, \Gamma_{\sigma(t)})$. □

The reduction indicated in Proposition 3.1 is implicitly used in the work of Morse (see, e.g., [13]), but not stated there in this form. It is also used in [7] and [8] in a stronger context.

In light of Proposition 3.1, we now examine the stability of the time-varying linear system $(\Sigma_{\sigma(t)}, \Gamma_{\sigma(t)})$ for each admissible switching signal $\sigma(t)$.

Recall that a time-varying linear system $\dot{x}(t) = A(t)x(t)$ is exponentially stable if there exist constants $k_1, k_2 > 0$ such that for all $t \geq \mu \geq 0$, $\|\Phi(t,\mu)\| \leq k_1 e^{-k_2(t-\mu)}$, where $\|\Phi(\cdot,\cdot)\|$ denotes the state transition matrix of the system. In the case at hand, $A(t) = A_{\sigma(t)\sigma(t)}$ is piecewise constant and by Assumption A2) for each $p \in \mathcal{P}$, there exist constants $a_p \geq 0$ and $\lambda_p \geq \gamma/2$ such that for all $t \geq 0$, $\|e^{A_{pp}t}\| \leq e^{(a_p - \lambda_p t)}$. Sufficient (conservative) conditions under which a linear time-varying system of this form is exponentially stable are given in the following lemma.

Lemma 3.2: Assume that condition A2) holds. If one of the following conditions is satisfied:

1) the finite dwell time satisfies $\tau_D > \sup_{p\in\mathcal{P}}\{\frac{a_p}{\lambda_p}\}$;
2) $\sup_{p\in\mathcal{P}} \|A_{pp}\| < (2\sup_{p\in\mathcal{P}}\{\frac{a_p}{\lambda_p}\}\exp[\sup_{p\in\mathcal{P}}\{a_p\}])^{-1}$;

then the time-varying linear system $(\Sigma_{\sigma(t)}, \Gamma_{\sigma(t)})$ is exponentially stable for any admissible switching signal $\sigma(t)$. Moreover, the constants in the exponential bound do not depend on the switching signal σ.

Proof: For the standard result, see, e.g., [3], [4], [6], [13], and [17]; also see Appendix 1. □

Fix an initial condition ϕ_0 of Ξ. Then, each initial condition of the form (w_1, z_1, ϕ_0) for the asymptotically stable LTI system $(\Sigma_p, \Gamma_p, \Xi)$ gives rise to a trajectory with a nonempty positive limit set $\Omega(w_1, z_1, \phi_0)$. Moreover, by the asymptotic stability of the above system this limit set depends only on ϕ_0. The next lemma gives a trivial result on the convergence of the state trajectory of $(\Sigma_{\sigma(t)}, \Gamma_{\sigma(t)}, \Xi)$ that will be useful later in the development.

Lemma 3.3: Assume that A2) and one of the conditions of Lemma 3.2 holds. Suppose that for a fixed initial condition ϕ_0 of Ξ, the LTI systems $(\Sigma_q, \Gamma_q, \Xi)$, $q \in Q \subseteq \mathcal{P}$ have a common ω-limit set $\bar{\Omega}$. Then if $\sigma(t)$ takes values in Q, and the initial condition of Ξ is fixed to be ϕ_0, the set $\bar{\Omega}$ is the unique ω-limit set for the time-varying linear system $(\Sigma_{\sigma(t)}, \Gamma_{\sigma(t)}, \Xi)$.

Proof: Without loss of generality, we can assume that the state trajectory $\phi(t)$ of Ξ is almost periodic. Hence, if Ω^ϕ denotes the set of limit points of $\phi(t)$, then $\phi_0 \in \Omega^\phi$. We begin by noting the following properties of $\bar{\Omega}$. First, for the fixed initial condition ϕ_0, $\bar{\Omega}$ is an invariant set of $(\Gamma_p, \Sigma_p, \Xi)$, $p \in Q$. Second, if $(z, w, \phi_0) \in \bar{\Omega}$, then the ω-limit set of the trajectory of $(\Gamma_p, \Sigma_p, \Xi)$ starting from (z, w, ϕ_0) is $\bar{\Omega}$.

Now for any initial time $\tau \geq 0$, $\bar{\Omega}$ is easily seen to be an invariant set of $(\Sigma_{\sigma(t)}, \Gamma_{\sigma(t)}, \Xi)$. For $(z_1, w_1, \phi_0) \in \bar{\Omega}$, let $(z_1(t), w_1(t), \phi(t)) \in \bar{\Omega}$ be the state trajectory of $(\Sigma_{\sigma(t)}, \Gamma_{\sigma(t)}, \Xi)$ from this initial condition at $t = 0$. Similarly, let $(z_2(t), w_2(t), \phi(t))$ denote the state trajectory of the system from any other initial condition (z_2, w_2, ϕ_0).

Set $e(t) = (z_1(t) - z_2(t), w_1(t) - w_2(t))$. Then by Lemma 3.2, $\lim_{t\to\infty} e(t) = 0$. Thus the ω-limit set of $(z_2(t)$, $w_2(t)$, $\phi(t))$ is the same as the ω-limit set of $(z_1(t)$, $w_1(t)$, $\phi(t))$. The latter set is $\bar{\Omega}$. □

We now use Proposition 3.1 and Lemma 3.2 to show the stability of the switched nonlinear system.

Proposition 3.4: If A2) and one of the conditions of Lemma 3.2 hold, then for every initial state of the nonlinear switched system (11)–(17):

1) if $\mathbf{e}_{\sigma(t)}(t)$ is bounded, then $\tilde{x}$ is bounded;
2) if R0) holds, then $\lim_{t\to\infty} \|(z(t), w(t)) - x_s^\sigma(t)\| = 0$ and $\lim_{t\to\infty} \|y(t) - y_s^\sigma(t)\| = 0$;
3) if R1) holds, then that the nonlinear closed-loop system is globally exponentially stable in the sense that there exist constants $C, \beta > 0$ such that with $r \equiv 0$ and for all initial conditions, the state trajectory $\tilde{x}(t)$ of (11) satisfies $\|\tilde{x}(t)\| \leq Ce^{-\beta t}\|\tilde{x}(0)\|$.

Proof: Fix an initial condition $\tilde{x}_0$, and let $\sigma(t)$ denote the resultant switching signal. Under A2) and either of the conditions of Lemma 3.2, there exists $\bar{K}$ such that the time-varying linear systems (21) and (22) are exponentially stable. Furthermore, the constants in the exponential bound do not depend on the initial condition $\tilde{x}_0$.

1) $\tilde{x}_1$ and $\tilde{x}_2$ are the state trajectories of an exponentially stable linear system to bounded inputs and are hence bounded. Thus $\tilde{x} = \tilde{x}_1 + \tilde{x}_2$ is bounded.
2) Under assumption R0), $e_{\sigma(t)}(t) \to 0$ as $t \to \infty$. Hence by (22) and exponential stability, $\lim_{t\to\infty} \|\tilde{x}_1(t)\| = 0$, or equivalently, $\lim_{t\to\infty} \|\tilde{x}_2(t) - \tilde{x}(t)\| = 0$. Now x_s^σ, given by (24), is just the second and third vector components of the state $\tilde{x}_2$ of (22). Hence $x_s^\sigma(t)$ converges to $(z(t), w(t))$ as claimed. Similarly, that the second limit is zero follows from the first limit and (21) and (19).
3) By R1), $e_{\sigma(t)}(t) \leq Ce^{-\alpha t}\|\tilde{x}(0)\|$. Hence by (21) and exponential stability, there exist constants $K_1, \alpha_1 > 0$ such that $\|\tilde{x}_1(t)\| \leq K_1 e^{-\alpha_1 t}\|\tilde{x}(0)\|$. Moreover, K_1 and α_1 do not depend on $\tilde{x}(0)$. Since the switching signal is fixed and $r \equiv 0$, $\tilde{x}_2(t)$ is the zero input response of the exponentially stable system (22) with the initial condition $\tilde{x}(0)$. Thus $\|\tilde{x}_2(t)\| \leq Ke^{-\lambda t}\|\tilde{x}(0)\|$. Since K and λ do not depend on σ, $\|\tilde{x}_2(t)\| \leq Ke^{-\lambda t}\|\tilde{x}(0)\|$ for any initial state of the nonlinear closed-loop system. It was shown above that $\tilde{x}(t)$ is the sum of $\tilde{x}_2(t)$ and $\tilde{x}_1(t)$. Hence $\tilde{x}(t)$ converges to zero at an exponential rate independent of the initial condition. □

A. Discussion of Assumptions R0)–R2)

We end this section with an additional discussion of assumptions R0)–R2). Our objective is to use the above stability result to show that there exist nontrivial performance measures and switching rules that will satisfy assumption R2). We do so by analyzing a particular choice of performance measure and switching rule but do not claim that the case analyzed is practical; it is simply intended to show that assumption R2) is not vacuous.

Let the plant transfer function be equal to that of one of the models. Then by Assumption A1) and elementary linear systems theory, at least one of the predictors, say p^*, satisfies $|e_{p*}(t)| < Ke^{-\delta t}$. Let the switching times be equally spaced and consider the performance measures

$$J(t,p) = |e_p(t)|^2 + \int_0^t |e_p(s)|^2\, ds$$

with the decision rule (10). First note that

$$\begin{aligned} J(t,p^*) &= |e_{p^*}(t)|^2 + \int_0^t |e_{p^*}(s)|^2\, ds \\ &\le K^2 e^{-2\delta t} + \frac{K^2}{2\delta}[1 - e^{-2t\delta}] \\ &\le K^2\left(1 + \frac{1}{2\delta}\right) \triangleq K_2. \end{aligned}$$

Thus if q_k is the selected controller at each decision time τ_k, i.e., $\sigma(\tau_k) = q_k$, then

$$J(\tau_k, q_k) = |e_{q_k}(\tau_k)|^2 + \int_0^{\tau_k} |e_{q_k}(s)|^2\, ds \le K_2.$$

Hence $|e_{q_k}(\tau_k)|^2 \le K_2$. From time τ_k to τ_{k+1} the error $e_{q_k}(t)$ can increase at most at an exponential rate determined by the maximum real part of the eigenvalues of the closed-loop system that results using controller q_k. Since the closed-loop eigenvalues vary continuously with $p \in \mathcal{P}$ and by the assumption that $\mathcal{P}$ is compact, there exists $\rho > 0$ such that for $t \in [\tau_k, \tau_{k+1})$ and all $q_k \in \mathcal{P}$, $|e_{q_k}(t)| \le K_3 e^{\rho(t-\tau_k)}|e_{q_k}(\tau_k)|$. It follows that for $t \in [\tau_k, \tau_{k+1})$

$$|e_{q_k}(t)|^2 \le K_2 K_3^2 e^{2\rho(\tau_{k+1}-\tau_k)}.$$

Thus $e_{\sigma(t)}(t)$ is a bounded function. Assuming that A2) and one of the conditions of Lemma 3.2 hold, it follows from Proposition 3.4 part 1) that $\tilde{x}$ is bounded and hence from the definition of e_p that for each $p \in \mathcal{P}$, e_p and $\dot{e}_p$ are bounded.

Now consider $p \in \bar{\sigma}$. By definition, there exists a subsequence of switching times τ_{k_j} and parameters q_{k_j} such that $q_{k_j} \to p$ as $j \to \infty$. Using the fact that J is continuous on $\mathcal{P}$ yields $\lim_{j\to\infty} |J(\tau_{k_j}, q_{k_j}) - J(\tau_{k_j}, p)| = 0$, and since $J(\tau_{k_j} q_{k_j})$ is bounded by K_2, it follows that $\int_0^\infty |e_p(s)|^2\, ds < \infty$. The fact that e_p and $\dot{e}_p$ are bounded and e_p is square integrable then yields by an application of a corollary to Barbalat's Lemma [1, p. 19] that $e_p \to 0$. So $\bar{\sigma} \in \mathcal{P}^*$, i.e., Assumption R2) holds.

We believe there are many switching rules and performance measures that will also satisfy the assumption. However, we leave the design of additional specific (practical) switching rules as an interesting subproblem.

IV. Sufficient Conditions for Tracking Performance

Having established sufficient conditions for stability, we now consider the tracking performance of the switched closed-loop system. The reduction and stability result of the previous section indicates that under the conditions of Lemma 3.2, if $e_{\sigma(t)} \to 0$, then asymptotically y behaves like y_s^σ. Hence we need only show that y_s^σ adequately tracks the reference signal r. Now according to Assumption A3) the controllers Γ_p are designed so that the time-invariant closed-loop systems (Γ_p, Σ_p) adequately track r. Thus it will be sufficient to determine conditions under which the system $(\Sigma_{\sigma(t)}, \Gamma_{\sigma(t)})$ inherits this property.

To this end, fix an admissible switching signal $\sigma(t)$. Then for a fixed initial state $(z(0), w(0), \phi(0))$ of the linear time-varying system $(\Sigma_{\sigma(t)}, \Gamma_{\sigma(t)}, \Xi)$ let Ω_s^σ denote the ω-limit set, i.e., the set of positive limit points, of the resultant state trajectory $(z_s^\sigma(t), w_s^\sigma(t), \phi(t))$. In terms of Ω_s^σ we now indicate sufficient conditions under which the output y_s^σ of the system $(\Sigma_{\sigma(t)}, \Gamma_{\sigma(t)}, \Xi)$ will adequately asymptotically track r.

Proposition 4.1: Assume that A2), A4), A5), and one of the conditions of Lemma 3.2 hold. Let $\sigma(t)$ be any admissible switching signal satisfying the following two conditions.

Condition C1) For each $p, q \in \bar{\sigma}$ and $(z, w, \phi) \in \Omega_s^\sigma$

$$(C_p - C_q)w = 0. \tag{25}$$

Condition C2) For each $p, q \in \bar{\sigma}$ and $(z, w, \phi) \in \Omega_s^\sigma$

$$(H_p - H_q)z + (S_p - S_q)C_p w + (T_p - T_q)g\phi = 0. \tag{26}$$

Then for each $p \in \bar{\sigma}$, $\lim_{t\to\infty} |y_s^\sigma(t) - y_{pp}(t)| = 0$.

To prove Proposition 4.1 we will make use of the following lemma.

Lemma 4.2: Assume that A2), A4), A5), and one of the conditions of Lemma 3.2 hold. Let σ be an admissible switching signal and $x_s^\sigma(t)$ and $y_s^\sigma(t)$ be given by (24) with initial condition (z_0, w_0). Similarly, let β be an admissible switching signal and $x_s^\beta(t)$ and $y_s^\beta(t)$ be given by (24) with σ everywhere replaced by β and with initial condition (z_1, w_1). If

$$\lim_{t\to\infty} d(\sigma(t), \beta(t)) = 0 \tag{27}$$

then $\lim_{t\to\infty} \|x_s^\sigma(t) - x_s^\beta(t)\| = 0$ and $\lim_{t\to\infty} |y_s^\sigma(t) - y_s^\beta(t)| = 0$.

Proof: Let $v_1(t) = x_s^\sigma(t) - x_s^\beta(t)$. Then

$$\begin{aligned} \dot{v}_1(t) &= A_{\sigma(t)\sigma(t)} x_s^\sigma(t) - A_{\beta(t)\beta(t)} x_s^\beta(t) \\ &\quad + \begin{pmatrix} LT_{\sigma(t)} - LT_{\beta(t)} \\ NT_{\sigma(t)} - NT_{\beta(t)} \end{pmatrix} r(t) \\ &= A_{\beta(t)\beta(t)} v_1(t) + (A_{\sigma(t)\sigma(t)} - A_{\beta(t)\beta(t)}) x_s^\sigma(t) \\ &\quad + \begin{pmatrix} LT_{\sigma(t)} - LT_{\beta(t)} \\ NT_{\sigma(t)} - NT_{\beta(t)} \end{pmatrix} r(t). \end{aligned} \tag{28}$$

By (27), (8), and Assumption A5), $\lim_{t\to\infty}(A_{\sigma(t)\sigma(t)} - A_{\beta(t)\beta(t)}) = 0$ and $\lim_{t\to\infty}(T_{\sigma(t)} - T_{\beta(t)}) = 0$. Then since x_s^σ and $r(t)$ are bounded, $\lim_{t\to\infty}(A_{\sigma(t)\sigma(t)} - A_{\beta(t)\beta(t)}) x_s^\sigma(t) = 0$ and

$$\lim_{t\to\infty} \begin{pmatrix} LT_{\sigma(t)} - LT_{\beta(t)} \\ NT_{\sigma(t)} - NT_{\beta(t)} \end{pmatrix} r(t) = 0. \tag{29}$$

Now, β is an admissible switching signal, and by assumption one of the conditions of Lemma 3.2 holds. Hence the system $\dot{d} = A_{\beta\beta} d$ is globally exponentially stable. Thus from (28), v_1 is the response of an exponentially stable system to inputs that converge to zero. It follows that

$$\lim_{t\to\infty} v_1(t) = \lim_{t\to\infty} \left(x_s^\sigma(t) - x_s^\beta(t)\right) = 0. \tag{30}$$

Finally

$$\begin{aligned} y_s^\sigma(t) - y_s^\beta(t) &= (0, C_{\sigma(t)}) x_s^\sigma(t) - (0, C_{\beta(t)}) x_s^\beta(t) \\ &= (0, C_{\sigma(t)} - C_{\beta(t)}) x_s^\sigma(t) \\ &\quad + (0, C_{\beta(t)})\big(x_s^\sigma(t) - x_s^\beta(t)\big). \end{aligned}$$

So by (30), (27), and Assumption A5) we have

$$\lim_{t\to\infty} \left|y_s^\sigma(t) - y_s^\beta(t)\right| = 0. \tag{31}$$

□

Proof of Proposition 4.1: Let $p \in \bar{\sigma}$ and $\beta(t)$ be an admissible switching signal. Then

$$\left|y_s^\sigma(t) - y_{pp}(t)\right| \le \left|y_s^\sigma(t) - y_s^\beta(t)\right| + \left|y_s^\beta(t) - y_{pp}(t)\right|. \quad (32)$$

Select $\beta(t) \in \bar{\sigma}$ so that β has the same switching times as $\sigma(t)$ and (27) is satisfied. To see that this is possible we argue as follows. For each integer $j \ge 1$ there exists a switching time τ_{k_j} such that for all $t > \tau_{k_j}$, $d(\sigma(t), \bar{\sigma}) \le 1/j$. Hence over each of the switching intervals $[\tau_i, \tau_{i+1})$, with $\tau_{k_j} \le \tau_i < \tau_{i+1} \le \tau_{k_{j+1}}$, we can select $\beta(t) \in \bar{\sigma}$ so that $d(\sigma(t), \beta(t)) < 2/j$. Thus (27) holds. It then follows by Lemma 4.2 that

$$\lim_{t\to\infty} \left|y_s^\sigma(t) - y_s^\beta(t)\right| = 0. \quad (33)$$

Now we examine the second term on the right-hand side (RHS) of (32). The signal $y_{pp}(t)$ is given by (7). Let $v_2(t) = x_{pp}(t) - x_s^\beta(t)$. Then $v_2(0) = 0$ and for $t \ge 0$

$$\begin{aligned}\dot{v}_2(t) &= A_{pp}x_{pp}(t) - A_{\beta(t)\beta(t)}x_s^\beta(t) + \begin{pmatrix} LT_p - LT_{\beta(t)} \\ NT_p - NT_{\beta(t)} \end{pmatrix} r(t) \\ &= A_{pp}v_2(t) + (A_{pp} - A_{\beta(t)\beta(t)})x_s^\beta(t) \\ &\quad + \begin{pmatrix} LT_p - LT_{\beta(t)} \\ NT_p - NT_{\beta(t)} \end{pmatrix} r(t). \end{aligned} \quad (34)$$

By C1), at every state $(z, w, \phi) \in \Omega_s^\sigma$, and for each $p, q \in \bar{\sigma}$

$$C_p w = C_q w. \quad (35)$$

In addition, by C2) for all $p, q \in \bar{\sigma}$ and $(z, w, \phi) \in \Omega_s^\sigma$, (26) holds. Then (8), (26), and (35) imply that for each $p, q \in \bar{\sigma}$ and $(x, z, w, \phi) \in \Omega$

$$(A_{pp} - A_{qq})\begin{pmatrix} z \\ w \end{pmatrix} + \begin{pmatrix} L(T_p - T_q) \\ N(T_p - T_q) \end{pmatrix} g\phi = 0. \quad (36)$$

In particular, since $\beta(t) \in \bar{\sigma}$, for each $p \in \bar{\sigma}$ and $(x, z, w, \phi) \in \Omega$

$$(C_p - C_{\beta(t)})w = 0 \quad (37)$$

$$(A_{pp} - A_{\beta(t)\beta(t)})\begin{pmatrix} z \\ w \end{pmatrix} + \begin{pmatrix} L(T_p - T_{\beta(t)}) \\ N(T_p - T_{\beta(t)}) \end{pmatrix} g\phi = 0. \quad (38)$$

Now $x_s^\beta(t)$ converges to $x_s^\sigma(t)$ [see (30)]. Thus $x_s^\beta(t)$ converges to Ω_s^σ and hence by (37), (38), and continuity

$$\begin{aligned}\lim_{t\to\infty}(A_{pp} - A_{\beta(t)\beta(t)})x_s^\beta(t) + \begin{pmatrix} L(T_p - T_{\beta(t)}) \\ N(T_p - T_{\beta(t)}) \end{pmatrix} r(t) = 0 \\ \lim_{t\to\infty}(0, C_p - C_{\beta(t)})x_s^\beta(t) = 0. \end{aligned} \quad (39)$$

Combining (34), (39), and the stability of the matrix A_{pp}, we conclude that

$$\lim_{t\to\infty} v_2(t) = \lim_{t\to\infty}\left(x_{pp}(t) - x_s^\beta(t)\right) = 0. \quad (40)$$

Write

$$\begin{aligned} y_s^\beta(t) - y_{pp}(t) &= (0, C_{\beta(t)})x_s^\beta(t) - (0, C_p)x_{pp}(t) \\ &= (0, C_{\beta(t)} - C_p)x_s^\beta(t) - (0, C_p)v_2(t). \end{aligned}$$

Hence using (39) and (40)

$$\lim_{t\to\infty}\left|y_s^\beta(t) - y_{pp}(t)\right| = 0. \quad (41)$$

Finally combining (33) and (41) yields $\lim_{t\to\infty}|y_s^\sigma(t) - y_{pp}(t)| = 0$. □

Condition C1) requires, roughly, that if two predictors are chosen "in the limit," then they agree on the ω-limit set. This is a natural condition that is easily seen to be implied, for example, by R2). Condition C2) is similar but more restrictive. It requires that if two controllers are chosen "in the limit," then they agree on the ω-limit set. Unlike C1), there is no *a priori* reason to suggest that this will be a natural consequence of the switching rule. Hence it is imposing an additional constraint on the model/controller pairs. Of course, the problem is that one may not know the ω-limit set in advance, in which case it may be necessary to check that (26) is satisfied on a larger set that is known to contain the possible ω-limit sets. Note that when the assumptions of Proposition 3.4 are satisfied, part 1) of the proposition implies that C2) can be verified by examining the possible ω-limit sets of the nonlinear switched system. In fact, for some simple cases C2) can be verified quite easily in this way as shown in the examples after the Proof of Theorem 4.3.

Our main result is that the standard assumptions together with R2) and C2) are sufficient to ensure that the output of the nonlinear switch system $y(t)$ asymptotically tracks the reference signal r. Indeed, the following theorem indicates that the asymptotic tracking performance of the closed-loop switched system is as good as one of the time-invariant linear systems (Σ_p, Γ_p), and by Assumption A3) this is adequate.

Theorem 4.3: Assume that A1)–A5) and either of the conditions of Lemma 3.2 are satisfied. Furthermore, assume that for each possible initial condition of the nonlinear switched system the resulting switching sequence is such that R2) and C2) are satisfied. Then the closed-loop switched control system (11)–(17) satisfies $\lim_{t\to\infty}|y(t) - y_{pp}(t)| = 0$ for some $p \in \mathcal{P}$ with $e_p(t) \to 0$. Hence asymptotically y adequately tracks r.

To prove Theorem 4.3 we will use the following lemma.

Lemma 4.4: Assume that A2)–A5), R2), and one of the conditions of Lemma 3.2 hold. Then for any initial state (x_0, z_0, w_0, ϕ_0) of the nonlinear closed-loop switched system the resultant state trajectory $(x(t), z(t), w(t), \phi(t))$ is bounded and has a nonempty ω-limit set Ω. Furthermore, for all $(x_1, z_1, w_1, \phi_1) \in \Omega$ and all $p, q \in \bar{\sigma}$, we have $C_p w_1 = C_q w_1$.

Proof: By assumption, the signals $\phi(t)$ and $r(t)$ are bounded. Since one of the conditions of Lemma 3.2 holds and R2) holds, it follows from Proposition 3.4 that the signals $z(t), w(t), x(t)$ are bounded. Thus the set of limit points Ω as $t \to \infty$ of the joint signal $(x(t), z(t), w(t), \phi(t))$, i.e., the ω-limit set of the trajectory, is nonempty.

By continuity, for each $p \in \mathcal{P}^*$ and $(x_1, z_1, w_1, \phi_1) \in \Omega$, $Cx_1 = C_p w_1$, i.e., any predictor $p \in \mathcal{P}^*$ has zero error on Ω. This in turn implies that for each $p, q \in \mathcal{P}^*$

$$C_p w_1 = C_q w_1 \quad (42)$$

i.e., all predictors in $\mathcal{P}^*$ agree on Ω. Since, by R2), $\bar{\sigma} \subseteq \mathcal{P}^*$, the same holds for $p, q \in \bar{\sigma}$. □

Proof of Theorem 4.3: Fix a pair of initial conditions $(\tilde{x}_0, \phi_0)$. This determines the switching signal $\sigma(t)$ and the bounded reference signal $r(t)$. By Lemma 4.4 the state trajectory $\tilde{x}(t)$ of (11) is bounded and the ω-limit set Ω is nonempty.

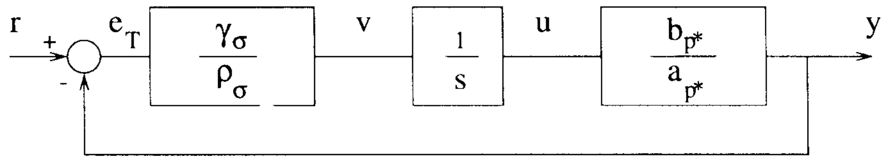

Fig. 2. Set-point control system.

By Lemma 4.4 for all $(x_1, z_1, w_1, \phi_1) \in \Omega$ and all $p, q \in \bar{\sigma}$, we have $C_p w_1 = C_q w_1$. By Proposition 3.4, $\Omega_s^\sigma = \Omega^{z,w,\phi}$. Hence for each $p, q \in \bar{\sigma}$ and each point $(z, w, \phi) \in \Omega_s^\sigma$, $C_p w = C_q w$, i.e., condition C1) holds.

Let $\tilde{x} = (x, z, w)$. We let $y(t)$ denote the output of the nonlinear closed-loop system with initial condition $\tilde{x}$, $y_{pp}(t)$ denote the output of the LTI system (Σ_p, Γ_p) driven by r and with initial condition (z, w), and $y_s^\sigma(t)$ denote the output of the linear time-varying system $(\Sigma_{\sigma(t)}, \Gamma_{\sigma(t)})$ for a fixed switching signal σ to the input r and with initial condition (z, w).

Let p be any element of $\bar{\sigma}$. Then under the fixed switching signal σ

$$|y(t) - y_{pp}(t)| \leq \left|y(t) - y_s^\sigma(t)\right| + \left|y_s^\sigma(t) - y_{pp}(t)\right|. \tag{43}$$

By Proposition 3.4, the first term on the RHS of (43) converges to zero as $t \to \infty$. We have shown that C1) holds and by assumption so does C2). Hence by Proposition 4.1, the second term on the RHS of (43) converges to zero as $t \to \infty$. Thus $\lim_{t\to\infty} |y(t) - y_{pp}(t)| = 0$ as claimed. □

As mentioned above, in general it may be difficult to check Condition C2) since it requires knowledge of the ω-limit sets of the switched closed-loop system. Nevertheless, in some simple but interesting cases it is possible to check the condition by direct computation as we demonstrate in the following examples.

Example 1: The set-point control problem illustrated in Fig. 2 is studied in [13]. We are given a family of SISO model transfer functions $b_p(s)/a_p(s)$ and a family of SISO controller transfer functions $\gamma_p(s)/\rho_p(s)$ such that for each $p \in \mathcal{P}$, the control system illustrated in Fig. 2 with $\sigma = p$ is stable and for any constant reference signal r, $\lim_{t\to\infty} |y(t) - r| = 0$.

As shown in Fig. 2, the actual plant input u is generated by an integrating subsystem, and the control signal v is generated by a switched controller Γ_σ driven by the tracking error $e_T(t) = r - y(t)$. In our notation, Σ is the cascade of the integrator and the actual plant. Thus the controller output and (augmented) plant input is denoted by v.

It is shown in [13] that a common state-space representation of the family of predictors and controllers takes the form

$$\dot{x}_C = A_C x_C + d_C y + b_C v + h_C e_T \tag{44}$$
$$v_q = f_q x_C + g_q e_T \tag{45}$$
$$e_p = c_p x_C - y. \tag{46}$$

Here A_C, b_C, d_C, h_C, and x_C are parameter independent and A_C is asymptotically stable. In our notation, (44)–(46) is equivalent to a common state realization of O_p and Γ_q with $M = F = A_C$, and $C_p = c_p$, $H_q = f_q$, $S_q = -g_q$, and $T_q = g_q$. Thus the model realization Σ_p is obtained by setting $y = c_p x_C$ and the controller realization Γ_q is obtained by setting $v = v_q$.

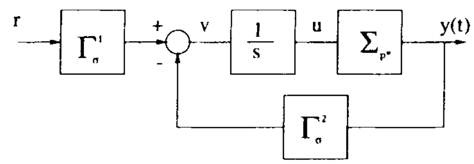

Fig. 3. Set-point control system for Example 2.

By the problem setup and the construction of A_C, Assumptions A1)–A3) hold. We assume that the parameterization of the models and controllers satisfies Assumptions A4) and A5).

That Condition C2) holds in this case can be verified by direct computation. First, it is easy to see that the ω-limit set for the system (Σ_p, Γ_p) is just the single equilibrium state $x_C^\infty = -A_C^{-1} d_C r$, and this is independent of p. Moreover, in steady state $y = r$, $v_p = 0$, and $e_p = 0$. So $f_p A_C^{-1} d_C = 0$ and $c_p A_C^{-1} d_C = -1$.

Let σ be a τ_D-admissible switching signal. Then by Lemma 3.3, under either of the conditions of Lemma 3.2, every state trajectory of the switched system $(\Sigma_{\sigma(t)}, \Gamma_{\sigma(t)})$ converges to the ω-limit set $\Omega(r) = x_C^\infty$. By Proposition 3.4 we have $\Omega^{z,w} = \Omega(r)$. Thus to check C2), we just need to check that (26) holds at the point x_C^∞. Using $f_p A_C^{-1} d_C = 0$ and $c_p A_C^{-1} d_C = -1$, gives

$$\begin{aligned} &H_p x_C^\infty + S_p C_p x_C^\infty + T_p r \\ &\quad = -f_p A_C^{-1} d_C r + g_p c_p A_C^{-1} d_C r + g_p r \\ &\quad = g_p \left(c_p A_C^{-1} d_C + 1\right) r \\ &\quad = 0. \end{aligned}$$

Hence for each $p, q \in \mathcal{P}$ $(H_p - H_q) x_C^\infty + (S_p - S_q) C_p x_C^\infty + (T_p - T_q) r = 0$, i.e., C2) is satisfied.

It follows from the above and Theorem 4.3 that any parameterization of the controllers for which A4) and A5) are satisfied and any switching rule satisfying R2) and having a sufficiently large dwell-time τ_D will ensure that the state of the closed-loop switched system is bounded and $\lim_{t\to\infty} |y(t) - r| = 0$.

Example 2: Consider the set-point control problem illustrated in Fig. 3. We assume the same conditions as the previous example.

In our notation, Σ is the cascade of the plant and integrator. Hence the control signal is v. The controller/predictor system is given by

$$\begin{aligned} \dot{z}(t) &= Fz(t) + Gy(t) + Lv(t) + Rr \\ v_p(t) &= H_p z(t) + S_p y(t) + T_p r \\ \dot{w}(t) &= Mw(t) + Nv(t) + Ky(t) \\ e_p(t) &= C_p w(t) - y(t). \end{aligned}$$

Consider the system (Σ_p, Γ_p). By assumption, this system is stable. Hence $\lim_{t\to\infty} v_p(t) \triangleq v_p^\infty = 0$ for all $p \in \mathcal{P}$, and the stationary point of the (Σ_p, Γ_p) system is given by

$$z_p^\infty = -F^{-1}\big(Gr + Lv_p^\infty + Rr\big) = -F^{-1}(G+R)r \triangleq z^\infty$$
$$w_p^\infty = -M^{-1}\big(Nv_p^\infty + Kr\big) = -M^{-1}Kr \triangleq w^\infty.$$

Note that the fixed point is independent of p. In addition

$$v_p^\infty = H_p z^\infty + (S_p + T_p)r = 0$$

and

$$e_p^\infty = C_p w^\infty - r = 0. \tag{47}$$

Let σ be any admissible switching signal. Then under any of the conditions of Lemma 3.1, every trajectory of the switched system $(\Sigma_{\sigma(t)}, \Gamma_{\sigma(t)})$ converges to the point: $\Omega(r) = (z^\infty, w^\infty)$. Thus to check C2) we need to show that (26) is satisfied at this point. Using (47), for all $p \in \mathcal{P}$, $H_p z^\infty + S_p C_p w^\infty + T_p r = H_p z^\infty + S_p r + T_p r = 0$, i.e., Condition C2) is satisfied.

It follows from the above and Theorem 4.3 that any parameterization of the controllers for which A4) and A5) are satisfied and any switching rule satisfying R2) and having a sufficiently large dwell-time τ_D will ensure that the state of the closed-loop switched system is bounded and $\lim_{t\to\infty} |y(t) - r| = 0$.

V. Asymptotic Exact Tracking

An interesting additional result can be obtained when we restrict attention to asymptotic exact tracking, i.e., we require $\lim_{t\to\infty} |y(t) - r(t)| = 0$. In this case we will assume that the following additional mild restriction holds.

Assumption A6) The models Σ_p have no zeros in common with the eigenvalues of the reference signal generator.

Our second main result is the following. In the case of asymptotic exact tracking in order to conclude that the nonlinear switched system will asymptotically exactly track the reference signal $r(t)$, it is only necessary to verify the structural conditions A1)–A6) and the switching rule constraints that R2) and one of the conditions of Lemma 3.2 hold. In particular, it is not necessary to verify that condition C2) holds. As we show in the proof of the theorem below, C2) will hold.

Theorem 5.1: Consider an asymptotic exact tracking problem for a reference signal given by (2). Assume that Conditions A1)–A6), R2), and one of the conditions of Lemma 3.2 hold. Then the closed-loop switched control system satisfies $\lim_{t\to\infty} |y(t) - r(t)| = 0$.

To prove Theorem 5.1 we will make use of the following preliminary lemmas. Assume that A1)–A6), R2), and one of the conditions of Lemma 3.2 hold. Fix an arbitrary initial condition for the switched closed-loop system and let σ be the switching signal from this initial condition. By Proposition 3.4 and the model for the reference signal, the trajectory $(x(t), z(t), w(t), \phi(t))$ is bounded and hence has a nonempty ω-limit set Ω. In addition, by assumptions A2) and A3), for each $p \in \mathcal{P}$, the closed-loop system $(\Sigma_p, \Gamma_p, \Xi)$ satisfies $\lim_{t\to\infty} |y_{pp}(t) - r(t)| = 0$, and for each initial state of the reference signal generator, the state of this system converges to a unique ω-limit set Ω_{pp}.

Lemma 5.2: For any $q \in \bar{\sigma}$ there exists a point $s^0 = (x^0, z^0, w^0, \phi^0) \in \Omega$ and a time $\tau > 0$ such that the trajectory of the closed-loop system starting from s^0 at $t = 0$ and using the fixed controller q stays in Ω over the time interval $[0, \tau)$.

Proof: Consider the joint signal $(\sigma(t), x(t), z(t), w(t), \phi(t))$. Sample this signal at the sequence of switching times τ_k, $k \geq 1$. The resultant sequence of samples $\{(\sigma_k, x_k, z_k, w_k, \phi_k), k \geq 1\}$ lies in a compact metric space and hence has a nonempty set of limit points Θ. Clearly, if $q \in \bar{\sigma}$, then there exists $s^0 \triangleq (x^0, z^0, w^0, \phi^0) \in \Omega$ such that $(q, x^0, z^0, w^0, \phi^0) \in \Theta$. Henceforth, let $\{(\sigma_j, x_j, z_j, w_j, \phi_j), j \geq 1\}$ denote a subsequence of the sampled trajectory consisting of points at switching times τ_{k_j} such that $\sigma_j \to q$, $x_j \to x^0$, $z_j \to z^0$, $w_j \to w^0$, and $\phi_j \to \phi^0$.

For each point $s_j \triangleq (x_j, z_j, w_j, \phi_j)$ consider the trajectory segment

$$\theta_j(t) = (x(t), z(t), w(t), \phi(t)), \qquad t \in \big[\tau_{k_j}, \tau_{k_j+1}\big)$$

of the closed-loop system that starts at s_j at time τ_{k_j} and has $\sigma(t) = \sigma_j$, for $t \in [\tau_{k_j}, \tau_{k_j+1})$. Each of these functions is a segment of a trajectory of an LTI system. Hence, by time shifting and restricting attention to an interval of time of duration τ_D, we can consider the trajectories to be defined over the time interval $[0, \tau_D)$. Similarly, let $\bar{\theta}^q(t) = (\bar{x}(t), \bar{z}(t), \bar{w}(t), \bar{\phi}(t)), t \in [0, \tau_D)$ denote the trajectory of the closed-loop (LTI) system starting from the initial condition s^0 and using $\sigma(t) = q$.

We claim that the sequence of functions θ_j converges pointwise over $[0, \tau_D)$ to the function $\bar{\theta}^q$ as $j \to \infty$. This follows by continuity with respect to the initial condition of the solutions of (2) and by the continuity of the solutions of the remaining linear ordinary differential equations (o.d.e.'s) with respect to the initial condition and the parameter p.

It follows that all points on the curve $\bar{\theta}^q$, $t \in [0, \tau_D)$ lie in Ω. □

Lemma 5.3: For each $p, q \in \bar{\sigma}$, if $(z, w, \phi) \in \Omega_{qq}$, then $C_p w = C_q w$.

Proof: Let $p, q \in \bar{\sigma}$. Then by Lemma 5.2 there exists a point $s^0 = (x^0, z^0, w^0, \phi^0) \in \Omega$ and a nonempty interval $[0, \tau)$ such that the trajectory of the switched system starting from s^0 at $t = 0$ and using the fixed controller q stays in Ω for the time interval $[0, \tau)$. Now, the ω-limit set of the system $(\Sigma_{\sigma(t)}, \Gamma_{\sigma(t)}, \Xi)$ is $\Omega^{z,w,\phi}$. Hence there exists a point $(z^0, w^0, \phi^0) \in \Omega^{z,w,\phi}$ and a nonempty time interval $[0, \tau)$ such that the trajectory of the LTI system $(\Sigma_q, \Gamma_q, \Xi)$ starting from (z^0, w^0, ϕ^0) at $t = 0$ remains in $\Omega^{z,w,\phi}$ over the time interval $[0, \tau)$. Denote this curve by $\bar{\theta}_q(t) = (\bar{z}_q(t), \bar{w}_q(t), \bar{\phi}(t))$, $t \in [0, \tau)$.

Since $\bar{\theta}_q(t)$ lies in $\Omega^{z,w,\phi}$, by Lemma 4.4, $(C_p - C_q)\bar{w}_q(t) = 0$ for all $p \in \bar{\sigma}$ and $t \in [0, \tau)$. But this trajectory segment is generated by an LTI system; hence $(C_p - C_q)\bar{w}_q(t) = 0$ for all $t \geq 0$. Now, in the limit the above trajectory converges to the unique ω-limit set Ω_{qq}. Thus by continuity, for every point $(z, w, \phi) \in \Omega_{qq}$, $C_p w = C_q w$. □

Lemma 5.4: For all $p, q \in \bar{\sigma}$, $\Omega_{pp} = \Omega_{qq}$.

Proof: In view of the assumptions on the reference signal generator, without loss of generality we can represent r as a finite sum of sinusoids. Each system (Σ_p, Γ_p) is LTI. Hence, by superposition it will be enough to prove the result assuming that r is a single sinusoid.

Suppose that $r(t) = r_o e^{i\omega_o t}$. Since the system (Σ_p, Γ_p) is exponentially stable, for all initial conditions the state trajectory settles into a periodic orbit of the form $(z_p(t), w_p(t)) = (\bar{z}_p(i\omega_o), \bar{w}_p(i\omega_o))e^{i\omega_o t}$. Here $\bar{z}_p(i\omega_o)$ and $\bar{w}_p(i\omega_o)$ are complex numbers that depend on the real constant ω_o. Similarly, $u_p(t)$ converges to a periodic signal of the form $u_p(t) = \bar{u}_p(i\omega_o)e^{i\omega_o t}$ and $y_{pp}(t)$ converges to $r(t)$.

Using (3), (5), and A1), we find that we have (48), as shown at the bottom of the page. Then using (48), the fact that $y_{pp} = C_p w_p$, and that in steady state $y_{pp} = r$, yields

$$r_o = C_p(i\omega_o I - M)^{-1} N \bar{u}_p(i\omega_o) + C_p(i\omega_o I - M)^{-1} K r_o. \tag{49}$$

Let $\mathcal{H}_p(s)$ denote the transfer function of Σ_p. Using the formula for $\mathcal{H}_p(s)$, Assumption A1), and the Matrix Inversion Lemma we find

$$\mathcal{H}_p(i\omega_o) = \frac{C_p(i\omega_o I - M)^{-1} N}{1 - C_p(i\omega_o I - M)^{-1} K}. \tag{50}$$

By A6), Σ_p has no zeros that are eigenvalues of the reference system. Hence the numerator in (50) is nonzero. Thus using (49) we can write

$$\bar{u}_p(i\omega_o) = \frac{1 - C_p(i\omega_o I - M)^{-1} K}{C_p(i\omega_o I - M)^{-1} N} r_o. \tag{51}$$

Let $p, q \in \bar{\sigma}$. Then by Lemma 5.3, $C_p \bar{w}_p(i\omega_o) = C_q \bar{w}_p(i\omega_o)$, i.e.,

$$\begin{aligned} &C_p(i\omega_o I - M)^{-1} N \bar{u}_p(i\omega_o) + C_p(i\omega_o I - M)^{-1} K r_o \\ &\quad = C_q(i\omega_o I - M)^{-1} N \bar{u}_p(i\omega_o) + C_q(i\omega_o I - M)^{-1} K r_o. \end{aligned}$$

Combining this with (49) we have

$$r_o = C_q(i\omega_o I - M)^{-1} N \bar{u}_p(i\omega_o) + C_q(i\omega_o I - M)^{-1} K r_o.$$

On rearrangement and comparison with (51) this yields

$$\bar{u}_p(i\omega_o) = \frac{1 - C_q(i\omega_o I - M)^{-1} K}{C_q(i\omega_o I - M)^{-1} N} r_o = \bar{u}_q(i\omega_o).$$

Thus for each $p, q \in \bar{\sigma}$, $\bar{u}_p(i\omega_o) = \bar{u}_q(i\omega_o)$. It then follows from (48) that for each $p, q \in \bar{\sigma}$, $(\bar{z}_p(i\omega_o), \bar{w}_p(i\omega_o)) = (\bar{z}_q(i\omega_o), \bar{w}_q(i\omega_o))$, i.e., $\Omega_{pp} = \Omega_{qq}$. □

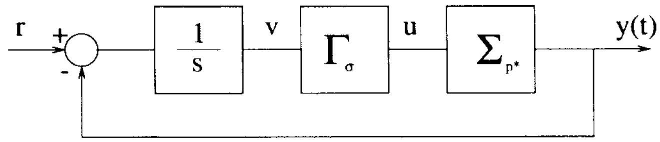

Fig. 4. Set-point control system, Example 3.

Proof of Theorem 5.1: The initial state (x_0, z_0, w_0, ϕ_0) of the closed-loop switched system determines: the state trajectory $\phi(t)$ of the reference signal generator, the reference signal $r(t)$, the switching signal $\sigma(t)$, and the state trajectories $z(t)$, $w(t)$, and $x(t)$ of the controller, predictor, and the plant, respectively. By the assumptions of the theorem and Proposition 3.4, all of these signals are bounded and the ω-limit set Ω of the trajectory $(x(t), z(t), w(t), \phi(t)), t \geq 0$, is nonempty. Let $\Omega^{z,w,\phi}$ (respectively, $\Omega^{z,w}$) denote the set of limit points of the signal $(z(t), w(t), \phi(t))$ [respectively, $(z(t), w(t))$].

Let $\sigma(t)$ be the fixed switching signal defined above. Then, by the assumptions of the theorem, Lemma 3.2, and Proposition 3.4, the ω-limit set of the time-varying system $(\Sigma_s(t), \Gamma_{\sigma(t)}, \Xi)$ is $\Omega^{z,w,\phi}$.

Let $\beta(t)$ be a switching signal taking values in $\bar{\sigma}$, having the same switching times as $\sigma(t)$, and satisfying $\lim_{t\to\infty} d(\sigma(t), \beta(t)) = 0$. By Lemma 4.2, for any initial condition, the trajectories of $(\Sigma_{\sigma(t)}, \Gamma_{\sigma(t)}, \Xi)$ and $(\Sigma_{\beta(t)}, \Gamma_{\beta(t)}, \Xi)$ converge as $t \to \infty$. By Lemma 5.4, for each $p, q \in \bar{\sigma}$, $\Omega_{pp} = \Omega_{qq} = \bar{\Omega}$. Thus, by Lemma 3.3, the system $(\Sigma_{\beta(t)}, \Gamma_{\beta(t)}, \Xi)$ has the globally attractive orbit $\bar{\Omega}$. Thus $(\Sigma_{\sigma(t)}, \Gamma_{\sigma(t)}, \Xi)$ has the globally attractive orbit $\bar{\Omega}$.

Finally we show that $\lim_{t\to\infty} |y(t) - r(t)| = 0$. By Assumption A3) and the requirement of asymptotic exact tracking, for each $p \in \bar{\sigma}$, $y_{pp}(t) = r(t)$ on $\bar{\Omega}$. However, by Lemma 5.3, for each $p, q \in \bar{\sigma}$, $C_p w = C_q w$ for each point $(z, w, \phi) \in \bar{\Omega}$. Thus $y_s^\sigma(t) = r(t)$ on $\bar{\Omega}$. By R2) and Proposition 3.4 part (1), this implies that $\lim_{t\to\infty} |y(t) - r(t)| = 0$. □

We illustrate Theorem 5.1 with the following example.

Example 3: Consider the set-point control problem illustrated in Fig. 4. Assume, as in Example 1, that we are given a set of SISO models and a set of SISO controllers, such that for each $p^* \in \mathcal{P}$ the controlled system with $\sigma = p^*$ is stable, and for any constant reference input r, $\lim_{t\to\infty} |y(t) - r| = 0$. The true plant Σ_{p^*} is one of the models and p^* is unknown. We assume that none of the models has a zero at 0. This example, although similar to Example 1, requires a different (and more difficult) analysis.

The plant input $u_{\sigma(t)}$ is generated by a switched controller $\Gamma_{\sigma(t)}$, whose input signal, $v(t)$, is generated by an integrating subsystem driven by the tracking error $r - y(t)$.

In our notation Γ is the cascade of the integrator and the controller. A common state-space realization of the system

$$\begin{pmatrix} \bar{z}_p(i\omega_o) \\ \bar{w}_p(i\omega_o) \end{pmatrix} = \begin{pmatrix} (i\omega_o I - F)^{-1} L & (i\omega_o I - F)^{-1}(G + R) \\ (i\omega_o I - M)^{-1} N & (i\omega_o I - M)^{-1} K \end{pmatrix} \begin{pmatrix} \bar{u}_p(i\omega_o) \\ 1 \end{pmatrix} \tag{48}$$

above is given by

$$\begin{aligned}\dot{z}(t) &= Fz(t) + Lu(t) + G(r - y(t))\\ u_p(t) &= H_p z(t)\\ \dot{w}(t) &= Mw(t) + Ky(t) + Nu(t)\\ e_p &= C_p w(t) - r.\end{aligned} \tag{52}$$

Assume that A1)–A5) are satisfied by the problem setup, that one of the conditions of Lemma 3.2 holds, and that the switching rule ensures that condition R2) holds. Stability and boundedness of the switched nonlinear system then follows by Proposition 3.4. Since we have assumed that the models do not have a zero at the origin, A6) is satisfied. Hence by Theorem 5.1, $\lim_{t\to\infty} |y(t) - r| = 0$. This latter result can also be shown by direct computation of the fixed points of the systems (Σ_p, Γ_p). However, in this case these fixed points depend on $p \in \mathcal{P}$, and the analysis method used in Examples 1 and 2 does not apply.

VI. Conclusion

We have presented a simple setting in which to analyze the stability and tracking performance of predictor-based controller switching rules. Our approach is to relate control performance to prediction performance by separating the output into an exogenous input response and an error term.

Our first main result, Theorem 4.3, gives a set of sufficient conditions under which good asymptotic tracking performance of the switched system is achieved. As a special case, this results leads to a simpler proof of a recent result of Morse [13], concerning a set-point control problem.

Our second main result, Theorem 5.1, shows that in the special case where asymptotic exact tracking is required and the plant has no zeros in common with the poles of the reference signal generator, the main assumption of Theorem 4.3 is in fact always satisfied so that the switched system achieves asymptotic exact tracking.

Appendix 1

Let $p_i = \sigma(\tau_i) \in \mathcal{P}$. For $\tau_{i-1} < \mu < \tau_i < \cdots < \tau_j < t < \tau_{j+1}$ the state transition matrix of $A_{\sigma\sigma}(t)$ is

$$\Phi(t,\mu) = e^{A_{p_j p_j}(t-\tau_j)} \prod_{k=i+1}^{j} e^{A_{p_k p_k}(\tau_k - \tau_{k-1})} \cdot e^{A_{p_i p_i}(\tau_i - \mu)}. \tag{53}$$

Since A_{pp}, $p \in \mathcal{P}$ are stable with stability margin γ, it is possible to find numbers $a_p > 0$ and $\lambda_p \in (0, \gamma)$ for which

$$|e^{A_{pp}t}| \le e^{(a_p - \lambda_p t)} \quad t \ge 0, \qquad p \in \mathcal{P} \tag{54}$$

(see [13] and [4]). In particular

$$|e^{A_{\sigma\sigma}(\tau)t}| \le e^{a}\, e^{-\frac{t}{\tau_0}} \quad t \ge 0, \qquad p \in \mathcal{P} \tag{55}$$

where

$$a = \sup_{p\in\mathcal{P}}\{a_p\}, \qquad \tau_0 > \sup_{p\in\mathcal{P}}\left\{\frac{a_p}{\lambda_p}\right\}.$$

From (53) and (54) it is easy to show that for any switching signal $\sigma : [0,\ \infty) \to \mathcal{P}$ with dwell time no smaller than τ_0, the state transition matrix of $A_{\sigma\sigma}$ satisfies

$$\|\Phi(t,\mu)\| \le e^{a - \lambda(t-\mu)} \quad \forall t \ge \mu \ge 0$$

where

$$\lambda = \inf_{p\in\mathcal{P}}\left\{\lambda_p - \frac{a_p}{\tau_0}\right\} \in (0,\ \lambda_p].$$

That is, $A_{\sigma\sigma}$ is exponentially stable with a decay rate λ (see [13]).

A different approach can be based on the conditions for stability of "slowly time varying systems"; see, e.g., [17]. Assume that $A_{pp}(t)$ are bounded by X, then we have the following.

Lemma 6.1 [3], [6]: For $A_{\sigma\sigma}$ of size $n \times n$ satisfying A2)

$$\|e^{A_{\sigma\sigma}(\tau)t}\| \le \left(\frac{2X}{\epsilon}\right)^{n-1} e^{(-\gamma+\epsilon)t} \quad \forall \tau, t \ge 0$$

for every $\epsilon \in (0,\ 2X)$.

Using (55), exponential stability is assured if the bound X is smaller than $(2e^{a}\tau_0)^{-1}$.

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