

Logarithmic Response and Climate Sensitivity of Atmospheric CO_2

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Abstract

The logarithmic response of global average surface warming to future increases of atmospheric carbon dioxide is derived analytically. The results include a simple and explicit formula for the *equilibrium climate sensitivity*. The values generated by this formula show deference to the low end of the current “best estimate” values. Policy implications of the findings are briefly discussed.

Keywords: global warming, carbon dioxide, saturation, climate sensitivity.

1 How Much Does the Earth Warm?

Atmospheric carbon dioxide is a greenhouse gas, because it absorbs infrared radiations emitted skyward by the surface of the earth. For many centuries before the industrial revolution, the amount of atmospheric carbon dioxide (to be denoted here by X_* (parts per million)) and the earth’s *average surface air* temperature (to be denoted here by T_{sa} ($^{\circ}K$)) had remained within relatively narrow limits. When carbon in the form of fossil fuels is dug up and pumped out from below ground and burned to provide energy for the world, additional carbon dioxide is injected into the earth’s carbon cycle. The amount of upward radiant energy absorbed (per unit time per unit surface area of the earth) by atmospheric carbon dioxide—over and above the

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amount previously absorbed before the industrial revolution—is called *radiative forcing*, to be denoted here by ΔF_{CO_2} (*watts/meter²*). The value of T_{sa} is expected to rise in response to this radiative forcing (so that the same amount of energy is radiated away from earth). This rise of T_{sa} above the pre-industrial revolution value $T_{sa,o}$ caused by atmospheric carbon dioxide—either directly or indirectly—is denoted by ΔT_{CO_2} (*°C*). How does ΔT_{CO_2} respond to the increase of X_* above the pre-industrial revolution value $X_{*,o}$ (approximately 285 parts per million)? Both *qualitative* and *quantitative* answers to this question shall be *derived* here.

Since the industrial revolution, X_* has increased by about 30%, and T_{sa} has risen by about 0.7°C. But “natural” T_{sa} excursions of this magnitude were not uncommon in the centuries before the industrial revolution.

The IPCC Estimate

In Climate Change 2007 [1], the *Intergovernmental Panel on Climate Change* (IPCC) says:

... The equilibrium climate sensitivity is a measure of the climate system response to sustained radiative forcing. It is not a projection but is defined as the global average surface warming following a doubling of carbon dioxide concentrations. It is likely to be in the range 2°C to 4.5°C with a best estimate of about 3°C, and is very unlikely to be less than 1.5°C. ...

The equilibrium climate sensitivity as defined above shall be denoted here by β . The IPCC also said the “assessed likely range” given above was the result of “analysis of climate models with constraints from observations”. Note that the IPCC specifically said β was *not a projection* (i.e. linear extrapolation is not advised). The IPCC “projection” of the dependence of ΔT_{CO_2} ’s on X_* was given only by a graph [2] which clearly showed that the response weakens as X_* increases.

It is intuitively obvious that the *qualitative* response of ΔT_{CO_2} to the increase of X_* must depend on the order of magnitude of some characteristic “optical thickness” of the atmosphere, and the *quantitative* response—characterized by the value of β —must depend on the quantitative details of the infrared radiation absorption properties of carbon dioxide. The IPCC estimates came from sophisticated models

which usually required massive amount of computations.

2 Energy Balance of Planet Earth

In the steady state, the appropriately averaged outward radiation energy flux escaping the earth must equal to the appropriately averaged constant amount of inward radiation energy flux arriving from the sun. The energy balance per unit area of the earth's surface is:

$$F_{outward} = \bar{\epsilon}_e(T_{sa})\sigma T_{sa}^4 - \Delta F_{CO_2} = F_{inward} \approx \text{constant}, \quad (1)$$

where $\Delta F_{CO_2} > 0$ is the incremental radiative forcing above the pre-industrial revolution level *directly* attributable to the increased atmospheric carbon dioxide, σ is the Stefan-Boltzmann constant, and $\bar{\epsilon}_e(T_{sa})$ is an *effective* emissivity of the earth—its T_{sa} -dependence shall be used to account for *all the indirect effects* of carbon dioxide on *other* greenhouse gases such as water vapor, including all the myriad complications of rising T_{sa} on clouds/ice albedo, etc. Replacing T_{sa} in Eq.(1) by $T_{sa,o} + \Delta T_{CO_2}$ and linearizing, one has:

$$\frac{\Delta T_{CO_2}}{T_{sa,o}} \approx \frac{\mu_o}{4} \frac{\Delta F_{CO_2}}{F_{outward,o}}, \quad F_{outward,o} = \bar{\epsilon}_{e,o}\sigma T_{sa,o}^4, \quad (2)$$

where $\bar{\epsilon}_{e,o} \equiv \bar{\epsilon}_e(T_{sa,o})$ and μ_o , which characterizes $\bar{\epsilon}_e(T_{sa})$ and is dimensionless, is defined by:

$$\mu_o \equiv \frac{1}{1 + \frac{1}{4} \left(\frac{d \ln \bar{\epsilon}_e}{d \ln T_{sa}} \right)_o} > 0. \quad (3)$$

Since a warmer atmosphere can hold more water vapor which is indeed the earth's dominant greenhouse gas, the resulting effective emissivity $\bar{\epsilon}_e$ is intuitively expected to be smaller, i.e. $(d \ln \bar{\epsilon}_e / d \ln T_{sa})_o < 0$. Thus the so-called water vapor feedback effect would be manifested by $\mu_o > 1$. Note that Eq.(2) says $4/\mu_o$ is the *effective* Stefan-Boltzmann exponent for T_{sa} of the earth.

For each planet in the solar system, the relevant F_{inward} is inversely proportional to the square of its distance from the sun. The observed planetary temperatures—including the earth—are approximately inversely proportional to the square root of their distances from the sun (Venus is an exception because of its well known overwhelming greenhouse effects). This observed scaling with distance from the sun

is classically explained in text books by taking the effective Stefan-Boltzmann exponent to be 4 for all the planets—i.e., $\mu_o \approx 1$. The value of earth's μ_o in Eq.(2) shall play a critical role in this paper.

Blackbody Radiation

Planck's blackbody radiation intensity (*watts/meter²* per wave length) formula is:

$$\mathcal{B}_\lambda(T_{sa}) \equiv \frac{2\pi hc^2}{\lambda^5(\exp(\frac{hc}{\lambda k T_{sa}}) - 1)}, \quad (4)$$

where h is the Planck constant, c is the speed of light, k is Boltzmann's constant, and λ is wave length.

For T_{sa} in the neighborhood of 300°K , the Wien Displacement Law says $\mathcal{B}_\lambda(300^\circ\text{K})$ peaks at roughly $\lambda \approx 10^{-5}$ *meters* (or around 10 microns). Carbon dioxide has a major absorption band centered around 15 microns. The lower and upper "outer edges" of this absorption band are at 11.5 and 19 microns, respectively. Carbon dioxide has three other infrared absorption bands, but their $T_{sa} \approx 300^\circ\text{K}$ blackbody radiation intensities are much smaller.

3 Radiative Transfer Equation

Let $\mathcal{J}_\lambda(z)$ denote the upward radiation intensity at altitude z . The appropriately averaged steady-state one-dimensional radiative transfer equation of the earth's atmosphere is:

$$\frac{d\mathcal{J}_\lambda}{dz} = \kappa_\lambda(z) [\epsilon_a(z)\mathcal{B}_\lambda(T(z)) - \mathcal{J}_\lambda], \quad (5)$$

where it had been assumed that the atmosphere emits as a grey body with some local emissivity $\epsilon_a(z)$ at its local temperature $T(z)$. The initial condition at $z = 0$ is: $\mathcal{J}_\lambda(0) \approx \epsilon_a(0)\mathcal{B}_\lambda(T_{sa})$. The amount of radiation that escape the earth is $\mathcal{J}_\lambda(\infty)$. Scattering and albedo effects are not included.

The absorption coefficient, $\kappa_\lambda(z)$, has the physical dimension of reciprocal length. By dimensional analysis, it is related to the number density $n_{CO_2}(z)$ of carbon dioxide in the atmosphere and its (collisional and Doppler broadened) absorption cross-section $\Psi_\lambda(p, T)$:

$$\kappa_\lambda(z) = n_{CO_2}(z)\Psi_\lambda(p, T). \quad (6)$$

Since carbon dioxide is a “well-mixed” greenhouse gas, this is approximated by:

$$\kappa_\lambda(z) = \bar{X}_* n_{AIR}(z) \bar{\Psi}_\lambda. \quad (7)$$

where \bar{X}_* is the average carbon dioxide mole fraction in the atmosphere, $n_{AIR}(z)$ is the number density of atmospheric air, and $\bar{\Psi}_\lambda$ is a suitably coarse-grained averaged (over the mess of many broadened lines) absorption cross section (now altitude independent by the the averaging) at wave length λ . The dimensionless *optical thickness of the atmosphere at sea level* is denoted by $\tau_\nu(\infty)$ and is defined by:

$$\tau_\lambda(\infty) \equiv \int_0^\infty \kappa_\lambda(z) dz \approx \frac{\bar{X}_*}{\bar{X}_{*,o}} N_{CO_2,o} \bar{\Psi}_\lambda. \quad (8)$$

where $N_{CO_2,o}$ is the total number of carbon dioxide molecules in a vertical column of unit area at the earth’s surface (before the industrial revolution):

$$N_{CO_2,o} \equiv \bar{X}_{*,o} \int_0^\infty n_{AIR}(z) dz \approx 5.2 \times 10^{25} / \text{meter}^2. \quad (9)$$

The atmosphere is said to be optically thin at sea level if $\tau_\lambda(\infty) \ll \ln_e 10 \approx 2.3$, and optically thick if $\tau_\lambda(\infty) \gg \ln_e 10$.

Note that for any $\kappa_\lambda(z) > 0$, Eq.(5) says $d\mathcal{J}_\lambda/dz$ is negative (or positive) whenever $\mathcal{J}_\lambda(z)$ is larger (or smaller) than the local $\epsilon_a(z)\mathcal{B}_\lambda(T(z))$. When $\kappa_\lambda(z)$ is locally sufficiently small, then the atmosphere is locally optically thin, and $d\mathcal{J}_\lambda/dz$ is then small *regardless* of the local values of $\epsilon_a(z)\mathcal{B}_\lambda(T(z))$. It is impossible for $\mathcal{J}_\lambda(\infty)$ —the amount of radiation that escapes—to be lower than the minimum value of $\epsilon_a(z)\mathcal{B}_\lambda(T(z))$ in the whole atmosphere if its initial value $\mathcal{J}_\lambda(0)$ is higher—which is in fact the case. Consequently, some radiation at any λ always manages to escape the earth even when the atmosphere is optically thick at sea level.

If the atmosphere were optically thin at sea level for *all* wave lengths in the absorption bands, ΔT_{CO_2} would be expected to vary linearly with $\bar{X}_*/\bar{X}_{*,o}$:

$$\Delta T_{CO_2} = \beta \left(\frac{\bar{X}_*}{\bar{X}_{*,o}} - 1 \right), \quad (10)$$

and β here would be the IPCC-defined equilibrium climate sensitivity. But the earth’s atmosphere is *not* optically thin at sea level with respect to *all* wave lengths in the carbon dioxide absorption bands.

There are ample data (e.g. direct radiation measurements by satellites) to conclude that the earth's atmosphere is optically thick at sea level for wave lengths in the middle of the 15 micron carbon dioxide absorption band, and that some finite amount of such radiations always manage to escape.

The Logarithmic Response

If the atmosphere were optically thick for *all* wave lengths in this major absorption band, one would expect ΔT_{CO_2} to be independent of $\bar{X}_*/\bar{X}_{*,o}$ —since absorption of all radiations in the band would be saturated. However, since $\bar{\Psi}_\lambda$ is (by definition) “negligibly small” at the “edges” of the band, it is not possible for the atmosphere to be optically thick for *all* the wave lengths inside the band. As $\bar{X}_*/\bar{X}_{*,o}$ is increased, a wave length slice at the edge of the band—which was previously optically thin—will become optically thick. Thus more radiation is absorbed by the new slice. This picture is understood and is generally accepted. Let $\lambda_{edge,o}$ denote the original edge of the absorption band (i.e. defined by $\Psi_{\lambda_{edge,o}} \equiv 2.3/N_{CO_2,o}$)—it separates the optically thick region from the optically thin regions when $\bar{X}_*/\bar{X}_{*,o} = 1$.

Fig. 1 shows the carbon dioxide data, $\log_{10} \bar{\Psi}_\lambda$ vs λ , for the 15 micron band. In the vicinity of $\lambda \approx \lambda_{edge,o}$, $\ln_e \bar{\Psi}_\lambda$ can be approximated by a Taylor series:

$$\ln_e \bar{\Psi}_\lambda = \ln_e \bar{\Psi}_{\lambda_{edge,o}} + \gamma(\lambda - \lambda_{edge,o}) + \dots \quad (11)$$

where γ is dimensional and is defined by:

$$\gamma \equiv \left(\frac{d \ln_e \bar{\Psi}_\lambda}{d\lambda} \right)_{\lambda_{edge,o}}. \quad (12)$$

Note that the γ 's have opposite signs at the two edges of the absorption band (and Fig. 1 says both $|\gamma|$'s are roughly the same). In the vicinity of $\lambda_{edge,o}$, $\tau_\lambda(\infty)$ can now be approximated by:

$$\begin{aligned} \tau_\lambda(\infty) &= N_{CO_2,o} \frac{\bar{X}_*}{\bar{X}_{*,o}} \bar{\Psi}_\lambda \\ &\approx N_{CO_2,o} \exp \left\{ \ln_e \frac{\bar{X}_*}{\bar{X}_{*,o}} + \ln_e \bar{\Psi}_{\lambda_{edge,o}} + \gamma(\lambda - \lambda_{edge,o}) + \dots \right\} \end{aligned}$$

$$= 2.3 \exp \left\{ \gamma \left(\lambda - \underbrace{\left(\lambda_{edge,o} - \frac{1}{\gamma} \ln_e \frac{\bar{X}_*}{\bar{X}_{*,o}} \right)}_{\text{new effective } \lambda_{edge}} \right) + \dots \right\}. \quad (13)$$

Eq.(13) says an increase of $\bar{X}_*/\bar{X}_{*,o}$ will move the location of the dividing effective λ_{edge} outward: a thin slice of thickness $\Delta\lambda$ previously in the optically thin region will be brought into the optically thick region. Eq.(13) says the value of $\Delta\lambda$ is:

$$\Delta\lambda = \frac{1}{|\gamma|} \ln_e \frac{\bar{X}_*}{\bar{X}_{*,o}}. \quad (14)$$

The incremental amount of radiation absorbed (over and above the pre-industrial revolution value by the movement of one edge) in response to increased atmospheric carbon dioxide is then:

$$\Delta F_{CO_2} = \eta \bar{\epsilon}_{e,o} \mathcal{B}_{\lambda_{edge}}(T_{sa}) \Delta\lambda, \quad (15)$$

where $\eta < 1$ is a dimensionless fudge factor introduced to account for *any other relevant physics*, such as the escape of radiant energy at any λ by thermal radiation $\epsilon_a(z) \mathcal{B}_\lambda(T(z))$ emitted at some high and cooler altitude z , and the effects of overlapping water vapor absorption bands.

Putting Eq.(2), Eq.(14) and Eq.(15) together, one finds both ΔF_{CO_2} and ΔT_{CO_2} respond logarithmically to $\bar{X}_*/\bar{X}_{*,o}$:

$$\Delta F_{CO_2} = \bar{\epsilon}_{e,o} \alpha \ln \frac{\bar{X}_*}{\bar{X}_{*,o}}, \quad (16)$$

$$\Delta T_{CO_2} = \frac{\beta}{\ln 2} \ln \frac{\bar{X}_*}{\bar{X}_{*,o}}. \quad (17)$$

The explicit (dimensional) formulas for α and β are:

$$\alpha = \frac{\eta^<}{|\gamma|^<} \mathcal{B}_{\lambda_{edge}^<}(T_{sa}) + \frac{\eta^>}{|\gamma|^>} \mathcal{B}_{\lambda_{edge}^>}(T_{sa}), \quad (18)$$

$$\beta = \mu_o \alpha \frac{\ln 2}{4\sigma T_{sa}^3}. \quad (19)$$

Here superscripts $<$ and $>$ are markers for the lower and upper edges, and contributions from both edges have been included. Note that the

only carbon dioxide specific parameters involved are the values of the λ_{edge} 's and the $|\gamma|$'s. Most importantly, Eq.(19) says α and β are related. The derivation assumed the atmosphere at sea level is optically thick in the middle of the carbon dioxide 15 micron absorption band, and optically thin beyond *both* edges. All “climate models” or “constraints from observations” involvements in the above derivation (and all un-accounted-for complications) are carried by the dimensionless and *physically meaningful* η 's and μ_o .

Eq.(16) for ΔF_{CO_2} was displayed in the first row of Table 6.2 of the IPCC Climate Change 2001 report [3], but the logarithm received no emphasis in the text. A footnote of the table indicated that the formula—including the numerical value of its coefficient—was known before 1990 [4, 5, 6]. In Climate Change 2007, Fig. SPM 8 [2] graphically shows the nonlinear response of ΔT_{CO_2} ($^{\circ}C$) versus X_* . Eq.(17) can approximate the middle curve of this graph with $\beta \approx 3^{\circ}C$ —which is the IPCC “best estimate.”

Numerical value for β

For carbon dioxide's 15 micron band, $\lambda_{edge}^< \approx 13$ microns, $\lambda_{edge}^> \approx 17$ microns, and $|\gamma|^< = |\gamma|^> = |\gamma| \approx 3.5/micron$ (extracted from Fig. 1 which used data from HITRAN [7]). Using $T_{sa} = 293^{\circ}K$ in Eq.(4), one obtains $\mathcal{B}_{\lambda_{edge}^<} (293) \approx 23$ watts/meter²-micron and $\mathcal{B}_{\lambda_{edge}^>} (293) \approx 15$ watts/meter²-micron. Using $\eta^< \approx \eta^> \approx \eta$ in Eq.(18) and Eq.(19), one obtains $\alpha \approx 11\eta$ watts/meter² and $\beta \approx 1.4\mu_o\eta$ $^{\circ}C$. If $\eta < 1$ and $\mu_o \approx 1$, the “theoretical” β value from Eq.(19) would be below the “very unlikely” $1.5^{\circ}C$ IPCC threshold.

Myhre *et. al.* performed detailed calculations in 1998 on “direct” radiative forcing (no water vapor feedback), displayed Eq.(18) as one of the “simplified expressions,” and referenced the 1990 IPCC report [4] which recommended $\bar{\epsilon}_{e,o}\alpha \approx 5.35$ watts/meter² [5, see its Table 3]. This numerical value was quoted in the 2001 IPCC report [3] as mentioned earlier. In order for Eq.(18) here to agree quantitatively with this IPCC-endorsed recommendation, one would need to choose $\eta \approx 0.5/\bar{\epsilon}_{e,o}$. This $\eta < 1$ choice is credible because it is totally consistent with both the radiation escape physics which underlaid Eq.(18), and the currently available (satellite) observations.

Using IPCC's $\bar{\epsilon}_{e,o}\alpha \approx 5.35$ watts/meter², Eq.(19) yields $\beta \approx 0.7\mu_o$ $^{\circ}C$ (with $\bar{\epsilon}_{e,o} \approx 1$). Thus the IPCC $4.5^{\circ}C \geq \beta \geq 2^{\circ}C$ likely range implies $6.4 > \mu_o > 3$, or $-2.7 > (d \ln \bar{\epsilon}_e / d \ln T_{sa})_o > -3.3$.

In other words, the IPCC's likely β range *needs the earth's effective Stefan-Boltzmann exponent of T_{sa} to be between 0.6 and 1.3* (instead of the classical 4).

Are such low effective T_{sa} exponents credible when many nearly transparent infrared windows are still available (mostly on the short wave length side of the blackbody peak)?

Eq.(2), Eq.(18) and Eq.(19) obviously need careful scrutiny to fully vindicate the various approximations used. The physics of $\eta < 1$ is understood. *The physics of large μ_o needed by the IPCC "likely" range of β must also be understood.* The simplicity of the present derivation should confer some deference for the lower end of the IPCC recommended β values.

4 Policy Implications

The possibility that β could be in the IPCC "very unlikely" lower range has immense policy consequences. Whether β is $4.5^\circ C$ or $0.7\mu_o^\circ C$ (how big can μ_o reasonably be?) can make all the difference to the future of the world.

For fixed β , Eq.(10) and Eq.(17) are not too different for $1 \leq \bar{X}_*/\bar{X}_{*,o} \leq 2$. For $\bar{X}_*/\bar{X}_{*,o} > 2$, however, their difference is significant. Let E denote the annual amount of (fossil fuels) carbon emitted into the atmosphere in units of billion tons (of carbon-equivalent) per year¹. At the beginning of the 21st century, $E \approx 8$ billion tons per year. The current main strategy to mitigate global warming is to reduce E to some sufficiently low level. Socolow and Lam [8] recently examined several published carbon cycle stabilization simulations, and found that all of them allow a finite E (to be denoted by E_{stab}) after \bar{X}_* is stabilized at $\bar{X}_{*,stab}$. For the next few centuries, their "good enough" correlation of the published results is:

$$E_{stab} = 3\left(\frac{\bar{X}_{*,stab}}{\bar{X}_{*,o}} - 1\right) \text{ billion tons of carbon per year.} \quad (20)$$

Putting $\bar{X}_* = \bar{X}_{*,stab}$ in Eq.(17) and using Eq.(20), one obtains an estimate of the approximate amount of global warming after E has been successfully held at E_{stab} :

$$\Delta T_{CO_2} = \frac{\beta}{\ln 2} \ln\left(1 + \frac{E_{stab}}{3}\right). \quad (21)$$

¹One ton of carbon-equivalent equals to 3.7 tons of carbon dioxide.

This simple relation between ΔT_{CO_2} and E_{stab} (for any β) is a useful tool for the assessment of various difficult policy trade-offs. Suppose the world could tolerate $\Delta T_{CO_2} \leq \beta$. Then $\bar{X}_{*,stab}/\bar{X}_{*,o} \approx 2$ and $E_{stab} \approx 3$ billion tons per year—the latter number is recognized to be a very difficult target to achieve. If the world could tolerate $\Delta T_{CO_2} \leq 2\beta$, then the logarithmic response says $\bar{X}_{*,stab}/\bar{X}_{*,o} \approx 4$ and $E_{stab} \approx 9$ billion tons per year—these are much more forgiving numbers than those estimated by linear extrapolations (for any β).

Both the physics that underlies the prospect that β may be nearer the lower end of the IPCC range, and the physics that underlaid the IPCC's β need for a large μ_o , deserve attentions.

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Figure Caption

Carbon dioxide 15 micron band, $\log_{10} \Psi_\lambda$ vs. λ (HITRAN data).

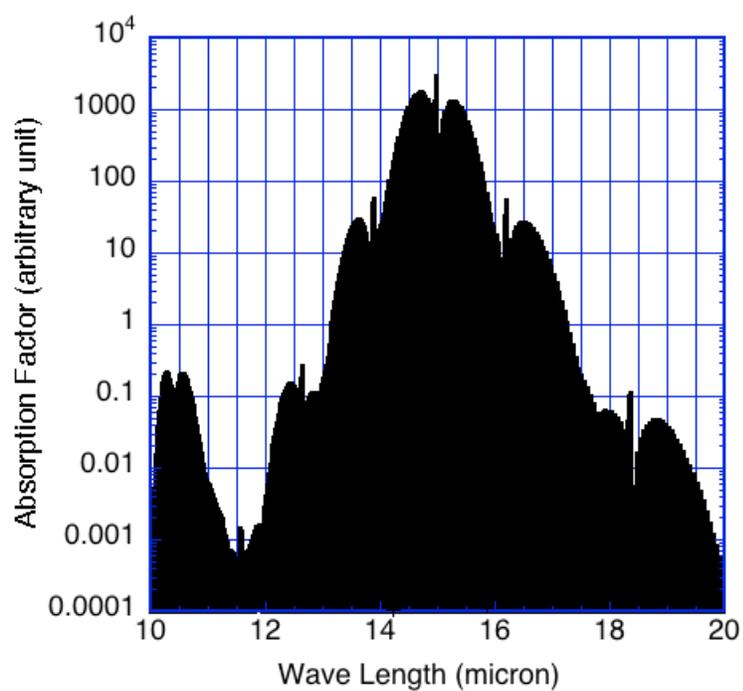


Figure 1: