Introduction to Econometrics (4th Edition)

by

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Solutions to Odd-Numbered End-of-Chapter Exercises: Chapter 17

(This version September 18, 2018)

17.1. Y_t follows a stationary AR(1) model, $Y_t = \beta_0 + \beta_1 Y_{t-1} + u_t$. The mean of Y_t is $\mu_Y = E(Y_t) = \frac{\beta_0}{1-\beta_1}$, and $E(u_t|Y_t) = 0.$

(a) The h-period ahead forecast of $Y_t, Y_{t+h|t} = E(Y_{t+h}|Y_t, Y_{t-1}, \quad)$, is

$$
\begin{aligned}
Y_{t+h|t} = E(Y_{t+h}|Y_t, Y_{t-1}, \quad) &= E(\beta_0 + \beta_1 Y_{t+h-1} + u_t|Y_t, Y_{t-1}, \quad)\\
&= \beta_0 + \beta_1 Y_{t+h-1|t} = \beta_0 + \beta_1(\beta_0 + \beta_1 Y_{t+h-2|t})\\
&= (1+\beta_1)\beta_0 + \beta_1^2 Y_{t+h-2|t}\\
&= (1+\beta_1)\beta_0 + \beta_1^2(\beta_0 + \beta_1 Y_{t+h-3|t})\\
&= (1+\beta_1+\beta_1^2)\beta_0 + \beta_1^3 Y_{t+h-3|t}\\
&=\\
&= \left(1+\beta_1+ \quad + \beta_1^{h-1}\right)\beta_0 + \beta_1^h Y_t\\
&= \frac{1-\beta_1^h}{1-\beta_1}\beta_0 + \beta_1^h Y_t\\
&= \mu_Y + \beta_1^h(Y_t - \mu_Y).
\end{aligned}
$$

(b) Substituting the result from part (a) into X_t gives

$$
\begin{aligned}
X_t = \sum_{i=0}^{\infty}\delta^i Y_{t+i|t} &= \sum_{i=0}^{\infty}\delta^i[\mu_Y + \beta_1^i(Y_t - \mu_Y)]\\
&= \mu_Y\sum_{i=0}^{\infty}\delta^i + (Y_t - \mu_Y)\sum_{i=0}^{\infty}(\beta_1\delta)^i\\
&= \frac{\mu_Y}{1-\delta} + \frac{Y_t - \mu_Y}{1-\beta_1\delta}.
\end{aligned}
$$

17.3. u_t follows the ARCH process with mean $E(u_t) = 0$ and variance $\sigma_t^2 = 1.0 + 0.5u_{t-1}^2$.

(a) For the specified ARCH process, u_t has the conditional mean $E(u_t|u_{t-1}) = 0$ and the conditional variance.

$$\text{var}(u_t|u_{t-1}) = \sigma_t^2 = 1.0 + 0.5u_{t-1}^2.$$

The unconditional mean of u_t is $E(u_t) = 0$, and the unconditional variance of u_t is

$$\begin{aligned}\text{var}(u_t) &= \text{var}[E(u_t|u_{t-1})] + E[\text{var}(u_t|u_{t-1})] \\ &= 0 + 1.0 + 0.5E(u_{t-1}^2) \\ &= 1.0 + 0.5\,\text{var}(u_{t-1}).\end{aligned}$$

The last equation has used the fact that $E(u_t^2) = \text{var}(u_t) + [E(u_t)]^2 = \text{var}(u_t)$, which follows because $E(u_t) = 0$. Because of the stationarity, $\text{var}(u_{t-1}) = \text{var}(u_t)$. Thus, $\text{var}(u_t) = 1.0 + 0.5\text{var}(u_t)$ which implies $\text{var}(u_t) = \frac{1.0}{0.5} = 2$.

(b) When $u_{t-1} = 0.2$, $\sigma_t^2 = 1.0 + 0.5 \times 0.2^2 = 1.02$. The standard deviation of u_t is σ_t = 1.01. Thus

$$\begin{aligned}\Pr(-3 \le u_t \le 3) &= \Pr\left(\frac{-3}{1.01} \le \frac{u_t}{\sigma_t} \le \frac{3}{1.01}\right) \\ &= \Phi(2.9703) - \Phi(-2.9703) = 0.9985 - 0.0015 = 0.9970.\end{aligned}$$

When $u_{t-1} = 2.0$, $\sigma_t^2 = 1.0 + 0.5 \times 2.0^2 = 3.0$. The standard deviation of u_t is σ_t = 1.732. Thus

$$\begin{aligned}\Pr(-3 \le u_t \le 3) &= \Pr\left(\frac{-3}{1.732} \le \frac{u_t}{\sigma_t} \le \frac{3}{1.732}\right) \\ &= \Phi(1.732) - \Phi(-1.732) = 0.9584 - 0.0416 = 0.9168.\end{aligned}$$

17.5. Because $Y_t = Y_t - Y_{t-1} + Y_{t-1} = Y_{t-1} + \Delta Y_t$,

$$\sum_{t=1}^{T} Y_t^2 = \sum_{t=1}^{T} (Y_{t-1} + \Delta Y_t)^2 = \sum_{t=1}^{T} Y_{t-1}^2 + \sum_{t=1}^{T} (\Delta Y_t)^2 + 2\sum_{t=1}^{T} Y_{t-1}\Delta Y_t.$$

So

$$\frac{1}{T}\sum_{t=1}^{T} Y_{t-1}\Delta Y_t = \frac{1}{T}\times\frac{1}{2}\left[\sum_{t=1}^{T} Y_t^2 - \sum_{t=1}^{T} Y_{t-1}^2 - \sum_{t=1}^{T} (\Delta Y_t)^2\right].$$

Note that $\sum_{t=1}^{T} Y_t^2 - \sum_{t=1}^{T} Y_{t-1}^2 = \left(\sum_{t=1}^{T-1} Y_t^2 + Y_T^2\right) - \left(Y_0^2 + \sum_{t=1}^{T-1} Y_t^2\right) = Y_T^2 - Y_0^2 = Y_T^2$ because $Y_0 = 0$. Thus:

$$\frac{1}{T}\sum_{t=1}^{T} Y_{t-1}\Delta Y_t = \frac{1}{T}\times\frac{1}{2}\left[Y_T^2 - \sum_{t=1}^{T} (\Delta Y_t)^2\right]$$

$$= \frac{1}{2}\left[\left(\frac{Y_T}{\sqrt{T}}\right)^2 - \frac{1}{T}\sum_{t=1}^{T} (\Delta Y_t)^2\right].$$

17.7.

$$\hat{\beta} = \frac{\sum_{t=1}^{T} Y_t X_t}{\sum_{t=1}^{T} X_t^2} = \frac{\sum_{t=1}^{T} Y_t \Delta Y_{t+1}}{\sum_{t=1}^{T} (\Delta Y_{t+1})^2} = \frac{\frac{1}{T}\sum_{t=1}^{T} Y_t \Delta Y_{t+1}}{\frac{1}{T}\sum_{t=1}^{T} (\Delta Y_{t+1})^2}.$$

Following the hint, the numerator is the same expression as (17.21) (shifted forward in time 1 period), so that $\frac{1}{T}\sum_{t=1}^{T} Y_t \Delta Y_{t+1} \xrightarrow{d} \frac{\sigma_u^2}{2}(\chi_1^2 - 1).$ The denominator is $\frac{1}{T}\sum_{t=1}^{T} (\Delta Y_{t+1})^2 = \frac{1}{T}\sum_{t=1}^{T} u_{t+1}^2 \xrightarrow{p} \sigma_u^2$ by the law of large numbers. The result follows directly.

17.9. (a) From the law of iterated expectations

$$
\begin{aligned}
E(u_t^2) &= E\left(\sigma_t^2\right) \\
&= E\left(\alpha_0 + \alpha_1 u_{t-1}^2\right) \\
&= \alpha_0 + \alpha_1 E\left(u_{t-1}^2\right) \\
&= \alpha_0 + \alpha_1 E\left(u_t^2\right)
\end{aligned}
$$

where the last line uses stationarity of u. Solving for $E(u_t^2)$ gives the required result.

(b) As in (a)

$$
\begin{aligned}
E(u_t^2) &= E\left(\sigma_t^2\right) \\
&= E\left(\alpha_0 + \alpha_1 u_{t-1}^2 + \alpha_2 u_{t-2}^2 + \quad + \alpha_p u_{t-p}^2\right) \\
&= \alpha_0 + \alpha_1 E\left(u_{t-1}^2\right) + \alpha_2 E\left(u_{t-2}^2\right) + \quad + \alpha_p E\left(u_{t-p}^2\right) \\
&= \alpha_0 + \alpha_1 E\left(u_t^2\right) + \alpha_2 E\left(u_t^2\right) + \quad + \alpha_p E\left(u_t^2\right)
\end{aligned}
$$

so that $E(u_t^2) = \dfrac{\alpha_0}{1 - \sum_{t=1}^{p} \alpha_i}$

(c) This follows from (b) and the restriction that $E(u_t^2) > 0$.

(d) As in (a)

$$
\begin{aligned}
E(u_t^2) &= E\left(\sigma_t^2\right) \\
&= \alpha_0 + \alpha_1 E\left(u_{t-1}^2\right) + \phi_1 E\left(\sigma_{t-1}^2\right) \\
&= \alpha_0 + (\alpha_1 + \phi_1) E\left(u_{t-1}^2\right) \\
&= \alpha_0 + (\alpha_1 + \phi_1) E\left(u_t^2\right) \\
&= \frac{\alpha_0}{1 - \alpha_1 - \phi_1}
\end{aligned}
$$

(e) This follows from (d) and the restriction that $E\left(u_t^2\right) > 0$.