Calibrated Uncertainty[†]

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July 2014

Abstract

We define a binary relation (*qualitative uncertainty assessment (QUA)*) to describe a decision-maker's subjective assessment of likelihoods on a collection of events. To model ambiguity, we permit incompleteness. Our axioms yield a representation according to which A is more likely than B if an only if a capacity, called *uncertainty measure (UM)* assigns a higher value to A than to B and a higher value A-complement than B-complement. We prove uniqueness, show that UMs are belief functions and describe how QUAs extended to utility functions on acts with a sophistication axiom.

[†] This research was supported by grants from the National Science Foundation.

1. Introduction

A binary relation on a collection of events describes the the likelihood assessments of a group of decision makers. These decision makers may perceive ambiguity; that is, they may be indifferent between betting on an A or $A^c := \Omega \backslash A$, indifferent between betting on B and on B^c, but may nevertheless prefer betting on A to betting on B. As a result of such perceived ambiguity, the binary relation may be incomplete; that is,it may be impossible to rank certain bets without specifying a decision maker's attitude to ambiguity. Our main result provides a cardinal representation of this binary relation.

Savage (1972) provides an analogous result for a *qualitative probability* (QP); that is, when decision makers perceive no ambiguity. It yields a cardinal measure of likelihood, a probability, that can be used in conjunction with either expected utility theory or non-expected utility alternatives. A statement such as *"voter i assigns probability* .3 *to the event that that party X will win the election,"* is meaningful even without knowing this decision maker's attitude toward gambles and even if we don't agree with her assessment. In particular, if this voter also assigns probability .3 to party Y winning the election and probability .4 to party Z winning, then virtually every theory of choice among gambles would conclude that she is indifferent between betting on a party X victory and a party Y victory, she prefers betting on a party Z victory to either of the previous two bets, and she prefers betting on a party Z loss (i.e., victory by either party X or party Y) to betting on a party Z victory.

Thus, a cardinal measure of uncertainty provides a separation between the individual's *perception of uncertainty* and *attitude towards* uncertainty. Two decision makers may perceive the same uncertainty in the prospect f, they may also perceive the same uncertainty in the prospect g; that is, they may assign the same probability distribution over prizes to the act f and the same probability distribution over prizes to act g, but nevertheless one may prefer f while the other prefers act g possibly because they have different levels of risk aversion.

In many economic models, the implicit or explicit assumption is that agents share a common uncertainty perception, that is, have the same priors and their current probability assessments differ only because they have received different information. To extend this

methodology to settings with perceived ambiguity requires a device for uncertainty measurement analogous to a probability measure. Such a device is not available in the presence of ambiguity. A nonadditive probability, that is, a capacity, plays a central role in Choquet expected utility theory (Schmeidler (1989)). However, it plays a role both in describing the decision maker's perception of uncertainty and her attitude towards it. An alternative to a capacity is a *set of probabilities*, as used in Maxmin Expected utility theory (Gilboa and Schmeidler (1989)). Indeed Ghirardato, Maccheroni and Marinacci (2004) and Ghirardato and Siniscalchi (2013) derive a set of "perceived priors" for a variety of models. As the authors note (see, for example, GMM page 137), this measure may confound ambiguity attitude and perception and, more importantly, is derived under the assumption that all deviations from expected utility theory are due to ambiguity.[1]

To circumvent these problems, we propose an alternative approach, analogous to that of Gilboa, Maccheroni, Marinacci and Schmeidler (2010). To illustrate the central idea, let E^1 and E^2 be two unambiguous events and let A be an ambiguous event. We interpret $\succeq$ as the common qualitative uncertainty assessment of a group of individuals and $E^1 \succeq A \succeq E^2$ means that even the most ambiguity averse person in this group prefers a bet on A to a bet on E^2 while even the most ambiguity loving member of the group prefers E^1 to A. By identifying a "tight" window of unambiguous events of this form, we determine the range of probabilities that A might have. Ambiguity renders the binary relation incomplete; there are events E and A such that the most ambiguity averse members of the group prefer a bet on E over a bet on A while the least ambiguity averse members have the reverse preference.

Our main theorem (Theorem 1) is a representation theorem for qualitative uncertainty assessment; it provides axioms such that a special type of capacity represents it. We call such a capacity an *uncertainty measures* and introduce them in section 2 below. Section 3 introduces the axioms for qualitative uncertainty assessments and states the representation theorem. Section 4 analyzes utility functions over monetary acts that are compatible with an uncertainty measure.

[1] See also Klibanoff, Mukherji and Seo (2014).

2. Cardinal Measurement of Uncertainty

2.1 Preliminaries

Throughout this paper, Ω is the state space and Σ is a σ-algebra of events. Henceforth, all sets considered are elements of Σ. A *set function* q is a mapping from any sub-σ-algebra of Σ to $[0,1]$ such that $q(\emptyset) = 0$, and $q(A) \leq q(B)$ whenever $A \subset B$. A set function q is *additive* if $q(A \cup B) + q(A \cap B) = q(A) + q(B)$; it is *continuous* if $A_{n+1} \subset A_n$ for all n implies $q\left(\bigcap A_n\right) = \lim q(A_n)$. It is not difficult to verify that an additive set function is continuous if and only if it is countably additive.

When we wish to be explicit about the domain of q, we write $(q, \mathcal{A})$. Given the set function $(q, \mathcal{A})$, the set $A \in \mathcal{A}$ is *null* if $q(A^c) = 1$ and *whole* if $B \subset A$ implies $B \in \mathcal{A}$. A set function is *complete* if every null set is whole. A complete, continuous, and additive set function q is a *probability measure* if $q(\Omega) = 1$ and a *subprobability measure* if $q(\Omega) < 1$. A probability measure $(\mu, \mathcal{A})$ is *nonatomic* if $A \in \mathcal{A}$, $\mu(A) > 0$ implies there is $B \subset A$, $B \in \mathcal{A}$ such that $0 < \mu(B) < \mu(A)$.

A set function q is a capacity if $q(\Omega) = 1$. Let $N = \{1, \ldots, n\}$. A capacity q is a *belief function* if $q\left(\bigcup_{i \in N} A_i\right) \geq \sum_{I \subset N, I \neq \emptyset} (-1)^{|I|+1} q\left(\bigcap_I A_i\right)$ for all n and for any collection of sets $A_1, \ldots, A_n$,

2.2 Uncertainty Measures

In this section, we will define a class of capacities, *uncertainty measures*, that represent the uncertainty perception of agents under the axioms provided in section 3. An uncertainty measure has two components: a probability measure $(\mu, \mathcal{E})$ that quantifies the uncertainty of unambiguous events (we refer to $(\mu, \mathcal{E})$ as the *risk measure*) and a subprobability measure (η, Σ) that quantifies the uncertainty of ambiguous events (we refer to (η, Σ) as the *ambiguity measure*). When an event E is μ-whole it means that E and every subset of E is unambiguous. In that case, ambiguity plays no role when quantifying the uncertainty of subsets of E. When E is not $\mu-$whole then some subsets of E are not $\mathcal{E}$-measurable. That is, there are $A, B \subset E$ such that $A \cup B = E$ but neither A nor B are unambiguous. In that case, the ambiguity measure (η, Σ) places bounds on the probabilities of A and B.

A probability measure $(\mu, \mathcal{E})$ and a subprobability measure (η, Σ) are *compatible* if $\mu(E) > \eta(E)$ for every μ-nonnull $E \in \mathcal{E}$, and $\eta(E) = 0$ for every μ-whole $E \in \mathcal{E}$. A capacity π is an *uncertainty measure* if there exists a non-atomic probability measure $(\mu, \mathcal{E})$ and a compatible, non-atomic subprobability measure (η, Σ) such that

$$\pi(A) = \max_{E \in [A]} \{\mu(E) + \eta(A \backslash E)\} \tag{1}$$

where $[A] = \{E \in \mathcal{E}, E \subset A\}$. Thus, the uncertainty measure of a set A is calculated by finding the maximal unambiguous subset of each event and adding to it the ambiguity measure of the residual. Since $(\mu, \mathcal{E})$ and (η, Σ) are compatible, it follows that (1) is equivalent to (2) below:

$$\pi(A) = \max_{E \in [A]} \mu(E) + \min_{E \in [A]} \eta(A \backslash E) \tag{2}$$

If π, μ, η satisfy the conditions above, we say that μ is a risk measure for π and η is an ambiguity measure for π.

Proposition 1, below, shows that for any uncertainty measure π there is a unique decomposition into risk and ambiguity measures. For any capacity π, the set A is π-*unambiguous* if $\pi(A) + \pi(A^c) = 1$ and $\mathcal{E}_\pi$ is the collection of all π-unambiguous sets.

Proposition 1: *(i) If $(\mu, \mathcal{E})$ is a risk measure for the uncertainty measure π, then $\mathcal{E} = \mathcal{E}_\pi$; (ii) every uncertainty measure has a unique risk measure and a unique ambiguity measure.*

Not all capacities can be uncertainty measures:

Proposition 2: *Every uncertainty measure is a belief function.*

In the following section, we consider a binary relation $\succeq$ on Σ. We will impose axioms on this binary relation that yield the following representation:

$$A \succeq B \text{ if and only if } \begin{cases} \pi(A) \geq \pi(B) & \text{and} \\ \pi(B^c) \geq \pi(A^c) \end{cases}$$

where π is an uncertainty measure.

3. Qualitative Uncertainty Assessments

The primitive is a binary relation $\succeq$ on Σ. We interpret $A \in \Sigma$ as a bet; one that the decision-maker wins if and only if a state in A occurs and interpret $A \succeq B$ to mean that the decision maker prefers the bet A to the bet B *regardless of her ambiguity attitude.* Thus, the binary relation $\succeq$ may be incomplete. This incompleteness is resolved once the decision maker's ambiguity attitude is taken into account.

As a benchmark, we state a continuous/countably additive version of Savage's QP representation theorem: call a binary relation $\succeq$ on Σ a QP if it is complete, transitive, *nondegenerate* (i.e., $\emptyset \not\succeq \Omega$) and satisfies the following additivity axiom:

Axiom A: *$(A \cup B) \cap C = \emptyset$ implies $A \succeq B$ if and only if $A \cup C \succeq B \cup C$.*

Let $\succ$ denote the strict part of $\succeq$. A finite collection of sets $C_1, \ldots, C_n$ is a partition if the sets are pairwise disjoint and $\bigcup C_k = \Omega$. Call a binary relation $\succeq$ *fine* if it satisfies the following axiom:

Axiom F: *$A \succ B$ implies there is a partition $C_1 \ldots C_n$ such that $A \succ B \cup C_n$ for all n.*

Finally, call a binary relation $\succeq$ continuous if it satisfies the following axiom.

Axiom C: *$A_n \succeq B$ and $A_{n+1} \subset A_n$ for all n implies $\bigcap A_n \succeq B$.*

Then, we say that a binary relation has a (nonatomic) probability representation if there exists a (nonatomic) probability measure (μ, Σ) such that $A \succeq B$ if and only if $\mu(A) \geq \mu(B)$. Savage provides a finitely additive probability representation of a QP that satisfies Axiom F. It is straightforward to see that with Axiom C, we get a non-atomic probability representation. Formally, we have:

Savage's Theorem: *A binary relation has a nonatomic probability representation if and only if it is a QP that satisfies Axioms C and F.*

Our goal is to provide a version of Savage's Theorem for decision makers who perceive ambiguity. We will use the following stylized version of an Ellsberg-type thought experiment to illustrate our model and the axioms. An urn has n balls, numbered from 1 to n. For each integer $i \in N := \{1, \ldots, n\}$, there is exactly one ball with number i. Each ball has one of k colors, that is, each ball i has some color $k \in K := \{c_1, \ldots, c_k\}$ and, therefore,

the state space is $\Omega = N \times K$. Let $A_j \subset \Omega$ denote the event that a ball of color c_j is drawn and E_i denote the event that ball number i is drawn. Nothing is known about the composition of colors in the urn or about the color of any particular ball i.

Suppose for a moment that $n = k = 10$. Let $A^j = \bigcup_{t=1}^{j} A_t$ and $E^i = \bigcup_{t=1}^{i} E_t$. Thus, E^i denotes the event that one of the first i balls will be drawn while A^j denotes the event that the drawn ball will have one of the colors in $\{c_1, \ldots, c_j\}$. How would be expect experimental subjects to rank A^j and E^i? If the subject is ambiguity averse, she might prefer betting on E^5 to betting on A^5. On the other hand, an ambiguity loving subject may prefer betting on A^5 to betting on E^5. Even when comparing two ambiguity averse subjects, it might be that one prefers E^5 to A^5 but would rather bet on A^5 than on E^4 while the other is more ambiguity averse; that is even prefers E^4 to A^5. Thus, simple comparisons of rankings bets by a single individual do not enable us to distinguish ambiguity perception from attitude towards ambiguity.

So, we propose an alternative approach analogous to that of Gilboa, Maccheroni, Marinacci and Schmeidler (2010). We interpret $\succeq$ as the common qualitative uncertainty assessment of a group of individuals. Thus, $E^7 \succeq A^5 \succeq E^3$ means that even the most ambiguity averse person in this group prefers A^5 to E^3 while even the most ambiguity loving member of the group prefers E^7 to A^5. By identifying a "tight" window of unambiguous events of this form, we determine the range of probabilities that A^5 might have. Specifically, assume that $E^7 \succeq A^5 \succeq E^3$ and $A^5 \not\succeq E^4$ and $E^6 \not\succeq A^5$. Thus, we conclude that the range of probabilities for A^5 is $[.3, .7]$. Hence, the comparison between A^5 and E^5 is indeterminate; i.e., $A^5 \not\succeq E^5$ and $E^5 \not\succeq A^5$. This indeterminacy does not reflect the incompleteness of a single agent's preferences; rather is reflects the fact that the group's common uncertainty perception by itself is not sufficient to rank the two bets E^5 and A^5.

Our main axiom, *calibration* encapsulates this notion of a common uncertainty perception. Let $W(A) = \{E \mid A \succeq E\}$. The binary relation $\succeq$ is *calibrated* if

$$A \succeq B \text{ if and only if } \begin{cases} W(B) \subset W(A) & \text{and} \\ W(A^c) \subset W(B^c) \end{cases}$$

We write $A \succ B$ when both of these inclusions are strict.

In our Ellsberg experiment, as in Ellberg's original thought experiments, the wording of the choice problem makes it clear which class of events are unambiguous and which ones are not. Events that correspond to outcomes of a symmetric game of chance (draws from an urn with a known composition) are unambiguous while events that correspond the decisions of the experimenter are ambiguous. In our model, we do not assume that the distinction between ambiguous events and unambiguous events is self-evident or exogenous. Instead, we provide a criterion for identifying unambiguous events and calibrate the ambiguous events with them.

Implicitly or explicitly, there are at least two different notion of an unambigous event in the literature. Some formal models and empirical/experimental work accept what we might call a "relative" notion of unambiguous events. In this approach, unambiguousness is a property of a collection of sets for which the more likely relationship is preserved whenever two unambiguous sets A, B are combined with a third disjoint unambiguous set C; that is, $A \succeq B$ if and only if $A \cup C \succeq B \cup C$ whenever $A \cap C = B \cap C = \emptyset$.[2] Other formulations assume (or imply) what we might call an "absolute" notion of unambiguousness. Here, unambiguousness is a property of a single set and a set E is deemed unambiguous if the more likely relationship is preserved when *any* A, B such that $(A \cup B) \cap E = \emptyset$ is combined with E.[3] We take this second approach.

Definition: *An event A is null if $B \succeq B \cup A$ for all B; E is unambiguous if $A \succeq B$ if and only if $A \cup E \succeq B \cup E$ whenever $(A \cup B) \cap E = \emptyset$.*

Let $\mathcal{E}$ denote the set of all unambiguous events. Henceforth, we use A, B, C, D to denote generic elements of Σ and E, F, G to denote generic elements of $\mathcal{E}$. The binary relation is *monotone* if $A \subset B$ implies $B \succeq A$ and $B \succ A$ if, in addition, $B \backslash A$ is non-null. The binary relation $\succeq$ is *non-degenerate* if Ω is non-null.

Axiom 1: *The binary relation $\succeq$ is monotone, non-degenerate and calibrated.*

It is easy to verify that a calibrated $\succeq$ must be transitive. However, we permit incompleteness to allow for the decision maker's ranking of bets to depend on her ambiguity

[2] This view is implicit in the notion of a source as in defined in Fox and Tversky (1995) and studied Gul and Pesendorfer (2014).

[3] See for example, Epstein and Zhang's (2001) notion of an unambiguous event and Gul and Pesendorfer's (2014) notion of an ideal event.

attitude. For any event A, let $\langle A \rangle$ denote the sets of unambiguous events that A preserves; that is,

$$\langle A \rangle = \{E \mid E \cap A \in \mathcal{E}, E \cap A^c \in \mathcal{E}\}$$

Suppose the first color in our urn example is red and the second color is blue. Then, $A = A_1 \cap E_j$ is the event that ball j is red whereas $B = A_2 \cap E_j$ is the event that ball j is green. These two events are similar because they both preserve the same unambiguous events; that is,

$$\langle A \rangle = \langle B \rangle = \{E \mid E \cap E_j = \emptyset\}$$

That is, $\langle A \rangle$ and $\langle B \rangle$ consist of all unambiguous events that do not contain E_j.

Definition: *A, B are similar ($A \cong B$) if $\langle A \rangle = \langle B \rangle$.*

When two events are similar, we expect the decision maker to be able to compare them solely based on her perception of ambiguity. To put it differently, when two sets are similar, they are equally ambiguous and hence the decision maker's attitude towards ambiguity plays no role in determining her betting preference over these events. In the example, we might expect decision makers to find A and B to be equally likely. Moreover, should they perceive one event to be more likely than the other, then this preference is unaffected by their ambiguity attitude. This is the first part of Axiom 2 below.

The second part of Axiom 2 requires unambiguous events to be similar. The two parts together ensure that the restriction of $\succeq$ to the set of unambiguous events is complete. A second consequence of Axiom 2 is that the set of unambiguous events is closed under unions and complements; that is, it is an algebra.

Axiom 2: *(i) $A \cong B$ implies $A \succeq B$ or $B \succeq A$. (ii) $E \cong F$.*

The familiar additivity axiom of qualitative probability requires that $A \succeq B$ if and only if $A \cup C \succeq B \cup C$ whenever A, C and B, C are both disjoint. Axiom 3 below weakens the additivity requirement of qualitative probability to permit ambiguity.

To see why we need a weaker form of additivity consider again our Ellsberg experiment. Consider a decision maker who finds the events $A = A_1 \cap E_1$ ("ball 1 will be drawn and it will be red") and $B = A_1 \cap E_2$ ("ball 2 will be drawn and it will be red") equally likely.

Consider the event $C = E_1 \backslash A_1$ ("ball 1 will be drawn and it will not be red.") Then $A \cup C = E_1$ ("ball one is drawn") is an unambiguous event while $B \cup C$ is not. Therefore, ambiguity averse agent might prefer $A \cup C$ to $B \cup C$ while an ambiguity loving agent might have the opposite preference.

Such a situation arises when $A \cup C$ creates a new unambiguous event while $B \cup C$ does not or $B \cup C$ destroys new unambiguous events while $A \cup C$ does not. For any two collections of unambiguous sets $\mathcal{A}, \mathcal{B} \subset \mathcal{E}$, let $\mathcal{A} - \mathcal{B} = \{E \in \mathcal{A} \,|\, E \cap F \sim \emptyset \text{ for all } F \in \mathcal{B}\}$ be the unambiguous sets in the collection $\mathcal{A}$ that $\mathcal{B}$ does not intersect. Note that in the above example:

$$\langle A \cup C \rangle - \langle A \rangle = \{E_1, \emptyset\}$$

and, therefore, adding C to A *creates* the new unambiguous set E_1. Conversely,

$$\langle B \rangle - \langle B \cup C \rangle = \{E_1, \emptyset\}$$

and, therefore, adding C to B *destroys* the unambiguous set E_1. Finally, adding C to B creates no new unambiguous sets (the first line of the display equation below) and adding C to A destroys no new unambiguous sets (the second line of the display equation below):

$$\langle B \cup C \rangle - \langle B \rangle = \{\emptyset\}$$
$$\langle A \rangle - \langle A \cup C \rangle = \{\emptyset\}$$

We require additivity to hold only when adding C to A creates the same new unambiguous sets as adding C to B and adding C to A destroys the same unambiguous sets as adding C to B. Formally,

Definition: *A, C and B, C conform if $(A \cup B) \cap C = \emptyset$, $\langle A \cup C \rangle - \langle A \rangle = \langle B \cup C \rangle - \langle B \rangle$ and $\langle A \rangle - \langle A \cup C \rangle = \langle B \rangle - \langle B \cup C \rangle$.*

When A, C and B, C conform, we are in a situation such that if A was more likely than B for all decision makers with a given perception of uncertainty, then Ellberg-style thought experiments do not suggest that additivity should be violated. For example, suppose $A = (E_1 \cup E_2) \cap A_1$ ("ball 1 or 2 will be drawn and it will be red") while $B = E_1 \cap (A_1 \cup A_2)$ ("ball 1 will be drawn and it will be red or green.") Notice that these

two events are not similar and, therefore, they need not be ranked. However, if there is agreement and, for example, all decision makers agree that A and B are equally likely then we require that they consider $A \cup C$ and $B \cup C$ equally likely where $C = E_3 \cap A_2$ ("ball 3 will be drawn and it is green.") To see that A, C and B, C conform, note that

$$\langle A \cup C \rangle - \langle A \rangle = \{\emptyset\}$$
$$\langle B \cup C \rangle - \langle B \rangle = \{\emptyset\}$$
$$\langle A \rangle - \langle A \cup C \rangle = \{E_3, \emptyset\}$$
$$\langle B \rangle - \langle B \cup C \rangle = \{E_3, \emptyset\}$$

Axiom 3: *If A, C and B, C conform, then $A \succeq B$ if and only if $A \cup C \succeq B \cup C$.*

Our axioms allow but do not require that the decision-maker perceives ambiguity. However, we do impose the existence of a rich set of unambiguous events. This assumption is implicit in any uncertainty model stated in the Anscombe-Aumann framework and in Savage's formulation of subjective expected utility. We also assume that every event, unambiguous or not, can be divided into finer, similar events. The following version of Axiom F ensures that this is so.

Axiom 4: *$A \succ B$ implies there is a partition $C_1, \ldots, C_k$ such that $C_n \backslash B \cong B$ and $A \succ B \cup C_n$ for all n.*

The following axiom is the counterpart of the continuity axiom (Axiom C) from the Savage framework. It ensures that unambiguous events are closed not countable (not just finite) unions. It also ensures that the risk measure and ambiguity measure of the representing uncertainty measure are indeed countably additive.

Axiom 5: *$A_{n+1} \subset A_n$, $A_n \succeq B$ for all n and either $B \in \mathcal{E}$ or $A_n \backslash A_{n+1} \in \mathcal{E}$ for all n, implies $\bigcap A_n \succeq B$.*

We call a binary relation that satisfies the five axioms above a *qualitative uncertainty assessment* (QUA). Let $\succeq$ be any binary relation on Σ. A capacity π represents $\succeq$ if

$$A \succeq B \text{ if and only if } \begin{cases} \pi(A) \geq \pi(B) \\ \pi(B^c) \geq \pi(A^c) \end{cases}$$

Theorem 1: *A binary relation $\succeq$ on Σ can be represented by an uncertainty measure if and only if it is a qualitative uncertainty assessment.*

Theorem 1 formalizes our notion of separating ambiguity attitude from ambiguity perception. It establishes that every QUA can be represented by a nonatomic uncertainty assessment.

4. Preferences over Acts and Sophistication

Let $X = [w, z]$ be a nondegenerate compact interval. Let $\mathcal{F}$ be the set of all simple, Σ-measurable functions on Ω; that is,

$$\mathcal{F} = \{f : \Omega \to X \mid f^{-1}(x) \in \Sigma \text{ and } f(\Omega) \text{ finite}\}$$

For any $f \in \mathcal{F}$, $x \in X$ and $A \in \Sigma$, let fAg denote $h \in \mathcal{F}$ such that $h(s) = f(s)$ for all $s \in A$ and $h(s) = g(s)$ for all $s \notin A$. We identify constant act that always yields x with x and hence write $x \in \mathcal{F}$ and xAh.

Let $\mathcal{L}$ be the set of cumulative distributions with a finite support in X. Given a non-atomic uncertainty assessment π, and $h \in \mathcal{F}$, define $(F, G) \in \mathcal{L} \times \mathcal{L}$ such that

$$F_h(x) = \sum_{y \leq x} \pi(h^{-1}(y))$$
$$G_h(x) = 1 - \sum_{y > x} \pi(h^{-1}(y))$$

We say that f dominates g if $F_f(x) \leq F_g(x)$, $G_f(x) \leq G_g(x)$ for all x and $F_f \neq F_g$ and $G_f \neq G_g$.

A complete and transitive binary relation $\succeq^*$ on $\mathcal{F}$ is an *act preference.* We say that $\succeq^*$ is *monotone* with respect to the the non-atomic uncertainty assessment π if $f \succ^* g$ whenever f dominates g. The act preference is continuous with respect to π if $g \succeq^* f_n \succeq h$ and F_{f_n} and G_{f_n} converge in distribution to F_f and G_f imply $g \succeq f \succeq h$. An act preference is *sophisticated* if it is monotone and continuous with respect to some some nonatomic uncertainty assessment π.

Given an uncertainty measure π, let $\phi : \mathcal{F} \to \mathcal{L} \times \mathcal{L}$ be the function $\phi(f) = (F_f, G_f)$ and let $\Phi = \phi(\mathcal{F})$ be the range of ϕ. We say that $\succeq^*$ has a double-lottery representation

if there is an uncertainty measure π and a continuous increasing function $V : \Phi \to I\!R$ such that $f \succeq^* g$ if and only if $V(F_f, G_f) \geq V(F_g, G_g)$. If (π, V) is a double-lottery representation for $\succeq^*$, we call π its uncertainty measure and V its utility function. We call a preference that has a double-lottery representation a double-lottery preference.

Proposition 3: *An act preference is sophisticated if and only if it is a double-lottery preference.*

It is easy to verify that each double-lottery preference has a unique uncertainty measure and its utility is unique up to a monotone transformation. Double-lottery preferences are analogous to the general class of probabilistically sophisticated preferences in Machina and Schmeidler (1992). These are preferences consistent with some qualitative uncertainty assessment and the monotonicity and continuity requirements described above.

Example: Let $\tau : [0,1] \to [0,1]$ be convex, strictly increasing and onto and suppose the agent's utility is $V(F,G) = \int x dG^\tau$ where $G^\tau(x) = 1 - \tau(1 - G(x))$. In that case, the agent is a rank dependent expected utility maximizer (with a linear utility index) when prospects are unambiguous. For general uncertain prospects, $f \in \mathcal{F}$ the agent disregards F_f and calculates the expected value of the transformed cdf G_f. Notice that the following functional form is an equivalent representation of this agent

$$U(f) = \min_{\nu \in C} E_\mu[f]$$

where $E_\nu[f]$ is the expectation of the random variable $f \in \mathcal{F}$ under the probability measure (ν, Σ) and C is the core of the convex capacity $\kappa = \tau \circ \pi$.[4] Thus, the maxmin representation of this agent yields a set of priors that is larger than the set of priors implied by the uncertainty measure π because it captures both the uncertainty perception (κ) and the non-linearity of the preference.

Next, we provide conditions under which ambiguity is the only source of deviations from expected utility theory. In that case, the utility function is $\alpha-$maxmin expected utility with a set of priors equal to the core of the uncertainty measure κ. To formalize

[4] The probability measure ν is in the core of κ if and only if $\nu(A) \geq \kappa(A)$ for all $A \in \Sigma$.

the idea that ambiguity is the only source of deviations from expected utility theory, we must add two of Savage's axioms.

Fix an uncertainty measure π and let $(\mu, \mathcal{E})$ be its risk measure. An act preference $\succeq^*$ satisfies the *sure-thing principle* with respect to π if, for all $E \in \mathcal{E}$,

$$fEh \succeq^* gEh \text{ implies } fEh' \succeq^* gEh' \qquad (STP)$$

Thus, STP imposes the sure thing principle for $\pi-$unambiguous events only. An act preference $\succeq^*$ satisfies Savage's axiom P4 if for all x, y, x', y' such that $x > y, x' > y'$ and all $A, B \in \Sigma$,

$$xAy \succeq xBy \text{ implies } x'Ay' \succeq x'By' \qquad (P4)$$

A function $V : \mathcal{L} \times \mathcal{L} \to I\!R$ is a *Double Expected Utility* (DEU) function if there exist continuous utility indices $u_1 : X \to I\!R, u_2 : X \to I\!R$ such that u_1 and u_2 are non-decreasing and $u_1 + u_2$ is strictly increasing and

$$V(F, G) = \int_X u_1 dF + \int_X u_2 dG \qquad (DEU)$$

We say that $\succeq^*$ has a DEU preference if it can be represented by a DEU. In the special case where u_1 and u_2 are affine transformations of one another we refer to the utility as a symmetric double expected utility. A function $V : \mathcal{L} \times \mathcal{L} \to I\!R$ is a *Symmetric Double Expected Utility* (SDEU) function if there exists continuous, strictly increasing utility index $u : X \to I\!R, \alpha \in (0, 1)$ such that

$$V(F, G) = (1 - \alpha) \int_X u dF + \alpha \int_X v dG \qquad (SDEU)$$

We say that $\succeq^*$ is a SDEU preference if it can be represented by a SDEU.

Proposition 4: *(i) An act preference is sophisticated and satisfies STP if and only if it is a DEU preference; (ii) a DEU preference satisfies P4 if and only if it is an SDEU preference.*

An SDEU preference with parameters u, α, π is an $\alpha-$maxim expected utility. Let C be the core of the capacity π. It is easy to show that the $\alpha-$maxmin representation of this preference is

$$U(f) = \alpha \min_{\nu \in C} E_\nu[f \circ u] + (1-\alpha) \max E_\nu[f \circ u]$$

where C is the core of the capacity π. In this case, the set of priors used in the representation coincides with the set of priors that characterize π, the uncertainty measure.

5. Appendix A: Preliminaries

For any σ-algebra $\mathcal{E}$ and $A \in \Sigma$, let $[A]_{\mathcal{E}} = \{E \in \mathcal{E} \,|\, E \subset \mathcal{E}\}$. When the choice of $\mathcal{E}$ is clear, we suppress it and write $[A]$. For any σ-continuous probability $(\mu, \mathcal{E})$, let

$$\mu_*(A) = \sup_{E \in [A]} \mu(E)$$

We call μ_* the inner probability of μ. Also, let $\mathcal{E}(A) = \{E \in \mathcal{E} \,|\, \mu(E) = \mu_*(A)\}$. Lemma A1, below, ensures that $\mathcal{E}(A)$ is nonempty for all A and hence we can replace the sup with max in the definition of μ_*.

For any $n \geq 1$, let $N = \{1, \ldots, n\}$ and let $\mathcal{N}$ denote the set of all nonempty subsets of N. Then, let $(\mu, \mathcal{E})$ be any σ-continuous probability measure and $\mathcal{A} = \{A_i \,|\, i \in N\}$ be any partition of Ω. The partition $\{E_\kappa \,|\, \emptyset \neq \kappa \subset N\}$ is a μ-split of $\mathcal{A}$ if $E_\kappa \in \mathcal{E}$ for all κ and

$$\begin{gathered} \bigcup_{\kappa \in \mathcal{K}} E_\kappa \subset \bigcup_{i \in K} A_i \\ \mu\left(\bigcup_{\kappa \in \mathcal{K}} E_\kappa\right) = \mu_*\left(\bigcup_{i \in K} A_i\right) \end{gathered} \tag{1}$$

for all $\emptyset \neq K \subset N$ and $\mathcal{K} = \{\kappa \in \mathcal{N} \,|\, \kappa \subset K\}$.

If $A_1 = A$ and $A_2 = A^c$, we say that and E_1, E_2, E_{12} is a μ-split of (A, A^c).

Lemma A1: *Let $(\mu, \mathcal{E})$ be a σ-continuous probability. Then, every partition $\mathcal{A}$ has a μ-split. Moreover, if $\{E_\kappa \,|\, \emptyset \neq \kappa \subset N\}$ is a μ-split of $\mathcal{A}$, then the partition $\{F_\kappa \,|\, \emptyset \neq \kappa \subset N\}$ is also a μ-split of $\mathcal{A}$ if and only if $\mu(E_\kappa \backslash F_\kappa) = 0$ for all κ.*

Proof: Suppose $(\mu, \mathcal{E})$, $N = \{1, \ldots, n\}$ and $\mathcal{A} = \{A_i \,|\, i \in N\}$ satisfy the hypotheses of the Lemma. The partition $\{E_\kappa \,|\, \emptyset \neq \kappa \subset N\}$ is a μ-split of $\mathcal{A}$ if $E_\kappa \in \mathcal{E}$.

First, we will show that for A_i, $\mathcal{E}(A_i) \neq \emptyset$. Let $E^i \in [A]$ be a sequence such that $\lim \mu(E^i) = \mu_*(A)$. By definition such a sequence exists. Then, since μ is a probability measure, $E = \bigcup E^i$ is the desired set.

Let $E^\emptyset = \emptyset$ and choose $E^\kappa \in \mathcal{C}\left(\bigcup_{i\in\kappa} A_i\right)$ for all $\kappa \in$ and let

$$E_\kappa = E^\kappa \backslash \bigcup_{\kappa \neq \hat{\kappa} \subset \kappa} E^{\hat{\kappa}}$$

for all $\kappa \in \mathcal{N}$. It is straightforward but somewhat tedious to verify that $\{E_\kappa \,|\, \kappa \in \mathcal{N}\}$ is the desired partition.

Now, if $\{E_\kappa\} \subset \mathcal{E}$ and $\{F_\kappa\} \subset \mathcal{E}$ are two partitions and $\mu(E_\kappa \backslash F_\kappa) > 0$, then $\mu_*(A_\kappa) \geq \mu_(E_\kappa \cup F_\kappa) > \mu(E_\kappa) = \mu_*(A_\kappa)$, a contradiction. $\square$

If (E_1, E_2, E_{12}) is a μ-split of (A, A^c), then by Lemma A1, E_1, E_{12} is unique up to a set of measure 0. Therefore, we call $E(A) = E_1$ *the core* of A and $F(A)$ *the boundary* of A. Note that $F(A) = F(A^c)$ and $(E(A), E(A^c), F(A))$ is a μ-split of (A, A^c).

Lemma A2: *If $(\mu, \mathcal{E})$ is a probability measure, then μ_* is a belief function.*

Proof: Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be any partition of Ω. Assume without loss of generality that $\emptyset \notin \mathcal{A}$. Let $\hat{\mathcal{A}}$ be the finite collection of sets that can be expressed as the unions of elements in $\mathcal{A}$. Let

$$\gamma(A) = \mu(E_\kappa)$$

for all $A = \bigcup_{i\in\kappa\in\mathcal{N}} A_i$. Clearly, $\gamma(A) \geq 0$ for all $A \in \hat{\mathcal{A}}$. Lemma A1 ensures that

$$\mu_*(A) = \sum_{\substack{B\in\hat{\mathcal{A}} \\ B\subset A}} \gamma(A)$$

for all $A \in \mathcal{A}$. Dempster (1967) shows that a capacity on the algebra of sets generated by some partition $\mathcal{A}$ is a belief function if and only if there exists a $\gamma \geq 0$ satisfying the above display equation. Hence, the restriction of μ_* to $\hat{\mathcal{A}} \cup \{\emptyset\}$ is a belief function. Since the partition $\mathcal{A}$ was arbitrary, this proves that μ_* is a belief function. $\square$.

Proof of Proposition 1

Suppose, π is an uncertainty measure let (μ, η) and (η, Σ) satisfy the appropriate properties. In particular,

$$\pi(A) = \max_{E \in [A]} \mu(E) + \min_{E \in [A]} \eta(A \backslash E)$$

for all A.

Since μ is complete, every μ-null set is μ-whole. Hence, $\eta(E) = 0$ whenever $\mu(E) = 0$ and therefore $\mu(E) \geq \eta(E)$ for all $E \in \mathcal{E}$. Next, we claim that for all $E \in [A]$, $\mu(E) = \max_{F \in [A]} \mu(F)$ implies $\pi(A) = \mu(E) + \eta(A \backslash E)$.

To see this, choose any $E \in [A]$ such that $\mu(E) = \max_{F \in [A]} \mu(F)$. Suppose there is $\hat{E} \in [A]$ such that $\eta(A \backslash \hat{E}) < \eta(A \backslash E)$. Hence, $\eta(A \backslash (\hat{E} \cup E)) < \eta(A \backslash E)$ and therefore $\eta(\hat{E} \cup E) > \eta(E)$ which means $\eta(\hat{E} \backslash E) > 0$ and hence $\mu(\hat{E} \backslash E) > 0$, contradicting the fact that $\mu(E) = \max_{F \in [A]} \mu(F)$.

Next, we will show that π is superadditive; that is, $\pi(A \cup B) \geq \pi(A) + \pi(B)$ whenever $A \cap B = \emptyset$. To see this, suppose $\pi(A) = \mu(E) + \eta(A \backslash E)$ and $\pi(B) = \mu(E) + \eta(A \backslash F)$ for some $E \in [A]$ and $F \in [B]$. Then, since $E \cup F \in [A \cup B]$, we have

$$\pi(A \cup B) \geq \mu(E \cup F) + \eta((A \cup B) \backslash (E \cup F)) = \mu(E) + \mu(F) + \eta(A \backslash F) + \eta(B \backslash F) = \pi(A) + \pi(B)$$

as desired.

Next, we claim that $\mathcal{E} = \mathcal{E}_\pi$. To see this, suppose $A \in \mathcal{E}_\pi$, then

$$1 = \pi(A) + \pi(A^c) = \mu(E) + \eta(A \backslash E) + \mu(F) + \eta(A^c \backslash F)$$

for some $E \in [A]$ and $F \in [A^c]$. Hence, $1 = \mu(\hat{E}) + \eta(\hat{E}^c)$ where $\hat{E} = (E \cup F)$. If $\eta(\hat{E}^c) > 0$, we have $\mu(\hat{E}^c) > \eta(\hat{E}^c)$ and hence $\mu(\hat{E}) + \mu(\hat{E}^c) > 1$ contradicting the fact that μ is a probability. So, we must have $\mu(\hat{E}^c) = 0$. Then, the completeness of μ ensures that $A \backslash E \in \mathcal{E}$ and hence $A \in \mathcal{E}$.

Suppose $E \in \mathcal{E}$. Then, $\pi(E) \geq \mu(E)$ and $\pi(E^c) \geq \mu(E^c)$. Since π is superadditive, $1 = \pi(E \cup E^c) \geq \pi(E) + \pi(E^c) \geq \mu(E) + \mu(E^c) = 1$. Hence, $E \in \mathcal{E}_\pi$ proving that $\mathcal{E} = \mathcal{E}_\pi$.

It follows that $\mu(E) = \pi(E)$ for all $E \in \mathcal{E}_\pi$ proving that the probability measure of any uncertainty measure is unique. Let (F, F^c) be any essential partition and choose

$B \subset F^c$ such that $\mu_*(B) = \mu_*(F^c \backslash B) = 0$. Hence, for any $E \subset F^c$, $\pi(E \cap B) = \eta(E \cap B)$ and $\pi(E \backslash B) = \eta(E \cap B)$. Hence, $\eta(E) = \pi(E \cap B) + \pi(E \backslash B)$. Then, for any $E \in \mathcal{E}$, $\eta(E) = \eta(E \cap F) + \eta(E \cap F^c) = \eta(E \cap F^c) = \pi(E \cap B) + \pi(E \backslash B)$. Hence, any two ambiguity measures for π must agree on $\mathcal{E}$.

Take any A and $E \in [E]$ such that $\mu(E) = \mu_*(E)$ and note that $\pi(A) = \mu(E) + \eta(A \backslash E) = \mu(E) + \eta(A) - \eta(E)$. Hence, $\eta(A) = \pi(A) - \eta(E)$. Since any two ambiguity measures for π must agree on $\mathcal{E}$, it follows that they must agree everywhere. This proves Proposition 1. □

5.1 Proof of Proposition 2

Let π be an uncertainty measure with probability measure μ and ambiguity measure η. If $\eta(\Omega) = 0$, then $\pi = \mu_*$ and hence, Lemma A2 ensures that π is a belief function. If $\eta(\Omega) > 0$, then let $a = 1 - \eta(\Omega)$. Since, μ dominates η, $\eta(\Omega) < \mu(\Omega)$ and hence $a \in (0, 1)$. Also, $\mu(E) = 0$ implies $\eta(E) = 0$. To see this, note that if $\mu(E) = 0$ and $\eta(E) > 0$ for some E, then $\pi(\Omega) > \mu(E_c) + \eta(E) > 1$, contradicting the fact that π is a capacity.

Let $\hat{\eta}(A) = \eta(A)/(1 - a)$ for all $E \in \mathcal{E}$. Clearly, Since μ dominate η, the σ-continuity of μ ensures the σ-continuity of the restriction of η to $\mathcal{E}$ and hence of the σ-continuity of the restriction of $\hat{\eta}$ to $\mathcal{E}$. Hence, $\hat{\mu} = (\mu - \hat{\eta})/a$ is a probability measure on $\mathcal{E}$ and therefore, by Lemma A2, its inner probability, $\hat{\mu}_*$, is a belief function.

Clearly $\mu(E \backslash F) = 0$ for all $E, F \in \mathcal{E}(A)$ and hence $\eta(E \backslash F) = 0$ for all $E, F \in \mathcal{E}(A)$ and therefore,

$$\pi(A) = \mu(E) + \eta(A \backslash E) = \mu_*(A) + \eta(A) - \eta(E)$$

for all $E \in \mathcal{E}(A)$. Then,

$$\pi = a\hat{\mu}_* + (1 - a)\eta$$

Clearly a convex combination of a belief function and a probability is a belief function. This completes the proof of Proposition 2. □

6. Appendix B: Proof of Theorem 1

Calibration implies *transitivity* ($A \succeq B$ and $B \succeq C$ implies $A \succeq C$) and *consistency* ($A \succeq B$ implies $B^c \succeq A^c$). For any two sets X, Y, let $X \Delta Y := (X \cup Y) \backslash (X \cap Y)$. For any two collection of events $\mathcal{A}$, $\mathcal{B}$, let $\mathcal{A} \sqcup \mathcal{B} = \{A \cup B \,|\, A \in \mathcal{A}, B \in \mathcal{B}\}$. Finally, let $[A] = \{E \subset A\}$.

Lemma B1:

(i) $\langle A \rangle = [A] \sqcup [A^c]$.

(ii) $\mathcal{E}$ *is an algebra.*

(iii) $A \in \mathcal{E}$ *if and only if* $A \cong \Omega$.

(iv) $A \cap E = \emptyset$ *implies* $A \cup E \cong A$.

(v) $E, F \in \mathcal{E}$ *and* $E \not\succeq F$ *implies* $F \succ E$.

(vi) If $(A \cup B) \cap C = \emptyset$, $A \cong B$ *and* $A \cup C \cong B \cup C$, *then* $A \succeq B$ *if and only if* $A \cup C \succeq B \cup C$.

(vii) If $B \succeq A_n$, $A_n \subset A_{n+1}$, $A_{n+1} \backslash A_n \in \mathcal{E}$ *for all* n, *then* $B \succeq \bigcup A_n$.

Proof: (i) Suppose $E \in \langle A \rangle$. Then, $E \cap A \in [A]$ and $E \cap A^c \in [A^c]$ and hence $E = (E \cap A) \cup (E \cap A^c) \in [A] \sqcup [A^c]$. Next, assume that $F_1 \in [A]$ and $F_2 \in [A^c]$ and let $B = F_1 \cup F_2$. Then, B is the union of two disjoint elements of $\mathcal{E}$. Therefore, it is enough to show that if $E \cap F = \emptyset$, then $E \cup F \in \mathcal{E}$. Suppose $A \succeq B$ and $(A \cup B) \cap (E \cup F) = E \cap F = \emptyset$. Then, $A \cup E \succeq B \cup E$ and hence $(A \cup E) \cup F \succeq (B \cup E) \cup F$ as desired.

(ii) Clearly, $\Omega \in \mathcal{E}$ and $\emptyset \in \mathcal{E}$. Then, Axiom 2(ii) and part (i) imply $[E] \sqcup [E^c] = [\Omega] \sqcup [\emptyset]$ and hence $\Omega \in [E] \sqcup [E^c]$ which yields $E^c \in \mathcal{E}$. To complete the proof that $\mathcal{E}$ is an algebra, we need to show that $E \cap F \in \mathcal{E}$. By Axiom 2(ii) and part (i), $[E] \sqcup [E^c] = [F] \sqcup [F^c]$. Clearly, $E \in [E] \cup [E^c]$ and therefore $E \in [F] \cap [F^c]$ which implies $E \cap F \in [F]$ and hence $E \cap F \in \mathcal{E}$.

(iii) Since Ω is unambiguous, one direction follows from Axiom 2(ii) and part (i). For the other direction, assume $A \cong \Omega$ and hence $\Omega \in [A] \sqcup [A^c]$ by part (i); that is, $A \in [A]$ and therefore $A \in \mathcal{E}$.

(iv) We will show that when $A \cap E = \emptyset$, $[A] \sqcup [A^c] = [A \cup E] \sqcup [A^c \cap E^c]$ and appeal to part (i). Suppose $F_1 \in [A]$ and $F_2 \in [A^c]$. Then, $F_1 \subset A \cup E$, $F_2 = F_3 \cup F_4$ where $F_3 := F_2 \cap E$ and $F_4 = F_2 \cap E^c$. Since $\mathcal{E}$ is an algebra (by part (ii)), $F_1 \cup F_3 \in [A \cup E]$ and $F_4 \in [A^c \cap E^c]$ and hence $F_1 \cup F_2 = F_1 \cup F_3 \cup F_4 \in [A \cup E] \sqcup [A^c \cap E^c]$.

For the converse, take $F_1 \in [A \cup E]$ and $F_2 \in [A^c \cap E^c]$. Then, let $F_3 = F_1 \cap A$ and $F_4 = F_1 \cap E$. Since $\mathcal{E}$ is an algebra, $F_4 \in \mathcal{E}$ and therefore, $F_3 = F_1 \backslash F_4 \in \mathcal{E}$. But $F_1 \cap A = F_3$. Hence, $F_3 \in [A]$ and $F_2, F_4 \in [A^c]$. Hence, by part (ii), $F_2 \cup F_4 \in [A^c]$ and therefore $F_1 \cup F_2 = F_3 \cup F_4 \cup F_2 \in [A] \sqcup [A^c]$.

(v) By Axiom 2, we have $F \succeq E$ and, by calibration, we have $W(E) \subset W(F)$ and $W(F^c) \subset W(E^c)$. Moreover, since $E \not\succeq F \in W(F)$, the first inclusion is strict. It remains to show that the second inclusion is strict as well. Since, by consistency, $E^c \succeq F^c$ implies $F \succeq E$ it follows that $E^c \notin W(F^c)$, as desired.

(vi) If $A \cong B$, $A \cup C \cong B \cup C$ and $(A \cup B) \cap C = \emptyset$, then A, C and B, C conform and, therefore, the desired result follows from Axiom 3.

(vii) Suppose $B \succeq A_n$, $A_n \subset A_{n+1}$ and $A_{n+1} \backslash A_n \in \mathcal{E}$ for all n. Then, by consistency, $A_n^c \succeq B^c$ and also $A_{n+1}^c \subset A_n^c$, $A_n \backslash A_{n+1}^c \in \mathcal{E}$ for all n. Hence, A_n^c converges to $\bigcap A_n^c$ and by Axiom 5 $\bigcap A_n^c \succeq B^c$. Hence, again by consistency, $B \succeq \bigcup A_n$. □

Lemma B2: *$\mathcal{E}$ is a σ-algebra and there exists a nonatomic probability measure μ on $\mathcal{E}$ such that $E \succeq F$ if and only if $\mu(E) \geq \mu(F)$.*

Proof: First, we show that $\mathcal{E}$ is a σ-algebra. Let $E = \bigcup F_i$. Define $E_n = \bigcup_{i=1}^n F_i$ and note that $E = \bigcup E_i$. By Lemma B1(ii), $E_n \in \mathcal{E}$ for all n. Suppose $A \succeq B$ and $A \cap E = B \cap E = \emptyset$. Then, $A \cup E_n \succeq B \cup E_n$ for all n and, by monotonicity, $A \cup E \succeq A \cup E_n \succeq B \cup E_n$ for all n. Hence, transitivity implies $A \cup E \succeq B \cup E_n$ for all n. Then, Lemma B1 (vii) yields $A \cup E \succeq B \cup E$.

For the converse, assume $A \cup E \succeq B \cup E$ and $(A \cup B) \cap E = \emptyset$. By consistency, $B^c \cap E^c \succeq A^c \cap E^c$. Hence, $(B^c \cap E^c) \cup E_n \succeq (A^c \cap E^c) \cup E_n$. Monotonicity and transitivity yield $(B^c \cap E^c) \cup E \succeq (A^c \cap E^c) \cup E_n$. Then Lemma B1 (vii) implies $(B^c \cap E^c) \cup E \succeq (A^c \cap E^c) \cup E$; that is, $B^c \succeq A^c$ and hence by consistency $A \succeq B$ as desired. This proves that $\mathcal{E}$ is a σ-algebra.

Let $\succeq_{\mathcal{E}}$ be the restriction of $\succeq$ to $\mathcal{E}$. Lemma B1(v) ensures that $\succeq_{\mathcal{E}}$ is complete and transitive and also that $\succ$ has the usual interpretation. By definition, $E \succeq F$ if and only if $E \cup F' \succeq E' \cup F'$ whenever $(E \cup F) \cap F' = \emptyset$. Hence, $\succeq$ restricted to $\succeq_{\mathcal{E}}$ is a qualitative probability. By Lemma B1(iii), Axiom 4 restricted to $\succeq_{\mathcal{E}}$ yields Savage's small event continuity axiom. Repeating Savage's proof for the σ-algebra $\mathcal{E}$ instead of the σ-algebra of

all subsets of Ω yields a finitely additive, convex valued probability μ that represents $\succeq_{\mathcal{E}}$. Then, the restriction of Axiom 5 to sets in $\mathcal{E}$ yields establishes that μ is in fact countably additive.

Next, we will show that μ is complete; that is, $C \subset E$ and $\mu(E) = 0$ implies $C \in \mathcal{E}$. Assume $C \subset E$ and $\mu(E) = 0$. First, we will show that E is null. Then, since μ represents $\succeq_{\mathcal{E}}$, $E \sim \emptyset$. By Lemma B1(iv), $A \backslash E \cong A \cup E$ and therefore, by Lemma B1(vi), $A \cup E \sim A \backslash E$ and hence by monotonicity, $A \succeq A \cup E$, proving that E is null. Monotonicity ensures that any subset of a null set is also null and hence C is null. Then, for any A, B such that $A \succeq B$, we have $A \cup C \sim A \succeq B \sim B \cup C$ and hence $C \in \mathcal{E}$.

Finally, we will show that μ is nonatomic. Suppose $\mu(E) > 0$. Then, since μ represents $\succeq_{\mathcal{E}}$, $E \succ \emptyset$. Then, by Axiom 4, there exists a partition $E_1, \ldots, E_n$ of Ω such that $E_i \cong E$ for all i and $E \succ E_i$. By Lemma B1(iii), $E_i \in \mathcal{E}$ and hence by Lemma B1(ii), $E_i \cap E \in \mathcal{E}$ for all i. If $E_i \cap E$ were null for all i, then we would get $\emptyset \succeq E$, contradicting the fact that $\mu(E) > 0$ and μ represents $\succeq_{\mathcal{E}}$. Hence, $E \cap E_i \in \mathcal{E}$ is nonnull for some i, say $i = 1$ $\mu(E \cup E_1) > 0$ and therefore, $\mu(E) > \mu(E \cap E_1) > 0$ as desired. $\square$

For the remainder of this proof we fix the non-atomic probability measure $(\mu, \mathcal{E})$. We let $\mathcal{N}(A)$ denote the set of all null subsets of A, $\mathcal{E}_o(A) = \{E \subset A \,|\, \mu(E) = 0\}$. Let $E(A)$ be the core of A, $F(A)$ the boundary of A and let $(E(A), F(A), E(A^c))$ be a $\mu-$split of A.

Definition: *(i) E is blank if there exists $A \subset E$ such that $\mu_*(A) = \mu_*(E \backslash A) = 0$; (ii) (E, E^c) is an essential partition for μ if $E \in \mathcal{E}$ and is whole and E^c is blank.*

Lemma B3:

(i) $\mathcal{N}(A) = \mathcal{E}_o(A)$

(ii) $\langle A \rangle = \{E \,|\, \mu(E \cap F(A)) = 0\}$.

(iii) If $A \subset E$ and $B \subset E^c$, then $F(A \cup B) = F(A) \cup F(B)$; $\langle A \cup B \rangle = \langle A \rangle \sqcup \langle B \rangle$; $\langle A \cup B \rangle - \langle A \rangle = \{\emptyset\}$ and $\langle A \rangle - \langle A \cup B \rangle = [F(B)] \cup \mathcal{N}(\Omega)$.

(iv) Every probability measure has an essential partition.

(v) If (E, E^c) and (F, F^c) are two essential partitions for μ, then $\mu(E \Delta F) = 0$.

Proof: (i) While proving that μ is complete (Lemma B2), we have shown that $\mu(E) = 0$ implies E is null. It follows that $\mathcal{E}_o(A) \subset \mathcal{N}(A)$. For the converse, assume C is null.

Then, $A' \cup C \sim A'$ for all A' and hence $A' \succeq B'$ implies $A' \cup C \sim A' \succeq B' \sim B' \cup C$ and by transitivity $A' \cup C \succeq B' \cup C$ proving that $C \in \mathcal{E}$. But if $C \in \mathcal{E}$ is null then clearly, $\mu(C) = 0$. It follows that $\mathcal{N}(A) \subset \mathcal{E}_o(A)$.

(ii) If $E = E_1 \cup E_2$ and $E_1 \in [A]$ and $E_2 \in [A^c]$, then clearly $\mu(E \cap F(A)) = 0$. Conversely, if $\mu(E \cap F(A)) = 0$, then $E = E_3 \cup E_4$ for some $E_3 \subset [A] \cup [A^c]$ and E_4 such that $\mu(E_4) = 0$. Then, since μ is complete, $E_4 \cap A \subset [A]$, $E_4 \cap A^c \in [A^c]$ and hence $E \in [A] \sqcup [A^c] = \langle A \rangle$.

(iii) For the first assertion, first note that $[A \cup B] = [A] \cup [B]$. This follows since $E' \in [A \cup B]$ implies $E \cap E' \in [A]$, $E^c \cap E \in [B]$ and, therefore, $E' \in [A] \cup [B]$. The converse is obvious. Similarly, since $A^c \cap B^c = ((E \backslash A) \cup E^c) \cap ((E^c \backslash B) \cup E) = (E \backslash A) \cup (E^c \backslash B)$ it follows that $[A^c \cap B^c] = [E \backslash A] \cup [E^c \backslash B]$ by the preceding argument. The last two observations imply

$$E(A \cup B) = E(A) \cup E(B)$$

$$E(A^c \cap B^c) = E(A^c) \cap E(B^c)$$

It follows that $F(A \cup B) = (E(A \cup B) \cup E(A^c \cap B^c))^c = (E(A) \cup E(A^c))^c \cup (E(B) \cup E(B^c))^c = F(A) \cup F(B)$. Part (ii) and the first assertion yield the second assertion from which $\langle A \cup B \rangle - \langle A \rangle = \{\emptyset\}$ follows as does the fact that $\langle A \rangle - \langle A \cup B \rangle = \langle A \rangle - (\langle A \cap \langle B \langle)$. Then, by the part (ii), $\langle A \rangle - \langle A \cup B \rangle = \{E \,|\, \mu(E \cap F(A)) = 0 \text{ and } \mu(E \cap F(B)^c) = 0\}$. By the first assertion, $F(A) \subset F(B)^c$ and therefore $\langle A \rangle - \langle A \cup B \rangle = \{E \,|\, \mu(E \cap F(B)^c) = 0\} = [F(B)] \cup \mathcal{N}(\Omega)$ as desired.

(iv) Let $\mathcal{B} = \{E \in \mathcal{E} \,|\, E \text{ is blank}\}$ and $b = \sup_{E \in \mathcal{B}} \mu(E)$. First, we will show that a countable union of blank sets is a blank set. Suppose $E = \bigcup E_n$ and define $F_1 = E_1$, $F_n = E_n \backslash \bigcup_{i<n} E_n$. Hence, the sets F_n are pairwise disjoint and $\bigcup F_n = F$. Choose $A_n \subset F_n$ for all n so that $\mu_*(A_n) = \mu_*(F_n \backslash A_n) = 0$. Suppose there exists $E' \subset F$ such that $\mu(E') > 0$ and (i) $E' \cap (\bigcup A_n) = \emptyset$) or (ii) $E' \cap (E \backslash (\bigcup A_n)) = \emptyset$. Then, the countable additivity (i.e., σ-continuity) of μ ensures that we have $\mu(E' \cap F_n) > 0$ for some n. Then, (i) yields $\mu_*(F_n \backslash A_n) > 0$ while (ii) yields $\mu_*(A_n) > 0$ both are contradictions.

Let $E_n \in \mathcal{B}$ be a sequence such that $\lim \mu(E_n) = b$. Then, $\mu(\bigcup E_n) = b$ and by the argument above, $\bigcup E_n \in \mathcal{B}$. Set $G = (\bigcup E_n)^c$. Assume there exists $A \subset G$ such that $A \notin \mathcal{E}$. Since μ is complete, part (i) implies that $\mu(F(A)) > 0$. Thus, $F(A)$ is a blank set

and, by the above argument, so is $F(A) \cup G^c$. But, $\mu(F(A)) \cup G^c) > b$, which contradicts the definition of b and proves that F is whole. So, (G, G^c) is the desired partition.

(v) The proof is straightforward and omitted. $\square$

If $\mathcal{E} = \Sigma$. Then, set $\pi = \mu$ and $\eta(A) = 0$ for all $A \in \mathcal{E}$ and note that π is the desired uncertainty measure. So, from now on, we will assume $\mathcal{E} \neq \Sigma$. Let (G, G^c) be an essential partition for μ. Since μ is complete and $\mathcal{E} \neq \Sigma$, $\mu(G^c) > 0$. For any subset E of G^c such that $\mu(E) > 0$ and let

$$\mathcal{C}_E = \{A \subset E \,|\, \mu_*(A) = \mu_*(E \backslash A) = 0\}$$

Since (G, G^c) is an essential partition, there is $B \subset G^c$ such that $\mu_*(B) = \mu_*(G^c \backslash B) = 0$. Setting $A_1 = B \cap F$ and verifying that $A_1 \in \mathcal{C}_E$ establishes that $\mathcal{C}_E \neq \emptyset$. Define,

$$\eta_E(A) = \sup\{\mu(E') \,|\, A \succeq E'\}$$

for all $A \in \mathcal{C}_E$. Call $(\mathcal{C}_E, \eta_E)$ a *local ambiguity measurement* (LAM). If $\mu(E) \leq 1/3$ we call $(\mathcal{C}_E, \eta_E)$ a *small LAM* (SLAM).

Lemma B4: *Let $A, B \in \mathcal{C}_E$ and $F, F' \subset E^c$. Then,*

(i) A is non-null.

(ii) $A \cup F \cong C$ if and only if $C = B \cup E'$ for some $B \in \mathcal{C}_E$ and some $E' \subset E^c$;

(iii) $A \subset C \subset E$, $B \cap C = \emptyset$ implies $B \in \mathcal{C}_E$;

(iv) $A \cup F \succ B \cup F'$ implies there is $C \in \mathcal{C}_E$ such that $B \cap C = \emptyset$, $A \cup F \succ B \cup F' \cup C$ and $B \cup C \in \mathcal{C}_E$.

(v) $A \cup F \succ B \cup F'$, $\mu(E) \leq 1/3$, and either $F = \emptyset$ or $F' = \emptyset$ implies there is $E' \subset E^c$ such that $E' \cap (F \cup F') = \emptyset$, $\mu(E') > 0$ and $A \cup F \succ B \cup F' \cup E'$.

(v) If $E_1 \ldots, E_n$ are pairwise disjoint and $E = \bigcup_{i=1}^n E_i$, then, $\mathcal{C}_E = \mathcal{C}_{E_1} \sqcup \ldots \sqcup \mathcal{C}_{E_n}$.

Proof: (i) If A is null then $A \in \mathcal{E}$ by Lemma B3(i). Then, $A^c \cap E_1 \in \mathcal{E}$ and hence $\mu(A^c \cap E_1) = \mu(E_1) > 0$ contradicting the fact that $A \in \mathcal{C}_E$.

(ii) First we show that $A \cong B$ if $B \in \mathcal{C}_E$. To so so, we will show that $[A] \sqcup [A^c] = [B] \sqcup [B^c]$ and appeal to Lemma B1(i). Recall that $(E(A), E(A^c), F(A))$ is a μ-split of

(A, A^c). We begin by showing that $\mu(E^c \Delta F(A^c)) = 0$. Let $F^* = E^c \backslash E(A^c)$. If $\mu(F^*) > 0$ then, since $A \subset E$, $E^c \subset A^c$, we have $\mu(F^* \cup E(A^c)) > \mu(E(A^c))$ and $F^* \cup E(A^c) \subset [A^c]$, contradicting the fact that $E(A), E(A^c), F(A)$ is a μ-split for (A, A^c). If $\mu(E(A^c) \backslash E^c) > 0$, then we have $\mu(E(A^c) \cap E) > 0$. Since $E(A^c) \cap E \subset E \backslash A$, we have $\mu_*(E \backslash A) > 0$, contradicting the fact that $A \in \mathcal{C}_E$ and proving the assertion.

Then, by symmetry, $\mu(E^c \Delta E(B^c)) = 0$. Now, suppose $F_1 \in [A]$ and $F_2 \in [A^c]$. Let $F_3 = F_1 \cap B$, $F_4 = F_1 \cap B^c$, $F_5 = F_2 \cap E \cap B$, $F_6 = F_2 \cap E \cap B^c$, $F_7 = F_2 \cap E^c \cap B$ and $F_8 = F_2 \cap E^c \cap B^c$. Clearly,

$$F_1 \cup F_2 = F_3 \cup F_4 \cup F_5 \cup F_6 \cup F_7 \cup F_8$$

Since μ is complete and $\mu_*(A) = \mu_*(B) = 0$, we have $F_3, F_4, F_5, F_7 \in \mathcal{E}$. Since $F_2 \in [A^c]$, $\mu(F_2 \backslash E(A^c)) = 0$. Then, since $\mu(E^c \Delta E(A^c)) = 0$, we have $\mu(F_2 \backslash E^c) = 0$ and therefore $F_6 \in \mathcal{E}$. Also, since $\mu(E^c \Delta E(B^c)) = 0$, we can write $F = (E(B^c) \backslash F_9) \cup F_{10}$ for $F_9, F_{10} \in \mathcal{E}$ such that $\mu(F_9 \cup F_{10}) = 0$. Then, $F \cap B^c = (E(B^c) \cap F_9^c \cap B^c) \cup (F_{10} \cap B^c)$. Since $E(B^c) \subset B^c$, we have $E^c \cap B^c = (E(B^c) \cap F_9^c) \cup (F_{10} \cap B^c)$. Clearly, $E(B^c) \cap F_9^c \in \mathcal{E}$. Since μ is complete, we have $(F_{10} \cap B^c) \in \mathcal{E}$ and therefore, $E^c \cap B^c \in \mathcal{E}$ and hence $F_8 \in \mathcal{E}$. It follows that $F_1 \cup F_3 \cup F_5 \cup F_7 \in [B]$ and $F_2 \cup F_4 \cup F_6 \cup F_8 \in [B^c]$ and therefore $F_1 \cup F_2 \in [B] \sqcup [B^c]$ proving that $A \cong B$.

Next, we show that $C \cong A$ if and only if $C = B \cup F$ for some $B \in \mathcal{C}_E$. First, assume $C = B \cup F$ for some $F \subset E^c$. Then, by Lemma B1(iv), $B \cong B \cup E$ and, since $A \cong B$, it follows that $A \cong B \cup F = C$. For the converse, note first that $[A] \sqcup [A^c] = \mathcal{N}(\Omega) \sqcup [E^c]$. To see, this if $F_1 \in [A]$ and $F_2 \in [A^c]$, then $\mu(F_1) = 0$ and $\mu(F_2 \cap E) = 0$. Hence, F_1 and $F_2 \cap E$ are in $\mathcal{N}(\Omega)$ by Lemma B3(i). Since the union of null sets is null, $F_1 \cup (F_2 \cap E) \in \mathcal{N}(\Omega)$. Let $F_3 = F_2 \backslash E$ and note that $F_2 \in [E^c]$ and therefore, $F_1 \cup F_2 \in \mathcal{N}(\Omega) \sqcup [E^c]$. Similarly, if $F_1 \in [E^c]$ and $F_2 \in \mathcal{N}(\Omega)$, then, since μ is complete, $F_3 = F_2 \cap A \in [A]$ and $F_4 = F_2 \cap A^c \in [A^c]$ and also $F_1 \in [A^c]$. Hence, $F_1 \cup F_2 = F_3 \cup (F_1 \cup F_4) \in [A] \sqcup [A^c]$ as desired.

Now, suppose $C \cap E \notin \mathcal{C}_E$. Then, either there is $F \in C \cap E$ such that $\mu(F) > 0$ or $F \in [C^c \cap E]$ such that $\mu(F) > 0$. Since, $[A] \sqcup [A^c] = \mathcal{N}(\Omega) \sqcup [E^c]$, either of these establishes that $[A] \sqcup [A^c] \neq [C] \sqcup [C^c]$. If $C \cap E^c \notin \mathcal{E}$, then, let E_1, E_2, E_{12} be a μ-split

of the partition A_1, A_2 where $A_1 = C \cap E^c$ and $A_2 = A_1^c$. Then, $\mu(E_{12} \backslash E) = \mu(E_{12}) > 0$. Hence, $E'_{12} \backslash E \in [A] \sqcup [A^c]$ and $E'_{12} \backslash E \notin [C] \cup [C^c]$.

It remains to show that $A \cup F \cong B \cup F'$ whenever $F, F' \subset E^c$. By Lemma B1(iv), $A \cong A \cup E$ and $B \cong B \cup F$. The result now follows from $A \cong B$.

(iii) Note that if $C \in \mathcal{C}_E$, $B \subset E$ and $B \cap C = \emptyset$, then $B \subset E \backslash C$ and hence $\mu_*(B) \leq \mu_*(E \backslash C) = 0$. Similarly, $E \backslash B \subset E \backslash A$ and therefore, $\mu_*(E \backslash B) \leq \mu_*(E \backslash A) = 0$. Hence, $B \in \mathcal{C}_E$.

(iv) Assume $A \cup F \succ B \cup F'$. By Axiom 4 there exists a partition $C_1, \ldots, C_k$ of Ω such that $B \cong C_n \backslash (B \cup F)$ and $A \cup F \succ B \cup F' \cup C_n$ for all n. By part (ii) this implies that there are B_n, F_n such that $B_n \cup F_n = C_n \backslash (B \cup F)$ for $B_n \in \mathcal{C}_E$ and $F_n \cap E = \emptyset$. Note that $B_2 \cap B_1 = \emptyset$ and therefore $B \cup B_1 \in \mathcal{C}_E$ by part (iii). Since $B_1 \in \mathcal{C}_E$, part(i) implies it is non-null; then, monotonicity implies that $B_1 \cup B \cup F' \succ B \cup F'$ and, therefore, B_1 is the desired set.

(v) Argue as in the proof of part (iv) to get $B_n \cup F_n = C_n \backslash (B \cup F)$ such that $B_n \in \mathcal{C}_E$, $F_n \cap E = \emptyset$. Since F_n is a partition of E^c and $\mu(E^c) > 0$ we must have $\mu(F_n) > 0$ for some n. Then if $F' = \emptyset$, $E' = F_n$ is the desired set. Similarly, if $F = \emptyset$, then $A \succ B \cup F'$ implies $\mu(F') \leq \mu(E) < \mu(E^c)$. Hence, there must be F_n such that $\mu(F_n \backslash F') > 0$ and therefore, $E' = F_n \backslash F'$ is the desired set.

(vi) We will prove the results for $n = 2$. Then, the general case follows from an inductive argument. Suppose $A \in \mathcal{C}_E$. Then, for $i = 1, 2$, $\mu_*(A \cap E_i) \leq \mu_*(A) = 0$ and $\mu_*(E_i \backslash A) = \mu_*((E \backslash A) \cap E_i) \leq \mu_*(E \backslash A) = 0$ and hence $A \in \mathcal{C}_{E_1} \sqcup \mathcal{C}_{E_2}$. Conversely, if $A_i \in \mathcal{C}_{E_i}$ for $i = 1, 2$, then for any $F \subset A_1 \cup A_2$, $F \cap E_i \subset A_i$ and hence $\mu(F \cap E_i) = 0$. Therefore, $\mu(F) = \mu(F \cap E_1) + \mu(F \cap E_2) = 0$ and similarly, $\mu(F) = 0$ whenever $F \subset A_1^c \cup A_2^c$ and therefore $A_1 \cup A_2 \in \mathcal{C}_E$.

□

Lemma B5: *Let $(\mathcal{C}_E, \eta_E)$ be a LAM and $A \in \mathcal{C}_E$. Then,*

(i) $E \succ A \succ \emptyset$.

(ii) $\eta_E(A) \geq \mu(E)$ *if and only if* $A \succeq E$;

(iii) For all $\epsilon > 0$ *there is* $C \in \mathcal{C}_E$ *such that* $\eta_E(C) < \epsilon$, $A \cap C = \emptyset$ *and* $A \cup C \in \mathcal{C}_E$.

Proof: (i) Lemma B4(i) implies that A is non-null. Therefore, monotonicity implies that $A \succ \emptyset$. From the definition of $\mathcal{C}_E$ it follows that $E \backslash A \in \mathcal{C}_E$ and, therefore, by Lemma B4(i) and monotonicity, $E \succ A$.

(ii) If $\eta_E(A) \geq \mu(F)$ we can find a sequence $F_n \in \mathcal{E}$ such that $A \succeq F_n$ and $\lim \mu(F_n) \geq \mu(F)$. Since μ is nonatomic, we can choose this sequence so that $F_n \subset F_{n+1}$. Hence, by Lemma B1(vii), $A \succeq \bigcup F_n \succeq F$, establishing $A \succeq F$. The converse follows from calibration.

(iii) By part (i), $E \succ A$. Hence, Axiom 4, there is a partition $C_1, \ldots, C_n$ of Ω such that $\hat{B}_n := (C_n \cap E) \backslash A \in \mathcal{C}_E$, $A \cup \hat{B}_n \in \mathcal{C}_E$. Let $B_1 = \hat{B}_1$. Replacing A with $A \cup B_1$ and repeating the argument yields $B_2 \in \mathcal{C}_E$ such that $(A \cup B_1) \cap B_2 = \emptyset$, $B_2 \in \mathcal{C}_E$ and $A \cup B_1 \cup B_2 \in \mathcal{C}_E$. Continuing in this fashion, we get a sequence of pairwise disjoints set B_n such that $B_n \in \mathcal{C}_E$ and $B_n \in E_1 \backslash A$ for all n. By part (i) the sequence $\eta_E(B_n)$ is bounded. If $\lim \eta_E(B_n) = 0$, we are done. Otherwise, $\eta_E(B_{n_j}) \geq \epsilon > 0$ some subsequence B_{n_j}. Assume, without loss of generality, that this subsequence is B_n and let $A_n = \bigcup_{i \geq n} B_n$. Then, by Lemma B4(iii), $A_n \in \mathcal{C}_E$. Also, by monotonicity, $A_n \succeq E$ for any E such that $\mu(E) \in (0, \epsilon)$. Hence, by Axiom 5, $\bigcap A_n \succeq E$ for some such E. But $\bigcap A_n = \emptyset$ and hence we have a contradiction. $\square$

Lemma B6: *Let $(\mathcal{C}_E, \eta_E)$ be a SLAM, $A, B \in \mathcal{C}_E$ and $F, F' \in E^c$. Then,*

(i) $F, F' \subset E^c$ implies $A \cup F \succeq B \cup F'$ if and only if $\eta_E(A) + \mu(F) \geq \eta_E(B) + \mu(F')$;

(ii) $A \cup B \in C_E$ implies $\eta_E(A \cup B) = \eta_E(A) + \eta_E(B)$.

Proof: (i) First, we will show that $A \cup F \succeq A \cup F'$ if and only if $\mu(F) \geq \mu(F')$. Note that $F \cong F'$ and $A \cup F \cong A \cup F'$ by Lemma B4(iii). Since μ represents $\succeq$ restricted to $\mathcal{E}$ it follows that $F \succeq F'$ if and only if $\mu(F) \geq \mu(F')$. Lemma B1(vi) implies that $F \succeq F'$ if and only if $A \cup F \succeq A \cup F'$ and, therefore, proves the assertion.

Next, we show that $A \cup F \succeq B$ implies $\eta_E(A) + \mu(F) \geq \eta_E(B)$. Choose $F^* \subset E^c$ such that $\eta_E(B) = \mu(F^*)$. Then, $B \sim F^*$ by Lemma B5(ii). If $\mu(F) \geq \mu(F^*)$, we get $\eta_E(A) + \mu(F) \geq \eta_E(B)$ immediately. Otherwise, since μ is non-atomic, we may assume $F \subset F^*$. Since $F \backslash F^* \in \mathcal{E}$, $A \succeq F^* \backslash F^{**}$. By Lemma B5(ii) this implies $\eta_E(A) \geq \mu(F^* \backslash F^{**}) = \eta_E(B) - \mu(F^{**})$ as desired.

Next, we show that $A \succeq B \cup F'$ implies $\eta_E(A) \geq \eta_E(B) + \mu(F')$. By Lemma B5(i) and (ii), it follows that $\eta(A) < 1/3$. Therefore $\mu(F') < 1/3$ and, since μ is non-atomic, we

can choose $F^* \subset E^c, F^* \cap F' = \emptyset$ such that $\eta_E(B) = \mu(F^*)$. Then, $B \sim F^*$ and hence $B \cup F' \sim F^* \cup F'$ and therefore $\eta_E(A) \geq \mu(F^*) + \mu(F') = \eta_E(B) + \mu(F')$.

Conversely, suppose $\eta_E(A) + \mu(F) \geq \eta_E(B)$. If $A \cup F \not\succeq B$, then, since $A \cup F \cong B$ by Lemma B4(ii), we must have $B \succ A \cup F$. Then, by Lemma B4(v), there is $F'' > 0$ such that $(A \cup F) \cap F'' = \emptyset$, $\mu(F'') > 0$ and $B \succ A \cup F' \cup F''$. Then, the part of the this lemma that we have already proven yields $\eta_E(B) \geq \eta_E(A) + \mu(F \cup F'')$. Hence, $\eta_E(B) > \eta_E(A) + \mu(F)$, a contradiction.

Next, suppose $\eta_E(A) \geq \eta_E(B) + \mu(F')$ and $A \not\succeq B \cup F'$. Then, arguing as above, we conclude that $B \cup F' \succ A$ and hence, by Lemma B4(v), $B \cup F' \succ A \cup F''$ for some $F'' \subset E^c$ such that $\mu(F'') > 0$. Then, since $\mu(F') < 1/3$ by the argument above, we can assume that F' and F'' are nested. If $F' \subset F''$, then we get $B \succ A \cup (F'' \backslash F')$ and then, part of the this lemma that we have already proven yields $\eta_E(B) > \eta_E(A)$, a contradiction. Similarly, if $F'' \subset F'$, then we have $B \cup (F' \backslash F'') \succ A$ and hence $\eta_E(B) + \mu(F') > \eta_E(B) + \mu(F' \backslash F'') \geq \eta_E(A)$, a contradiction.

Finally, to complete the proof of part (i) assume F, F' are nested. Since μ is nonatomic, we can do so without loss of generality. Then, if $F \subset F'$, $A \cup F \succeq B \cup F'$ if and only if $A \succeq B \cup (F' \backslash F)$. If $F' \subset F''$ then $A \cup F \succeq B \cup F'$ if and only if $A \cup (F \backslash F') \succeq B$. Then, the arguments given above prove part (i).

(ii) We first show that $\eta_E(A) + \eta_E(B) \geq \eta_E(A \cup B)$. Choose $F^* \subset E^c$ such that $\mu(F^*) = \eta_E(B)$. By part (i), $A \cup F^* \succeq A \cup B$ if and only if $\eta_E(A) + \eta_E(B) = \eta_E(A) + \mu(F^*) \geq \eta_E(A \cup B)$. Hence, it suffices to show that $A \cup F^* \succeq A \cup B$. If $A \cup F^* \not\succeq A \cup B$ then, since $A \cup F^* \cong A \cup B$ (by Lemma B4(ii)) it follows from Lemma B1(v) that $A \cup B \succ A \cup F^*$. Then, by Lemma B4(iv), $A \cup B \succ A \cup C \cup F^*$ for some $C \in \mathcal{C}_E$ such that $A \cap C = \emptyset$ and $A \cup C \in \mathcal{C}_E$ and, since $A \cong C \cup F^*$, $B \succ C \cup F^*$. From part (i) it now follows that $\eta_E(B) > \mu(F^*)$, a contradiction.

Finally, we show that $\eta_E(A \cup B) \geq \eta_E(A) + \eta_E(B)$. If not, then choose $F^* \subset E_E^c$ such that $\eta_E(A \cup B) = \mu(F^*) + \eta_E(A)$. It follows that $A \cup F^* \succeq A \cup B$ by part(i). Choose $C \in \mathcal{C}_E$ such that $\eta_E(C) < \eta_E(B) + \eta(A) - \eta_E(A \cup B)$, $A \cap C = \emptyset$ and $A \cup C \in \mathcal{C}_E$. By Lemma B5(iv) this is possible. By monotonicity, $A \cup F^* \not\succeq A \cup C \cup F^*$ and, since $A \cup F^* \cong A \cup C \cup F^*$, Lemma B1(v) implies that $A \cup C \cup F^* \succ A \cup F^*$. Then, Lemma

B1(v) and (vi) imply $C \cup F^* \succ B$. Then, part (i) implies $\eta_E(C) + \mu(F^*) \geq \eta_E(B) = \mu(F^*) + \eta_E(A) + \eta_E(B) - \eta_E(A \cup B) > \mu(F^*) + \eta_E(C)$, a contradiction.

(iii) By Lemma B5(iii), we can choose a set $C \in \mathcal{C}_E$ such that $C \subset B$, $A \cup C \in \mathcal{C}_E$ and $\eta_E(C) < \epsilon$. Hence, $B \backslash C, A \cup (B \backslash C)) \in \mathcal{C}_E$ and by Lemma B8, $\eta_E(B) = \eta_E(B \backslash C) + \eta_E(C)$ and also $\eta_E(A \cup (B \backslash C) = \eta_E(A) + \eta_E(B \backslash C)$. Hence, $\eta_E(A) + \eta_E(B) < \eta_E(A \cup (B \backslash C)) + \epsilon$. Therefore, $\eta_E(A) + \eta_E(B) - \epsilon < \sup_{D \in \mathcal{C}_E} \eta_E(D)$. Since ϵ is arbitrary, we have $\eta_E(A) + \eta_E(B) \leq \sup_{D \in \mathcal{C}_E} \eta_E(D) = \eta(E_E)$. □

Lemma B7: *Let $(\mathcal{C}_{E_i}, \eta_{E_i})$ for $i = 1, \ldots, n$ be LAMs such that $E_i \cap E_j = \emptyset$ for $i \neq j$. Then, $\eta_E(\bigcup A_i) = \sum \eta_{E_i}(A_i)$ whenever $E = \bigcup E_i$ and $A_i \in \mathcal{C}_i$ for all i.*

Proof: We will prove the result for $n = 2$ and $\mu(E_1) \leq 1/3$; then the general case follows from an inductive argument. Choose $E_i^* \subset E_i$ such that $\eta_{E_i}(A_i) = \mu(E_i^*)$. Let $B_i = E_i \backslash A_i$ and choose $F_i^* \subset E_i$ such that $\eta_{E_i}(B_i) = \mu(F_i^*)$. Finally, let $F_i = E \backslash E_i^*$.

To prove $\eta_E(A_1 \cup A_2) \geq \eta_{E_1}(A_1) + \eta_{E_2}(A_2)$, note that $A_1 \cup A_2 \in \mathcal{C}_E$ by Lemma B4(v). By Lemma B5(ii), $A_i \succeq E_i^*$. Lemma B3(iii) implies that A_1, A_2 and E_1^*, A_2 conform and, therefore, $A_1 \cup A_2 \succeq E_1^* \cup A_2$. Since E_2^* is unambiguous it follows that $E_1^* \cup A_2 \succeq E_1^* \cup E_2^*$ and, by transitivity, $A_1 \cup A_2 \succeq E_1^* \cup E_2^*$. Lemma B5(ii) now implies $\eta_E(A_1 \cup A_2) \geq \mu(E_1^*) + \mu(E_2^*) = \eta_{E_1}(A_1) + \eta_{E_2}(A_2)$.

For the converse, assume $\eta_E(A_1 \cup A_2) - \delta' \geq \mu(E_1^* \cup E_2^*) = \eta_{E_1}(A_1) + \eta_{E_2}(A_2)$ for some $\delta' > 0$. We first establish that

$$\eta_{E_1}(A_1) + \eta_{E_1}(B_1) + \eta_{E_2}(A_2) + \eta_{E_2}(B_2) \leq \mu(E_1 \cup E_2) - \delta'$$

To prove this assertion, first note that by Lemma B5(i) and (ii) there exist $\hat{E}_1 \subset E_1, \hat{E}_2 \subset E_2$ such that $A_1 \cup A_2 \succeq \hat{E}_1 \cup \hat{E}_2$ and $\mu(\hat{E}_1) + \mu(\hat{E}_2) = \mu(E_1^* \cup E_2^*) + \delta'$. By calibration, it follows that $W((A_1 \cup A_2)^c) \subset W((\hat{E}_1 \cup \hat{E}_2)^c)$, which, in turn, implies that $W(B_1 \cup B_2) \subset W((E_1 \backslash \hat{E}_1) \cup (E_2 \backslash \hat{E}_2))$. Therefore, $\eta_E(B_1 \cup B_2) \leq \mu(E_1 \backslash \hat{E}_1) + \mu(E_2 \backslash \hat{E}_2)$. Since $\eta_{E_1}(B_1) + \eta_{E_2}(B_2) \leq \eta_E(B_1 \cup B_2)$ it follows that

$$\begin{aligned}\eta_{E_1}(B_1) + \eta_{E_1}(A_1) + \eta_{E_2}(B_2) + \eta_{E_2}(A_2) &\leq \mu(E_1 \backslash \hat{E}_1) + \mu(E_2 \backslash \hat{E}_2) + \mu(E_1^*) + \mu(E_2^*) \\ &= \mu(E_1 \cup E_2) - \delta'\end{aligned}$$

as desired.

Next, observe that

$$A_1 \cup A_2 \succeq A_1 \cup E_2^* \succeq E_1^* \cup E_2^*$$

$$A_1 \cup A_2 \succeq A_2 \cup E_1^* \succeq E_1^* \cup E_2^*$$

by the argument above. Further, since $A_1 \succeq E_1^*$ if and only if $A_1 \cup E_2^* \succeq E_1^* \cup E_2^*$ it follows that $A_1 \cup E_2^* \succeq E_1^* \cup E_2^*$ and (by an analogous argument) $A_2 \cup E_1^* \succeq E_1^* \cup E_2^*$ with

$$W(A_1 \cup E_2^*) = W(E_1^* \cup E_2^*) = W(A_2 \cup E_1^*)$$

By calibration, we have $A_1 \cup A_2 \succ E_1^* \cup E_2^*$. If $A_1 \cup A_2 \succ A_1 \cup E_2$, then, by Lemma B5(i), we can choose $E' \subset E_2$ such that $A_1 \cup A_2 \succeq A_1 \cup (E_2^* \cup E')$. Hence, by Axiom 3, $A_2 \succeq E_1^* \cup E'$ which is impossible. The same argument also rules out $A_1 \cup A_2 \succ A_2 \cup E_1$. Then, by calibration, we conclude that

$$W(A_1 \cup A_2) \supset W(A_1 \cup E_2^*) = W(E_1^* \cup E_2^*)$$

$$W(B_1 \cup B_2) = W(B_1 \cup F_2) \subset W(F_1 \cup F_2)$$

$$W(A_1 \cup A_2) \supset W(A_2 \cup E_1^*) = W(E_1^* \cup E_2^*)$$

$$W(B_1 \cup B_2) = W(B_2 \cup F_1) \subset W(F_1 \cup F_2)$$

with all inclusions strict. Since $W(B_1 \cup F_2) = W(B_2 \cup F_1)$, the definition of η and Axiom 3 imply that $\eta_{E_1}(B_1) + \mu(F_2) = \eta_{E_2}(B_2) + \mu(F_1)$ which, in turn, implies that

$$\mu(E_1) - (\eta_{E_1}(B_1) + \eta_{E_1}(A_1)) = \delta = \mu(E_2) - (\eta_{E_2}(B_2) + \eta_{E_2}(A_2)) \qquad (\dagger)$$

Note that $\delta \geq \delta'/2$.

Let $E_1' \subset E_1$ such that $\mu(E_1') = \delta/8$. Define $E_1'' = E_1 \backslash E_1'$ and let $A_1' = A_1 \cap E_1', A_1'' = A_1 \cap E_1''$, $B_1' = B_1 \cap E_1', B_1'' = B_1 \cap E''$. Note that $\eta(A_1') \leq \delta/8, \eta(B_1') \leq \delta/8$. Furthermore, $E_1' \cup A_1'' \succeq A_1$. Hence, part (i) of Lemma B6 implies that $A_1'' \succeq F$ if $\mu(F) \leq \eta_{E_1}(A_1) - \delta/8$ and, therefore, $\eta_{E_1}(A) \geq \eta_{E_1}(A_1'') \geq \eta_{E_1}(A_1) - \delta/8$. An analogous argument shows that $\eta_{E_1}(B_1) \geq \eta_{E_1}(B_1'') \geq \eta_{E_1}(B_1) - \delta/8$. Therefore,

$$0 \leq \mu(E_1') - (\eta_{E_1'}(B_1') + \eta_{E_1'}(A_1')) < \delta/4 < \mu(E_1'') - (\eta_{E_1''}(B_1'') + \eta_{E_1''}(A_1''))$$

Hence, (†) cannot hold for A_1', A_1'' and for A_1', A_2. We conclude that $\eta_{E_1'}(A_1') + \eta_{E_1''}(A_1'') = \eta_E(A)$ and $\eta_{E_1'}(A_1') + \eta_{E_2}(A_2) = \eta_{E_1' \cup E_2}(A_1' \cup A_2)$.

Let $A_2'' = A_2 \cup A_1'$ and let $E_2'' = E_2 \cup E_1'$. Note that $A_1'' \cup A_2'' = A_1 \cup A_2$ and hence

$$\eta_{E_1''}(A_1'') + \eta_{E_2''}(A_2'') = \eta_{E_1}(A_1) + \eta_{E_2}(A_2)$$
$$\eta_E(A_1'' \cup A_2'') = \eta_E(A_1 \cup A_2)$$

Thus, we can repeat the argument above with A_1'', A_2'' replacing A_1, A_2 for the same value of δ. Since $\mu(E_1'') = \mu(E_1) - \delta/8$, repeated application of this argument then yields the desired contradiction. □

Lemma B8: *Let $(\mathcal{C}_E, \eta_E)$ be a LAM and suppose $A, B, A \cup B \in \mathcal{C}$. Then, $\eta_E(A \cup B) = \eta_E(A) + \eta_E(B)$.*

Proof: Lemma B6 (ii) proves the result for all SLAMs. To prove the result for an arbitrary LAM, divide E into three sets E_1, E_2, E_3 such that $\mu(E_i) = \mu(E)/3$. Let $A_i = E_i \cap A$ and $B_i = E_i \cap B$ for $i = 1, 2, 3$. Then, $A_i \in \mathcal{C}_i$ by Lemma B4(v). By the argument above and by Lemma B7, $\eta_E(A) + \eta_E(B) = \eta_{E_1}(A_1) + \eta_2(A_2) + \eta_3(A_3) + \eta_{E_1}(B_1) + \eta_{E_2}(B_2) + \eta_{E_3}(B_3) = \eta_{E_1}(A_1 \cup B_1) + \eta_{E_2}(A_2 \cup B_2) + \eta_{E_3}(A_3 \cup B_3)$. Applying Lemma B7 and Lemma B6(ii) yields $\eta_{E_1}(A_1 \cup B_1) + \eta_{E_3}(A_2 \cup B_2) + \eta_3(A_3 \cup B_3) = \eta_E(A_1 \cup B_1 \cup A_2 \cup B_2 \cup A_3 \cup B_3) = \eta_E(A \cup B)$. □

Let (G, G^c) be an essential partition of Ω. Define the measure η on Σ as follows:

$$\eta(A) := \hat{\eta}(E(A \cap G^c)) + \hat{\eta}(A \cap G^c \cap F(A \cap G^c))$$

where

$$\hat{\eta}(A) := \begin{cases} \eta_E(A) & \text{if } A \in \mathcal{C}_E, E \in G^c \\ \sup_{A \in \mathcal{C}_E} \eta_E(A) & \text{if } A = E \subset G^c \\ 0 & \text{if } A \text{ is null} \end{cases}$$

First, we will show that η is well-defined. In particular, $\hat{\eta}$ is well-defined; that is, $A \in \mathcal{C}_E \cap \mathcal{C}_F$ for two LAMs $(\eta_E, \mathcal{C}_E)$ and $(\eta_F, \mathcal{C}_F)$ implies $\eta_E(A) = \eta_F(A)$. Note that if $(\eta_E, \mathcal{C}_E)$ and $(\eta_F, \mathcal{C}_F)$ are LAMs with a common element A, then $A \cap E \cap F \in \mathcal{C}_{E^*}$ for $E^* = E \cap F$ and $E \backslash E^*$ and $F \backslash E^*$ are null. Hence, $\eta_{E^*}(A \cap E^*) = \eta_E(A)$ and symmetrically, $\eta_{E^*}(A \cap E^*) = \eta_F(A)$ and therefore, $\eta_E(A) = \eta_F(A)$.

Then, let $E_0 = F(A \cap G^c) \cap G^c$. If $\mu(E_0) = 0$, then $A \cap G^c \cap E$ is null for any $E \subset G^c$ such that $\mu(E \Delta E_0) = 0$. If $\mu(E_0) > 0$ and $\mu(E \Delta E_0) = 0$, then, $A \cap G^c \in \mathcal{C}_{E_0}$ if and

only if $A \cap G^c \in \mathcal{C}_E$ and $\eta_E(A \cap E) = \eta_{E_0}(A \cap E_0)$. Similarly, if $E(A \cap G^c)$ is null and $\mu(E(A \cap G^c)\Delta E) = 0$ for some $E \subset G^c$, E is also null and hence $\hat{\eta}(E(A \cap G^c)) = \hat{\eta}(E) = 0$. Finally, if $E_0 = E(A \cap G^c)$, $\mu(E_0) > 0$, $E \subset G^c$, $C \in \mathcal{C}_{E_0}$ and $\mu(E_0 \Delta E_1) = 0$, then $C \cap E \in \mathcal{C}_E$ and $\eta_{E_0}(C) = \eta_E(C \cap E)$.

Lemma B9: $\eta(A) = \eta(E(A)) + \eta(A \cap F(A))$.

Proof: $\eta(A) = \hat{\eta}(E(A \cap G^c)) + \hat{\eta}(A \cap G^c \cap F(A \cap G^c))$ by definition. Note that $\eta(E(A)) = \hat{\eta}(E(E(A) \cap G^c)) + \hat{\eta}(E(A \cap G^c) \cap G^c \cap F(E(A \cap G^c) \cap G^c)) = \hat{\eta}(E(A) \cap G^c))$ since $E(E(A) \cap G^c) = E(A) \cap G^c$ and $F(E')$ is null for all $E' \in \mathcal{E}$. Similarly, $\hat{\eta}(E(A \cap G^c) = \hat{\eta}(E(E(A \cap G^c))) + \hat{\eta}(E(A \cap G^c) \cap G^c \cap F(E(A \cap G^c) \cap G^c))$. It is also easy to verify that $E(A) \cap G^c = E(A \cap G^c)$ and therefore, $\eta(E(A)) = \hat{\eta}(E(A \cap G^c)$ as desired.

Next, consider $\eta(B)$ for $B := A \cap F(A)$. Then, $\eta(B) = \hat{\eta}(E(B \cap G^c)) + \hat{\eta}(B \cap G^c \cap F(B \cap G^c))$. But $E(B \cap G^c) = E(B) = E(A \cap F(A)) = E(A) \cap F(A) = \emptyset$. Also, $F(B \cap G^c) = F(A \cap F(A) \cap G^c) = F(A \cap G^c)$. Hence, $\hat{\eta}(B \cap G^c \cap F(B \cap G^c)) = \hat{\eta}(A \cap F(A) \cap G^c \cap F(A \cap G^c)) = \hat{\eta}(A \cap G^c \cap F(A \cap G^c))$ since $A \cap F(A) \cap G^c \cap F(A \cap G^c) = A \cap G^c \cap F(A \cap G^c)$. Thus, $\eta(E(A)) + \eta(A \cap F(A)) = \hat{\eta}(E(A \cap G^c)) + \hat{\eta}(A \cap G^c \cap F(A \cap G^c))$ and hence $\eta(A) = \eta(E(A)) + \eta(A \cap F(A))$. $\square$

Lemma B10: *Let $(\eta_E, \mathcal{C}_E)$ be a LAM. If $A, B \in \mathcal{C}_E$, $A \cap B = \emptyset$ and $A \cup B = E$, then $\eta(A) = \eta_E(A)$, $\eta(B) = \eta_E(B)$ and $\eta_E(A) + \eta_E(B) = \eta(A \cup B) = \eta(E)$.*

Proof: If $A \in \mathcal{C}_E$ for some E, then $A \cap G^c \in E_2$ for $E_2 := E \cap G^c$ and $E(A)$ is null. So, $\eta(A) = \hat{\eta}(A) = \eta_{E_2}(A \cap G^c) = \eta_E(A)$ and similarly, $\eta(B) = \eta_E(B)$. So, we can assume without loss of generality, that $E \subset G^c$.

By Lemma B5(iii), we can choose a set $C \in \mathcal{C}_E$ such that $C \subset B$, $A \cup C \in \mathcal{C}_E$ and $\eta_E(C) < \epsilon$. Hence, $B \backslash C, A \cup (B \backslash C)) \in \mathcal{C}_E$ and by Lemma B8, $\eta_E(B) = \eta_E(B \backslash C) + \eta_E(C)$ and also $\eta_E(A \cup (B \backslash C) = \eta_E(A) + \eta_E(B \backslash C)$. Hence, $\eta_E(A) + \eta_E(B) < \eta_E(A \cup (B \backslash C)) + \epsilon$. Therefore, $\eta_E(A) + \eta_E(B) - \epsilon < \sup_{D \in \mathcal{C}_E} \eta_E(D)$. Since ϵ is arbitrary, we have $\eta_E(A) + \eta_E(B) \leq \sup_{D \in \mathcal{C}_E} \eta_E(D) = \eta(E_E)$

Next, choose any $D \in \mathcal{C}_E$ such that $\eta_E(D) > \eta(E) - \epsilon$. Let $C_1 = A \cap D, C_2 = B \cap D, C_3 = A \cap (E \backslash D), C_4 = B \cap (E \backslash D)$ and $C_5 = E^c$ and let $\{E_\kappa\}$ for $\emptyset \neq \kappa \subset \{1, \ldots, 5\}$ be a μ-split of $C_1, \ldots, C_5$. Since $E^c \in \mathcal{E}$, E_κ is null for all κ such that $5 \in \kappa, \kappa \neq \{5\}$.

Hence, we ignore these elements of the μ-split and assume $E_5 = E^c$; that is, we can find an alternative μ-split in which all these sets are empty and $E_5 = E^c$. Let $F_1, \ldots, F_m$ be the nonnull elements of the μ-split E_κ for κ such that $5 \notin \kappa$ and, therefore, $F_1, \ldots, F_m$ is a partition of E. Since $A, B, D \in \mathcal{C}_E$, Lemma B4(v) implies that $A \cap F_i, B \cap F_i, D \cap F_i \in \mathcal{C}_{F_i}$ for all i. By Lemma B7, $\eta_E(A) = \sum_i \eta'_{F_i}(A \cap F_i)$, $\eta_1(B) = \sum_i \eta'_{F_i}(A \cap F_i)$ and $\eta_1(D) = \sum_{F_i} \eta'_i(D \cap F_i)$. Since $D \subset A \cup B$ it follows that $D \cap F_i = (D \cap A \cap F_i) \cup (D \cap B \cap F_i)$ for all i. So, $\eta'_i(D \cap F_i) \leq \eta'_i(A \cap F_i) + \eta'_i(B \cap F_i)$ for all i. Hence, $\eta_E(D) \leq \eta_E(A) + \eta_E(B)$ and therefore, $\sup_{D \in \mathcal{C}_E} \eta_E(D) \leq \eta_E(A) + \eta_E(B)$. □

Lemma B11: *$A \subset E$ and $B \cap E = \emptyset$ implies $\eta(A \cup B) = \eta(A) + \eta(B)$.*

Proof: Lemma B3(iii) implies that $E(A \cup B) = E(A) \cup E(B)$ and $F(A \cup B) = F(A) \cup F(B)$. Since $\eta(A) = \eta(A \cap G^c)$, we can assume without loss of generality that $A, B \subset G^c$. Hence we have $\eta(A \cup B) = \eta(E(A) \cup E(B)) + \eta((A \cap F(A)) \cup (B \cap F(B)))$. The argument we made to show that η is well defined ensures that $\eta(C \cup D) = \eta(D)$ whenever C is null, and, therefore, we may assume that all four sets $E(A), E(B), F(A), F(B)$ are nonnull.

Then, $\eta((A \cap F(A)) \cup (B \cap F(B)) = \eta(A \cap F(A)) + \eta(B \cap F(B))$ follows from Lemma B7. Take $C \in \mathcal{C}_{E(A)}$ and $D \in \mathcal{C}_{E(B)}$. Let $F = E(A) \cup E(B)$. Then, by Lemma B4(v), $C \cup D \in \mathcal{C}_F$ and Lemma B10 implies $\eta(E(A) \cup E(B)) = \eta(C \cup D) + \eta((E(A) \backslash C) \cup (E(B) \backslash D))$. By Lemma B7, $\eta(C \cup D) = \eta(C) + \eta(D)$ and $\eta((E(A) \backslash C) \cup (E(B) \backslash D)) = \eta(E(A) \backslash C) + \eta(E(B) \backslash D)$. Again, by Lemma B10, $\eta(C) + \eta(E(A) \backslash C) = \eta(E(A))$ and $\eta(D) + \eta(E(B) \backslash D) = \eta(E(B))$ and therefore, $\eta(E(A) \cup E(B)) = \eta(E(A) \cup E(B))$, proving that $\eta(A \cup B) = \eta(A) + \eta(B)$. □

Lemma B12: *η is a nonatomic measure.*

Proof: (i) First, we will show that η is additive. Suppose $A_1 \cap A_2 = \emptyset$. Let $A = A_1 \cup A_2$, $A_3 = A^c$ and let $\{E_\kappa\}$ be a μ-split of A_1, A_2, A_3. Let K_i the set of all κ such that $i \in \kappa$. Then, $E(A) = E(A_1) \cup E(A_2) \cup E_{12} = E_1 \cup E_2 \cup E_{12}$. Also, $F(A) = E_{13} \cup E_{23} \cup E_{123}$ and by Lemma B9, $\eta(A) = \eta(E(A)) + \eta(A \cap F(A))$. So, by Lemma B11,

$$\eta(E(A)) = \eta(E(A_1)) + \eta(E(A_2)) + \eta(E_{12})$$

$$\eta(A \cap F(A)) = \eta(A \cap E_{13}) + \eta(A \cap E_{23}) + \eta(A \cap E_{123})$$

But $\eta(A \cap E_{13}) = \eta(A_1 \cap E_{13})$, $\eta(A \cap E_{23}) = \eta(A_2 \cap E_{23})$ and $\eta(A \cap E_{123}) = \eta(A_1 \cap E_{123}) + \eta(A_2 \cap E_{123})$. Finally, $\eta(E_{12}) = \eta(A_1 \cap E_{12}) + \eta(A_2 \cap E_{12})$ and again by Lemma B11,

$$\begin{aligned}
\eta(A_1 \cap F(A_1)) &= \eta(A_1 \cap (E_{13} \cup E_{12} \cup E_{123})) \\
&= \eta(A_1 \cap E_{13}) + \eta(A_1 \cap E_{12}) + \eta(A_1 \cap E_{123}) \\
\eta(A_2 \cap F(A_2)) &= \eta(A_2 \cap (E_{23} \cup E_{12} \cup E_{123})) \\
&= \eta(A_2 \cap E_{13}) + \eta(A_2 \cap E_{23}) + \eta(A_2 \cap E_{123})
\end{aligned}$$

Hence, $\eta(A_1 \cup A_2) = \eta(A_1) + \eta(A_2)$ as desired.

Next, we will show that it is countably additive. For that, it is enough the establish that $A_{n+1} \subset A_n$, $\bigcap A_n = \emptyset$ implies $\lim \eta(A_n) = 0$. Let $E_n = E(A_n)$ and $F_n = F(A_n)$. Clearly, $E_{n+1} \subset E_n$ for all n. If $\lim \mu(E_n) > 0$, then countable additivity of μ yields $\mu(\bigcap E_n) = \lim \mu(E_n) > 0$. Hence, $\emptyset \neq \bigcap E_n \subset \bigcap A_n$, a contradiction. So, by Lemma B9, it remains to show that $\lim \eta(A_n \cap F_n) = 0$.

Let $G_n = E_n \cup F_n$ and note that $G_n = (E(A_n^c))^c$ and hence $G_{n+1} \subset G_n$ for all n. Let $G^* = \bigcap G_n$. Since $\mu(\bigcap E_n) = 0$ and $\bigcap G_n = G^*$, for all $\epsilon > 0$, there exists N such that $\mu(E_N) < \epsilon$ and $\mu(G_N \backslash G^*) < \epsilon$. Note that

$$\begin{aligned}
\eta(A_n \cap F_n) &\leq \eta(A_n \cap (E_N \cup (F_n \backslash E_N)) \\
&= \eta(A_n \cap E_N) + \eta(A_n \cap (F_n \backslash E_N)) \\
&\leq \epsilon + \eta(A_n \cap (G_n \backslash E_N)
\end{aligned}$$

Since $G_n \backslash E_N = G^* \cup (G_n \backslash G^*) \backslash E_N = (G^* \backslash E_N) \cup ((G_n \backslash G^*) \backslash E_N) \subset (G^* \backslash E_N) \cup (G_n \backslash G^*)$, Lemma B5(ii) implies that

$$\begin{aligned}
\eta(A_n \cap F_n) &\leq \epsilon + \eta(A_n \cap (G^* \backslash E_N)) + \eta(A_n \cap (G_n \backslash G^*) \\
&\leq 2\epsilon + \eta(A_n \cap (G^* \backslash E_N))
\end{aligned}$$

for $n \geq N$. But $\eta(C) = \eta(C \cap G^c)$ for all C and therefore we can assume without loss of generality that $A_n, G^* \subset G^c$. Let $E' = G^* \backslash E_N$. If $\mu(E') = 0$, we have $\eta(A_n \cap F_n) \leq 2\epsilon$ and since ϵ is arbitrary, we have $\lim \eta(A_n \cap F_n) = 0$. If $\mu(E') > 0$, consider the LAM $(\eta_{E'}, \mathcal{C}_{E'})$. Note that $G^* \subset G_n = E_n \cup F_n$ and $E_n \subset E_N$, $G^* \backslash E_N \subset F_n$ for all $n \geq N$. Therefore, $C_n := A_n \cap (G^* \backslash E_N) \in \mathcal{C}_{E'}$. Since $\bigcap C_n = \emptyset$, if $\lim \eta_{E^n}(C_n) > 0$, we have $C_n \succeq E$ for

some E such that $\mu(E) > 0$. Hence, $\emptyset = \bigcap C_n \succeq E$, a contradiction. So, $\lim \eta_{E'}(C_n) = 0$ and therefore $\lim \eta(A_n \cap (G^* \backslash E_N) = \lim \eta_{E'}(C_n) = 0$ as desired.

To see that η is nonatomic, note that by Lemma B9, $\eta(A) > 0$ implies $\eta(E(A)) = \eta(E(A) \cap G^c) > 0$ or $\eta(A \cap F(A)) = \eta(A \cap G^c \cap F(A \cap G^c)) > 0$. In the first case, the desired result follows from the fact that μ is nonatomic and Lemma B5(i) and (ii). In the former case, the desired result follow from Lemma B5(iii). □

Let

$$\pi(A) = \max_{E \in [A]} \mu(A) + \min_{E \in [A]} \eta(A \backslash E)$$

Lemma B13:

(i) If $A \subset E$ and $B \cap E = \emptyset$, then $\pi(A \cup B) = \pi(A) + \pi(B)$.

(ii) If $\mu(E) = \eta(E)$, then $\mu(F) = \eta(F)$ for all $F \subset E$.

(iii) If $\mu(E) = \eta(E)$, $A \cup B \subset E$, $A \cap B = \emptyset$, then $\pi(A \cup B) = \pi(A) + \pi(B)$.

Proof: (i) This follows from Lemma B11 and the additivity of μ.

(ii) If E is null, this is obvious. So assume $\mu(E) > 0$. If $\mu(E) = \mu(F)$, then, $\mu(F) = \mu(E) = \eta(E) = \eta(F)$ and we are done. So, assume $\mu(F) \in (0, \mu(E))$. Then, by definition $\mu(E) = \eta(E) = \eta(E \cap G^c)$ and by Lemma B5(i) and (ii), $\eta(E \cap G^c) \leq \mu(E \cap G^c) = \mu(E) - \mu(E \backslash G^c)$. So, $\mu(E \backslash G^c) = 0$. Hence, assume without loss of generality, that $E \subset G^c$ and pick $A \in \mathcal{C}_E$ and let $B = E \backslash A \in \mathcal{C}_E$. Let $E^* = E \backslash F$. Hence, $\mu(E) = \mu(F) + \mu(E^*) \geq \eta_F(F) + \eta_{E^*}(E^*)$ by Lemmas B5(i) and B5(ii). Then, by Lemma B10, $\mu(E) \geq \eta_F(A \cap F) + \eta_F(B \cap F) + \eta_{E^*}(A \cap E^*) + \eta_{E^*}(B \cap E^*)$. Since, by Lemma B11, $\mu(E) \geq \eta(A) + \eta(B) = \eta(E) = \mu(E)$, it follows that $\mu(F) = \eta_F(A \cap F) + \eta_F(B \cap F) = \eta_F(F)$, as desired.

(iii) Assume, without loss of generality, that $A \cup B \subset E \subset G^c$. Then, let $E_1 = E(A), E_2 = E(B)$ and $E_3 = E(A \cup B) \backslash (E_1 \cup E_2)$. From the definition of π and the fact that $E(A \cup B) = E_1 \cup E_2 \cup E_3$ it follows that

$$\pi(A \cup B) = \mu(E_1) + \mu(E_2) + \mu(E_3) + \eta((A \cup B) \cap F(A \cup B))$$

$$\pi(A) = \mu(E_1) + \eta(A \cap E_3) + \eta(A \cap F(A))$$

$$\pi(B) = \mu(E_2) + \eta(B \cap E_3) + \eta(B \cap F(B))$$

Let $E_5 = E(F(A \cup B)\backslash B)$ and $E_6 = E(F(A \cup B)\backslash B)$. Then, $F(A \cup B) = E_5 \cup E_6$, $A \cap F(A) \in \mathcal{C}_{E_5}$ and $B \cap F(B) \in \mathcal{C}_{E_6}$. Let $E_7 = E_5 \cap E_6$, $E_8 = E_5 \backslash E_7$ and $E_9 = E_6 \backslash E_7$. Arguing as above, we get $\eta((A \cup B) \cap F(A \cup B)) = \sum_{i=7}^{9} \eta(A \cap E_i) + \sum_{i=7}^{9} \eta(B \cap E_i) = \eta(A \cap F(A)) + \eta(B \cap F(B))$ and $\mu(E(A \cup B)) = \mu(E_1) + \mu(E_2) + \eta(A \cap E_3) + \eta(B \cap E3) = \mu(E(A)) + \mu(E(B))$, as desired. $\square$

Lemma B14:

(i) $\pi(A) = \max\{\mu(E) \,|\, A \succeq E\}$.

(ii) π *represents* $\succeq$.

(iii) $(\mu, \mathcal{E})$ *and* (η, Σ) *are compatible.*

Proof: (i) If $A \in \mathcal{E}$, then $\pi(A) = \mu(E)$ and hence the result is obvious. Suppose $A \notin \mathcal{E}$ Then, $A = E(A) \cup (A \cap F(A))$ and $\pi(A) = \mu(E(A)) + \eta(A \cap F(A))$ and $F(A) \subset G^c$. Let $a = \sup\{\mu(E) \,|\, A \succeq E\}$ and we can construct E_n such that $E_n \subset E_{n+1}$, $\bigcup E_n = E$, $\mu\left(\bigcup_{n=1}^{k} E_n\right) < a$ and $\lim\left(\bigcup E_n\right) = a$. Then, $A \succeq \bigcup_{n=1}^{k} E_n$ for all k and hence $\left(\bigcup_{n=1}^{k} E_n\right)^c \succeq A^c$ by consistency and therefore $E^c = (\bigcap E_n)^c \succeq A^c$ by Axiom 5. Hence, $A \succeq E$, again by consistency. Note that $\mu(E(B) \cup F(B)) > \mu(E) > \mu(E(B))$. Therefore, we may choose E such that $E(A) \subset E \subset E(A) \cup F(A)$. Since $A \succeq E$, we have $A \cap F(A) \succeq E \backslash E(A)$. Therefore, $\eta(A \cap F(A)) \geq \mu(E) - \mu(E(A))$ and $\pi(B) \geq a$.

If $\pi(A) > a$, then we can find F such that $F \subset E(A) \cup F(A)$, $\mu(F) > a$ and $E(A) \cup (A \cap F(A)) \succeq F$. Hence $A \cap F(A) \succeq F \backslash E(A)$. That is, $\eta(A) > \pi(A) - \mu(E(A))$, a contradiction. This completes the proof of (i).

For (ii), suppose $A \succeq B$, then by consistency $W(B) \subset W(A)$ and $W(A^c) \subset W(B^c)$. Take E, F such that $\pi(B) = \mu(E)$ and $\pi(A^c) = \mu(F)$. By part (i), E and F are well-defined. Then, by calibration, $A \succeq E$ and $B^c \succeq F$ and hence $\pi(A) \geq \mu(E) = \pi(B)$ and $\pi(B^c) \geq \mu(F) = \pi(A^c)$. For the converse, suppose $\pi(A) \geq \pi(B)$ and $\pi(B^c) \geq \pi(A^c)$. Then, take F such that $\mu(F) = \pi(A)$. By part (i), $A \succeq F$. Then, $B \succeq E$ implies $\mu(F) \geq \mu(E)$ and hence $A \succeq F \succeq E$. Hence, $W(B) \subset W(A)$. The same argument establishes that $W(A^c) \subset W(B^c)$ and hence $A \succeq B$ by calibration.

For (iii), note that $\eta(E) = \eta(E \cap G^c)$ by construction. If $\eta(E \cap G^c) = 0$, we are done. Otherwise, let $E_0 = E \cap G^c$. By definition, $\eta(E \cap G^c) = \eta_{E_0}(E_0)$. Lemma B5(i) and B5(ii),

$\eta_{E_0}(E_0) \leq \mu(E_0)$. To conclude the proof, we will show that this inequality is strict. Pick any $A \in \mathcal{C}_{E_0}$. Then, by Lemma B10, $\eta_{E_0}(E_0) = \eta_{E_0}(A) + \eta_{E_0}(B)$ for $B := E_0 \backslash A$. To conclude the proof, we will show that if $\eta_{E_0}(A) = \mu(E_0) - \eta_{E_0}(B)$, then $A \in \mathcal{E}$, which would contradict the fact that $A \in \mathcal{C}_{E_0}$.

To prove this, we will first show that $\eta_{E_0}(A) = \mu(E_0) - \eta_{E_0}(B)$ and $(C \cup D) \cap A = \emptyset$ implies $[\pi(C) \geq \pi(D), \pi(D^c) \geq \pi(C^c)]$ if and only if $[\pi(C \cup A) \geq \pi(D \cup A), \pi(D^c \cap A) \geq \pi(C^c \cap A^c)]$ and appeal to part (ii) of this lemma. Note that $\pi(C \cup A) = \pi(((C \cup A) \cap E_0) \cup (C \cup A) \cap E_0^c) = \pi((C \cup A) \cap E_0) + \pi((C \cup A) \cap E_0^c)$ by Lemma B13(i). Clearly $(C \cup A) \cap E_0^c = A \cap E_0$ and hence $\pi(C \cup A) = \pi(C \cap E_0^c) + \pi(C \cap E_0) + \pi(A \cap E_0)$ by Lemma B13(iii). Again by Lemma B13(i), we have (a) $\pi(C \cup A) = \pi(C) + \pi(A)$. Similarly, $\pi(C^c) = \pi(E_0^c \cap C^c) + \pi(C^c \backslash E^c) = \pi(A) + \pi(B \cap C^c) + \pi(C^c \backslash E^c) = \pi(A) + \pi(A^c \cap C^c)$; that is, (b) $\pi(C^c) = \pi(A) + \pi(A^c \cap C^c)$. But, (a) and (b) imply $[\pi(C) \geq \pi(D), \pi(D^c) \geq \pi(C^c)]$ if and only if $[\pi(C \cup A) \geq \pi(D \cup A), \pi(D^c \cap A) \geq \pi(C^c \cap A^c)]$ as desired $\square$

7. Appendix C: Proofs of Propositions 3 and 4

7.1 Proof of Proposition 3

An act preference $\succeq^*$ is monotone* if it is monotone and $F_f \leq F_g$, $G_f \geq G_g$ implies $f \succeq^* g$. First, we will show that a sophisticated and monotone preference is monotone*. If $\mu(f^{-1}(z)) = 1$, then $f \sim^* z$. To see this, note that since $\succeq^*$ is monotone, $z \succ^* wE_nf \succ^* w$ whenever E_n E_n^c are both nonnull. Hence by continuity, $z \succeq^* f$. Next, suppose $z \succ^* f$. Then, by continuity, we must have $y < z$ such that $y \succ^* f$. But then by monotonicity and transitivity, $y \succ^* wE_nf \succ^* w$ whenever E_n E_n^c are both nonnull. By choosing E_n such that $\lim \mu(E_n) = 0$ we can ensure F_{g_n} and G_{g_n} converge to F_z, G_z, where $g_n = wE_nf$. Thus, by continuity $y \succeq^* z$, a contradiction.

Let $u(z) = 1$ and $u(w) = 0$. Then, for $x \in (w, z)$, let $u(x) = \sup\{\mu(E') \,|\, x \succeq^* zE'w\}$ where μ is the unique probability measure of π. If $u(x) = 0$, then choose E_n such that $\pi(E_n) = 1/(n+1)$. Let $f_n = zE_nw$. Then, since $\succeq^*$ monotone*, $z \succeq^* f_n \succeq^* x$. Clearly, F_{f_n} converges to F_w and G_{f_n} converges to F_w and hence by continuity $w \succeq^* x$ contradicting the fact that $\succeq^*$ is monotone. A symmetric argument ensures that $u(x) \in (0, 1)$.

Since μ is a probability measure, there exist E such that $\mu(E) = \sup\{\mu(E') \,|\, x \succeq^* zEw\}$. An argument similar to the one in the preceding paragraph ensures that $x \sim^* zEw$. Then, the fact that $\succeq^*$ is monotone implies u is strictly increasing.

Next, we will show that u is continuous. Suppose x_n is a weakly increasing sequence converging to x. Choose E_n so that $u(x_n) = \mu(E_n)$. Then, $\mu(E_n)$ is an increasing sequence and therefore, we can assume without loss of generality, that $E_n \subset E_{n+1}$. Let $E = \bigcup E_n$ and let $y = \inf\{x' \,|\, x' \succeq^* zEw\}$. Hence, $y \geq x_n$ and therefore $y \geq x$. If $y > x$, then for $y' \in (x, y)$ we have E' such that $\mu(E') = u(y')$ and $y' \succeq^* x_n \sim^* zE_nw$ for all n. Hence, by the continuity of $\succeq^*$, $y' \succeq^* zEw$, contradicting the definition of y. So, $y = x$. That is, $u(x) = \lim u(x_n)$. A symmetric argument establishes that $u(x) = \lim u(x_n)$ for any decreasing sequence as well. Hence, u is continuous.

Let $U(f) = u(x)$ such that $x \sim^* f$. The continuity and strict increasingness of u together with the fact that $\succeq^*$ is monotone* ensures that $U(f)$ is well defined. Then, the fact that $\succeq^*$ is monotone* yields $f \sim^* g$ if $F_f = F_g$ and $G_f = G_g$. Define, $V(F_f, G_f) = U(f)$. By the argument above, V is well-defined. To see that V represents $\succeq^*$, note that $f \succeq^* g$ implies $x \succeq^* y$ for x, y such that $u(x) = U(f)$ and $u(y) = U(g)$. Hence, $x \geq y$ and therefore $u(x) \geq u(y)$ and $V(F_f, G_f) \geq V(F_g, G_g)$. Conversely, if $V(F_f, G_f) \geq V(F_g, G_g)$, then $u(x) \geq u(y)$ for some x, y such that $x \sim^* f$ and $y \sim^* g$ and therefore $x \succeq^* y$ and hence $f \succeq^* g$. Monotonicity of V follows from the monotonicity of $\succeq^*$ and the continuity of V follows from the continuity of u. □

7.2 Proof of Proposition 4

For Lemmas C1 and C2 below, fix an uncertainty measure π. Let μ be the risk measure of π and let η be the corresponding ambiguity measure of π.

Lemma C1: *Let $(F^j, G^j) \in \Phi$ for $j = 1, \ldots, 4$. For all $\lambda \in (0, 1)$, there are $f_j \in \mathcal{F}, E \in \mathcal{E}$ such that $\mu(E) = \lambda$, and for all $x \in [w, z]$:*

$$\lambda F^j(x) = \sum_{x' \leq x} \pi(f_j^{-1}(x') \cap E), j = 1, 2$$

$$\lambda G^j(x) = \sum_{x' > x} (\lambda - \pi(f_j^{-1}(x') \cap E)), j = 1, 2$$

$$(1 - \lambda) F^j(x) = \sum_{x' \leq x} \pi(f_j^{-1}(x') \cap E^c), j = 3, 4$$

$$(1 - \lambda) G^j(x) = \sum_{x' > x} (1 - \lambda - \pi(f_j^{-1}(x') \cap E^c)), j = 3, 4$$

Proof: Let (G, G^c) be an essential partition for μ. If $\mu(G^c) = 0$ then $(F, G) \in \Phi$ implies $F = G$ and the lemma follows from the fact that μ is non-atomic. Hence, assume that $\mu(G^c) > 0$. Let $x_1, \ldots, x_n$ be the union of the supports of F^1, F^2, F^3, F^4 (note that F^j and G^j have the same support since $(F^j, G^j) \in \Phi$). Let $\hat{f}_j, j = 1, \ldots 4$ be acts such that $F_{\hat{f}_j} = F^j, G_{\hat{f}_j} = G^j, j = 1, \ldots 4$. Let $\{E^j_\kappa\}$ be a μ-split of $\{\hat{f}_j^{-1}(x_i)\}_{i=1}^n$ and let $\{G_\kappa\}$ be a common refinement of the partitions $\{E^j_\kappa\}, j = 1, \ldots, 4$. By Lyapunov's convexity theorem for vector measures (see, for example, Artstein (1990)) we can partition each G_κ into G^1_κ, G^2_κ such that $\mu(G^1_\kappa) = \lambda\mu(G_\kappa)$ and $\eta(G^1_\kappa) = \lambda\eta(G_\kappa)$. Let $a^j_{i\kappa} = \eta(\hat{f}_j^{-1}(x_i) \cap G_\kappa)$. To construct f_j for $j = 1, 2$, partition G^1_κ into n subsets $\{A^j_{1\kappa}, \ldots, A^j_{n\kappa}\}$, such that $\eta(A^j_i) = \lambda a^j_{i\kappa}$ and $1 > \eta(A^j_{i\kappa}) > 0$ implies $A^j_i \in \mathcal{C}_{G_\kappa}$. Such a partition exists because η is non-atomic. Let $f_j(\omega) = x_i$ for $\omega \in A^j_{i\kappa}$. To construct $f_j, j = 3, 4$, partition G^2_κ into n subsets $\{C^j_{1\kappa}, \ldots, C^j_{n\kappa}\}$, such that $\eta(C^j_i) = (1 - \lambda)a^j_{i\kappa}$ and $1 > a^j_{i\kappa} > 0$ implies $C^j_i \in \mathcal{C}_{G_\kappa}$. Let $f_j(\omega) = x_i$ for $\omega \in C^j_{i\kappa}$. Let $E = \bigcup_\kappa G^1_\kappa$ and let $f_1(\omega) = w = f_2(\omega)$ for $\omega \in E^c$ and $f_3(\omega) = w = f_4(\omega)$ for $\omega \in E$. It is straightforward to verify that $f_j, j = 1, \ldots, 4$ and E satisfy the desired properties. □

Let (G, G^c) be an essential partition for μ.

Lemma C2: *Assume $\mu(G^c) > 0$ and let $F, G \in \mathcal{L} \times \mathcal{L}$. There exists $f \in \mathcal{F}$ and $\lambda \in (0, 1)$ such that $\lambda(F, G) + (1 - \lambda)(F_f, G_f) \in \Phi$.*

Proof: Fix $Y = \{x_1, \ldots, x_n\}$. The act f is a binary partition act for Y if there is a partition $\{F^i_{jk}, A_{jk}, B_{jk}\}, i = 1, 2; j = 1, \ldots, n; k = 1, \ldots, n$ such that for $F^3_{jk} := A_{jk} \cup B_{jk}$ and for all $i = 1, 2, 3$ and all j, k,

$$F^i_{jk} \in \mathcal{E}$$

$$\mu(F^i_{jk}) > 0$$

$$F^2_{jk}, F^3_{jk} \in G^c$$

$$A_{jk}, B_{jk} \subset \mathcal{C}_{F^3_{jk}}$$

and if

$$f^{-1}(x_i) = \bigcup_{k=1}^n F^1_{ik} \cup \bigcup_{k=1}^n F^2_{ki} \cup \bigcup_{k=1}^n A_{ik} \cup \bigcup_{k=1}^n B_{ki}$$

Let $\mathcal{F}_Y$ be the binary partition acts for Y and note that Lyapunov's theorem (see, Artstein (1990)) ensures that $\mathcal{F}_Y$ is non-empty.

Claim: For any $f \in \mathcal{F}_Y$ there exists $\epsilon > 0$ such that $(F, G) \in \Phi$ if F and G have support Y and $|F_{\tilde{f}}(z) - F(z)| \leq \epsilon, |G_{\tilde{f}}(z) - G(z)| \leq \epsilon$ for all $z \in Y$.

Let $z > y$. We say the cdf F differs from the cdf F' by an $\epsilon-$shift from y to z (resp. z to y), if $F(x) = F'(x)$ for $x \leq y$ and for $x \geq z$ and $F(x) = F'(x) + \epsilon$ for $x \in [y, z)$, (resp. $F(x) = F'(x) - \epsilon$ for $x \in [y, z)$). Since Y is finite, the following assertion is sufficient to prove the Claim: For any $f \in F_Y$ there exists $\epsilon > 0$ such that for $\epsilon' \leq \epsilon$ and any $y, z \in Y$:

(i) there is $\tilde{f} \in \mathcal{F}_Y$ such that $G_{\tilde{f}} = G_f$ and $F_{\tilde{f}}$ and $F_{\tilde{f}}$ differ by an ϵ' shift from y to z.

(ii) there is $\tilde{f} \in \mathcal{F}_Y$ such that $F_{\tilde{f}} = F_f$ and $G_{\tilde{f}}$ and $G_{\tilde{f}}$ differ by an ϵ' shift from y to z;

We will prove (i) for $z > y$; an analogous argument proves (i) for $z < y$ and (ii). Let $\{F^i_{jk}, A_{jk}, B_{jk}\}$ be the partition corresponding to $f \in \mathcal{F}_Y$. Let $y = x_r, z = x_t$ and assume $z > y$. Let $\tilde{f} \in \mathcal{F}_Y$ be the act generated by the partition $\{\tilde{F}^i_{jk}, \tilde{A}_{jk}, \tilde{B}_{jk}\}$ where

$$\begin{aligned}
\tilde{F}^1_{jk} &= F^1_{jk}, \forall j, k \\
\tilde{F}^2_{jk} &= F^2_{jk}, \forall j, k \neq (t, r) \\
\tilde{A}_{jk} &= A_{jk}, \forall j, k \neq (t, r) \\
\tilde{B}_{jk} &= B_{jk} \forall j, k \neq (t, r) \\
F^2_{tr} &\subset \tilde{F}^2_{tr} \\
\tilde{F}^3_{tr} &\subset F^3_{tr} \\
F^2_{tr} \cup F^3_{tr} &= \tilde{F}^2_{tr} \cup \tilde{F}^3_{tr} \\
\mu(F^2_{tr}) + \delta &= \mu(\tilde{F}^2_{tr}) \\
\mu(F^3_{tr}) - \delta &= \mu(\tilde{F}^3_{tr}) \\
\eta(B_{tr}) &= \eta(\tilde{B}_{tr}) - \delta
\end{aligned}$$

Since η and μ are non-atomic, $\{\tilde{F}^i_{jk}, \tilde{A}_{jk}, \tilde{B}_{jk}\}$ with these properties exist for all $\delta = \mu(\tilde{F}^2_{tr} \backslash F^2_{tr})$ sufficiently small. Let

$$\epsilon' := \mu(\tilde{F}^2_{tr} \backslash F^2_{tr}) - \eta(\tilde{F}^2_{tr} \backslash F^2_{tr}) = \mu(F^3_{tr} \backslash \tilde{F}^3_{tr}) - \eta(\tilde{F}^2_{tr} \backslash F^2_{tr})$$

Since $\mu(E) > \eta(E)$ for all $E \in \mathcal{E}$ such that $\mu(E) > 0$ it follows that $\epsilon' > 0$ if $\delta > 0$. It is straightforward to verify that $\epsilon', \tilde{f}$ satisfy (i).

To complete the proof of the Lemma, let $(F, G) \in \mathcal{L} \times \mathcal{L}$ and let Y be the union of the supports of F and G. Let $\tilde{f} \in F_Y$. Then, by the claim above $\lambda(F, G) + (1-\lambda)(F_{\tilde{f}}, G_{\tilde{F}}) \in \Phi$ for λ sufficiently small. $\square$

Proof of Proposition 4(i): Let $M = (F, G)$ denote a generic element of $\mathcal{L} \times \mathcal{L}$. By Lemma C1, $M, M' \in \Phi, \lambda \in [0, 1]$ implies $\lambda M + (1 - \lambda)M' \in \Phi$ and hence Φ is a mixture space. By Proposition 3, there exists a continuous and strictly increasing $V : \Phi \to I\!R$ such that $f \succeq^* g$ if and only if $V(M_f) \geq V(M_g)$. The sure thing principle together with Lemma C1 imply that if $(M^j) \in \Phi, j = 1, \ldots 4$ then $V(\lambda M^1 + (1 - \lambda)M^3) \geq V(\lambda M^2 + (1 - \lambda)M^3)$ implies $V(\lambda M^1 + (1 - \lambda)M^4) \geq V(\lambda M^2 + (1 - \lambda)M^4)$. By a standard argument (see, Fishburn (1970), Lemma 14.3) this, together with the continuity of V, yields independence: for all $M, M', M'' \in \Phi$ and $\lambda \in (0, 1)$, $V(M) \geq V(M')$ implies $V(\lambda M + (1 - \lambda)M'') \geq V(\lambda M' + (1 - \lambda)M'')$.

We conclude that the induced preference on Φ satisfies the von Neumann-Morgenstern axioms and therefore has a linear representation. That is, there exists $W : \Phi \times \Phi \to I\!R$ such that $W(\lambda M + (1 - \lambda)M') = \lambda W(M) + (1 - \lambda)W(M')$ and $f \succeq^* g$ if and only if $W(M_f) \geq W(M_g)$.

If $\mu(G^c) = 0$ then $(F, G) \in \Phi$ implies $F = G$ and, by linearity, there exists a function $u : X \to I\!R$ such that $W(F, F) = \int u dF$. Next, assume $\mu(G^c) > 0$ and let $M \in \mathcal{L} \times \mathcal{L}$. Choose $\lambda \in (0, 1), \hat{M} \in \Phi$ such that $\lambda M + (1 - \lambda)\hat{M} \in \Phi$. By Lemma C2 this can be done. Define $W^* : \mathcal{L} \times \mathcal{L} \to I\!R$ as follows:

$$W^*(M) = \frac{1}{\lambda}\Big(W(\lambda M + (1 - \lambda)\hat{M}) - (1 - \lambda)W(\hat{M})\Big)$$

First, note that by linearity $W^*(M) = W(M)$ for $M \in \Phi$. Second, note that W^* is well defined (that is, independent of the choice of λ and $\hat{M}$.) To see this, let $\lambda M + (1-\lambda)\hat{M}_1 \in \Phi$ and $\lambda' M + (1 - \lambda')\hat{M}_2 \in \Phi$ then

$$\lambda'\Big(W(\lambda M + (1-\lambda)\hat{M}_1) - (1-\lambda)W(\hat{M}_1)\Big) = \lambda\Big(W(\lambda' M + (1-\lambda')\hat{M}_2) - (1-\lambda')W(\hat{M}_2)\Big)$$

because W is linear. Next, we observe that W^* is linear. Fix $M_1, M_2 \in \mathcal{L} \times \mathcal{L}$ and $\alpha > 0$. Let $M^\alpha = \alpha M_1 + (1-\alpha)M_2$. Let $\lambda, \hat{M}_i \in \Phi$ be such that $M_i^\lambda := \lambda M_i + (1-\lambda)\hat{M}_i \in \Phi$ for $i = 1, 2$. By Lemma C1, $\hat{M}^\alpha := \alpha\hat{M}_1 + (1-\alpha)\hat{M}_2 \in \Phi$ and $M^{\alpha\lambda} := \alpha M_1^\lambda + (1-\alpha)M_2^\lambda \in \Phi$. Then, by linearity of W,

$$\begin{aligned}
\lambda W^*(M^\alpha) &= W(\lambda(M^\alpha) + (1-\lambda)\hat{M}^\alpha) - (1-\lambda)W(\hat{M}^\alpha) \\
&= \alpha(W(M_1^\lambda) - (1-\lambda)W(\hat{M}_1)) + (1-\alpha)(W(M_2^\lambda) - (1-\lambda)W(\hat{M}_2)) \\
&= \lambda\alpha W^*(M_1) + \lambda(1-\alpha)W^*(M_2))
\end{aligned}$$

Since W^* is linear on $\mathcal{L} \times \mathcal{L}$ it follows (from a standard argument, see Kreps (1988), Proposition 7.4) that there are functions $u_1 : X \to I\!R$ and $u_2 : X \to I\!R$ such that $W^*(F, G) = \int u_1 dF + \int u_2 dF$. Next, we prove that u_1 and u_2 are non-decreasing and continuous. Non-decreasingness follows from the monotonicity of $\succeq^*$. To prove continuity, normalize $u_1(w) = u_2(w) = 0, u_1(z) = 1-\alpha, u_2(z) = \alpha$ for $\alpha \in [0, 1]$. Then, $u_1(x)+u_2(x) = \mu(E)$ such that $x \sim^* zEw$. In the proof of Proposition 3 we have shown that $u = u_1 + u_2$ is continuous. Since u_1 and u_2 are non-decreasing it follows that u_1 and u_1 are continuous. That $u_1 + u_2$ is strictly increasing follows from the monotonicity of $\succeq^*$. □

Proof of Proposition 4(ii): Let (G, G^c) be an essential partition for π. If $\mu(G^c) = 0$ then the result is trivial. Hence, assume $\mu(G^c) = \delta > 0$. Let $A \subset \mathcal{C}_{G^c}$ and let $B = \mathcal{C}_{G^c} \backslash A$. Let u, v be the parameters of a double expected utility representation of $\succeq^*$ and let $x > w$ and normalize $u(w) = v(w) = 0, u(z) = 1-\alpha, v(z) = \alpha$. Let $a = (\mu(G^c) - \eta(A))$ and let $b = \eta(A)$ and note that $a > b$. Then, $zAw \sim zEw$ if $(1-\alpha)a + \alpha b = c$ and, therefore, P4 implies $au(x) + bv(x) = ((1-\alpha)a + \alpha b)(u(x) + v(x))$ for all x. This, in turn, implies that $\alpha u(x) = (1-\alpha)v(x)$ for all $x \in X$. □

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