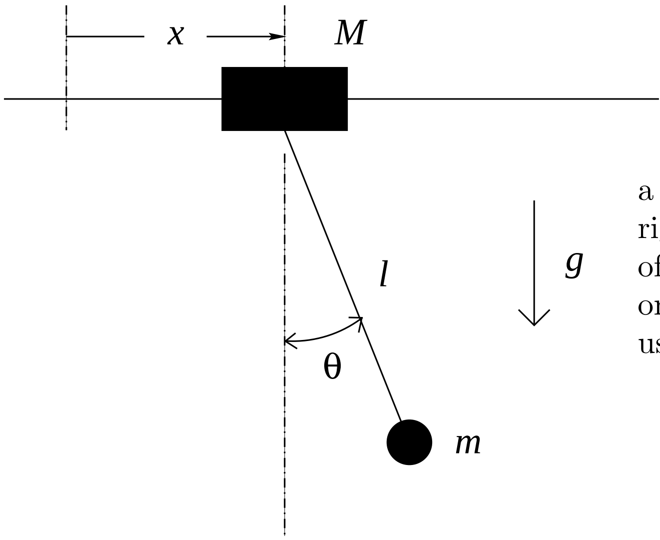
Physics 205—Final Exam Fall 2003
Solutions



1) (40 pts) A plane pendulum consists of a bob of mass m suspended by a massless rigid rod of length l that is hinged to a sled of mass M. The sled slides without friction on a horizontal rail. Gravity acts with the usual downward acceleration g.

a) Taking x and θ as generalized coordinates write the Lagrangian for the system.

Solution: We start by computing the Cartesian coordinates of the bob

$$x_1 = x + l\sin\theta \quad \Longrightarrow \dot{x}_1 = \dot{x} + \dot{\theta} l\cos\theta$$

$$y_1 = y + l\cos\theta \quad \Longrightarrow \dot{y}_1 = \dot{y} + \dot{\theta} l\sin\theta$$

thus

$$T = \frac{m}{2}\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{M}{2}\dot{x}^2 = (m+M)/over2 + \frac{m}{2}\left(2\dot{\theta}\dot{x}l\cos\theta + \dot{\theta}^2 l^2\right)$$

and

$$V = mgl\left(1 - \cos\theta\right)$$

hence

$$L = T - V = \frac{(m+M)}{2} + \frac{m}{2}\left(2\dot{\theta}\dot{x}l\cos\theta + \dot{\theta}^2 l^2\right) + mgl\cos\theta$$

where uninteresting constants have been dropped.

b) Use Lagrange's equations to derive the equations of motion for the system.

Solution: For x we have

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = (m+M)\ddot{x} + ml\ddot{\theta}\cos\theta - m\dot{\theta}^2 l\sin\theta = 0$$

and for θ

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = m\ddot{x}l\cos\theta + m\ddot{\theta}l^2 + mgl\sin\theta = 0$$

c) Use the equations from part (b) to find the frequency ω for *small* oscillations of the bob about the vertical. (Hint: You will need to make some approximations.)

Solution: For small oscillations, θ, x, and their derivatives will be small, so we can neglect terms containing $\dot{\theta}^2$ and $\dot{x}\dot{\theta}$. Also $\cos\theta \simeq 1$ and $\sin\theta \simeq \theta$. With this we obtain

$$(m+M)\ddot{x} + ml\ddot{\theta} \simeq 0 \Longrightarrow \ddot{x} \simeq -\left(\frac{m}{m+M}\right) l\ddot{\theta}$$

and

$$\ddot{x} + \ddot{\theta}l + g\theta \simeq 0 \Longrightarrow \left(1 - \frac{m}{m+M}\right) l\ddot{\theta} \simeq -g\theta$$

yielding

$$\omega_{\text{osc}} \simeq \sqrt{\frac{g}{\left(1\frac{m}{m+M}\right) l}} = \sqrt{\frac{g(m+M)}{lM}}$$

d) At time $t = 0$ the bob and the sled, which had previously been at rest, are set in motion by a sharp tap delivered to the bob. The tap imparts a horizontal impulse $\Delta P = F\Delta t$ to the bob. Find expressions for the values of $\dot{\theta}$ and $\dot{x}$ just after the impulse.

Solution: Here we conserve linear and angular momentum.

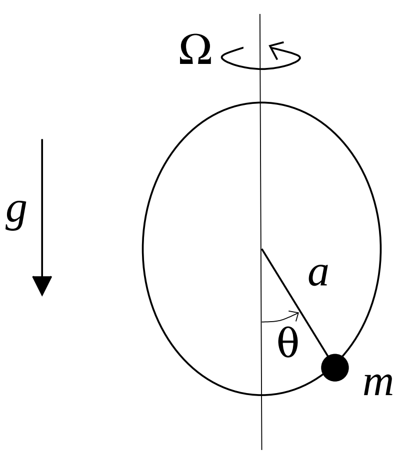
$$\Delta P = \sum \Delta p_i = m(\dot{x} + \dot{\theta}l) + M\dot{x}$$

and

$$\Delta L = \Delta P l = \sum L_i = ml^2\dot{\theta} \Longrightarrow \dot{\theta} = \frac{\Delta P}{ml}$$

Note that since

$$\Delta P = ml\dot{\theta} \Longrightarrow \dot{x} = 0$$



2) (45 pts) A bead of mass m slides without friction on a rotating circular hoop. The hoop is forced to rotate about a vertical axis along its diameter at a constant angular velocity Ω. The position of the bead can be described by the angle θ, which is the angle that a line running between the center of the hoop and the bead makes with the vertical.

a) What is the Lagrangian for the system?

Solution:

$$L = T - V = \frac{ma^2}{2}\left(\dot{\theta}^2 + \Omega^2 \sin^2\theta\right) - mga(1 - \cos\theta)$$

b) What is θ_0, the equilibrium position of the bead (i.e., the value for θ that allows $\dot{\theta} = 0$)?

Solution: Using

$$h = \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L = \text{constant}$$

we get

$$\frac{ma^2\dot{\theta}^2}{2} - ma\left[\frac{a\Omega^2 \sin^2\theta}{2} + g\cos\theta\right] = h$$

or

$$\frac{a^2\dot{\theta}^2}{2} - \left[\frac{a\Omega^2 \sin^2\theta}{2} + g\cos\theta\right] = \frac{h}{ma} \equiv h'$$

where we identify the first term with an effective kinetic energy and the second two terms with an effective potential. Finding

$$\left[\frac{\partial V_{\text{eff}}}{\partial \theta}\right|_{\theta_0} = -\left(a\Omega^2 \cos\theta \sin\theta - g\sin\theta\right) = 0$$

yields

$$\cos\theta_0 = \frac{g}{a\Omega^2}$$

c) What is the frequency for small oscillations about θ_0? You may assume that $g < a\Omega^2$.

Solution: Compute

$$k_{\text{eff}} = \left[\frac{\partial^2 V_{\text{eff}}}{\partial \theta^2}\right|_{\theta_0} = -\left[a\Omega^2\left(\cos^2\theta - \sin^2\theta\right) - g\cos\theta\right|_{\theta_0} = a\Omega^2 \sin^2\theta_0$$

or

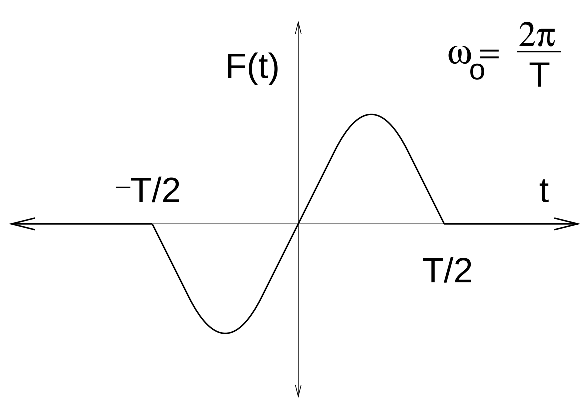
$$\omega_{\text{osc}}^2 = \frac{k_{\text{eff}}}{m_{\text{eff}}} = \frac{a\Omega^2\sin^2\theta_0}{a} = \Omega^2 \sin^2\theta_0$$

or

$$\omega_{\text{osc}} = \Omega \sin\theta_0$$

3) (30 pts) An undamped oscillator having frequency $\omega_0 = 2\pi/T$ is subjected to a driving force given by

$$F(t) = \begin{cases} 0, & \text{if } t < -T/2; \\ F_0 \sin(\omega_0 t), & \text{if } -T/2 < t < T/2; \\ 0, & \text{if } t > T/2. \end{cases}$$



as shown in the sketch. Calculate the displacement of the oscillator for times $t > T/2$.

Solution: Starting with the general Green function solution

$$x(t) = \int_{-\infty}^{t} dt' \frac{F(t')}{m\omega_1} e^{-\gamma(t-t')} \sin\left[\omega_1(t-t')\right]$$

where since there is no damping $\gamma = 0$ and $\omega_1 = \omega_0$. We thus have

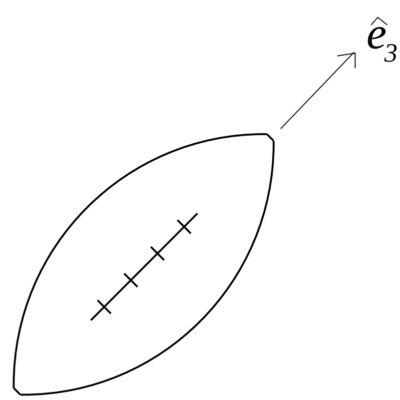
$$x(t) = \int_{-\infty}^{t} dt' \frac{F(t')}{m\omega_0} \sin\left[\omega_1(t-t')\right]$$

since the force is zero prior to $-T/2$ and after $T/2$ and we only want to know $x(t)$ for $t > T/2$, we can write

$$\begin{aligned} x(t) &= \int_{-T/2}^{T/2} dt' \frac{F_0 \sin(\omega_0 t')}{m\omega_0} \sin\left[\omega_0(t-t')\right] \\ &= \frac{F_0}{m\omega_0}\left[\sin\omega_0 t \int_{-T/2}^{T/2} \sin\omega_0 t' \cos\omega_0 t' dt' - \cos\omega_0 t \int_{-T/2}^{T/2} \sin^2\omega_0 t' dt'\right] \end{aligned}$$

where we have used the standard trig. ID for the sine of a difference. The first term in the integral vanishes and the second can be evaluated by noting that the average of $\sin^2\omega t$ over one period is 1/2. This yields

$$x(t) = -\frac{F_0}{m\omega_0}\cos\omega_0 t\left(\frac{T}{2}\right) = -\frac{F_0\pi}{m\omega_0^2}\cos\omega_0 t$$



4) (40 pts) A quarterback in American football throws the ball in such a way that it appears to "wobble" rather than spinning smoothly as it flies. For this problem assume that the football can be treated as an axisymmetric top having $I_1 = I_2 = 2I_3$, where the $\hat{e}_3$ axis is the long axis of the football. Assume further that the football is released with a rapid spin, i.e., $\omega_3 \neq 0$, where ω_3 is the component of the football's angular velocity along the $\hat{e}_3$ body axis.

a) What does the wobbling motion indicate? In particular what does it imply about the other ($\hat{e}_1$ and $\hat{e}_2$) components of ω? (No calculation is required here.)

Solution: This worked out to be a little bit more open ended than I had intended. The answer I was looking for was that it means that the angular velocity was not aligned with a body axis, meaning that it would also not be aligned with the angular momentum, which is a vector fixed in space (conservation of angular momentum with no external torques). I accepted most reasonable variations on this theme.

b) Expressing vectors with respect to the body fixed axes, $\hat{e}_1, \hat{e}_2, \hat{e}_3$, derive three equations that relate the components of $\vec{\omega}$ to one another and to the moments of inertia I_1, I_2, I_3.

Solution: Euler's equation is

$$\left(\frac{d\vec{L}}{dt}\right)_{\text{inertial}} = \left(\frac{d\vec{L}}{dt}\right)_{\text{body}} + \vec{\omega} \times \vec{L} = \vec{\Gamma}_{\text{external}} = 0$$

If we choose body axes that are principal axes then

$$\left(\frac{dL_i}{dt}\right)_{\text{body}} = \left(\frac{d}{dt} I_{ii}\omega_i\right)_{\text{body}} = I_i\dot{\omega}_i$$

combining this with Euler's equation and writing it by components yields

$$I_1\dot{\omega}_1 = \omega_2\omega_3(I_2 - I_3)$$

$$I_2\dot{\omega}_2 = \omega_3\omega_1(I_3 - I_1)$$

$$I_3\dot{\omega}_3 = \omega_1\omega_2(I_1 - I_2)$$

c) If the football is spinning rapidly, then one can assume that $\omega_3 \gg \omega_{1,2}$. Use this assumption to derive an approximate expression for $\vec{\omega}$ as a function of time. Find the period of the wobble in terms of $I_1 = I_2$, I_3, and ω_3.

Solution: Since the football is axisymmetric $I_1 = I_2 = I$ and $\dot{\omega}_3 = 0$. Defining $\Omega \equiv (I_3 - I)/I\omega_3$ yields

$$\dot{\omega}_1 = -\Omega\omega_2 \quad \text{and} \quad \dot{\omega}_2 = +\Omega\omega_1$$

which, when differentiated, can be solved to yield

$$\ddot{\omega}_1 = -\Omega^2\omega_1 \quad \text{and} \quad \ddot{\omega}_2 = -\Omega^2\omega_2$$

Thus both ω_1 and ω_2 vary harmonically with frequency

$$\Omega = \frac{\omega_3}{2}$$

and the period of the wobble is

$$T = \frac{2\pi}{\Omega} = \frac{4\pi}{\omega_3}$$