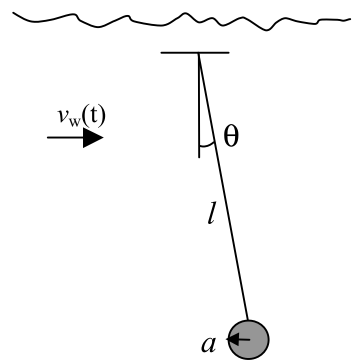
PHYS 203 **Princeton University** **Fall 2004**

Final Exam Solution

1. (20 points) **Underwater pendulum**



A ball of radius a and density ρ_b is suspended under water (density ρ_w) on a massless string of length l. You can assume that $a \ll l$. The ball experiences a drag force in the water $\vec{F}_d = -k(\vec{v}_b - \vec{v}_w)$. Initially the water is at rest (v_w=0).

a) (10 pt) Calculate the frequency of small oscillations and the damping coefficient for the motion of pendulum. Write down the general solution for the motion of the pendulum assuming that it is underdamped.

The torque on the pendulum for small angles is equal to

$$\tau = -(mg - \rho_w V g) l\,\theta - kl(l\dot{\theta} - v_w) = I\ddot{\theta} = ml^2\ddot{\theta}$$

For v_w=0, $\ddot{\theta} + \frac{k}{m}\dot{\theta} + \frac{g}{l}\left(1 - \frac{\rho_w}{\rho_b}\right)\theta = 0$. Let $\beta = \frac{k}{2m}$, $\omega_0^2 = \frac{g}{l}\left(1 - \frac{\rho_w}{\rho_b}\right)$

$$\ddot{\theta} + 2\beta\dot{\theta} + \omega_0^2\theta = 0$$

$$\theta = e^{-\beta t}[A\sin(\omega t) + B\cos(\omega t)],\ \ \omega = \sqrt{\omega_0^2 - \beta^2}$$

b) (10 pt) The water suddenly starts moving to the right with velocity $v_w = v_0$ at $t = 0$. Determine the subsequent motion of the pendulum $\theta(t)$. (Hint: What happens to the equilibrium position of the pendulum?)

Now $\ddot{\theta} + \frac{k}{m}\dot{\theta} + \frac{g}{l}\left(1 - \frac{\rho_w}{\rho_b}\right)\theta = \frac{kv_w}{ml}$

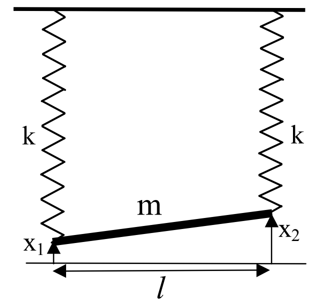
It can be reduced to the previous equation with a substitution $\theta' = \theta + \frac{kv_w}{mg(1 - \rho_w/\rho_b)}$

Starting from rest, the motion of the pendulum is given by

$$\theta(t) = \frac{kv_w}{mg(1 - \rho_w/\rho_b)}[1 - e^{-\beta t}\cos(\omega t)]$$

2. (35 points) **Suspended rod**

A rod of mass *m* and length *l* is suspended from two massless springs with a spring constant *k* as shown in the figure.



a) (10 pt) Write down the Lagrangian for the system in terms of displacements from equilibrium position of the two ends of the rod x_1 and x_2.

$L = \frac{1}{2} m\dot{x}_{cm}{}^2 + \frac{1}{2} I\omega^2 - \frac{1}{2} kx_1{}^2 - \frac{1}{2} kx_2{}^2$. *Note that if x_1 and x_2 are measured from the equilibrium position, the gravity force does not need to be included.*

$$x_{cm} = \frac{x_1 + x_2}{2}, \omega = \dot{\theta} = \frac{\dot{x}_2 - \dot{x}_1}{l}, I = \frac{1}{12} ml^2$$

$$L = \frac{m}{8}(\dot{x}_1{}^2 + 2\dot{x}_1\dot{x}_2 + \dot{x}_2{}^2) + \frac{m}{24}(\dot{x}_1{}^2 - 2\dot{x}_1\dot{x}_2 + \dot{x}_2{}^2) - \frac{1}{2} kx_1{}^2 - \frac{1}{2} kx_2{}^2$$

$$L = \frac{m}{6}(\dot{x}_1{}^2 + \dot{x}_1\dot{x}_2 + \dot{x}_2{}^2) - \frac{1}{2} kx_1{}^2 - \frac{1}{2} kx_2{}^2$$

b) (10 pt) Find the frequencies of the two normal modes and associated eigenvectors.

$$\begin{vmatrix} k - \omega^2 \frac{m}{3} & -\omega^2 \frac{m}{6} \\ -\omega^2 \frac{m}{6} & k - \omega^2 \frac{m}{3} \end{vmatrix} = 0$$

$$k^2 - \frac{2km\omega^2}{3} + \frac{\omega^4 m^2}{9} - \frac{\omega^4 m^2}{36} = 0; \qquad \omega^4 + \frac{8k}{m}\omega^2 - \frac{12k^2}{m^2} = 0$$

$$\omega^2 = \frac{\frac{8k}{m} \pm \sqrt{\frac{64k^2}{m^2} - \frac{48k^2}{m^2}}}{2}; \qquad \omega^2 = \frac{2k}{m}, \omega^2 = \frac{6k}{m}$$

Eigenvectors: $x_1 \frac{k}{3} - x_2 \frac{k}{3} = 0; \quad x_1 = x_2$

$$-x_1 k - x_2 k = 0; \quad x_1 = -x_2$$

c) (5 pt) Using the results from part b) or your intuition, describe the motion for each of the normal modes.

Normal modes correspond to pure translation of the rod and pure rotation about the center of mass.

d) (10 pt) Find the frequencies of the two normal modes using Freshman physics (i.e. no Lagrangians) and verify your results in part b)

Translation of CM: $F = -2kx_{cm} = m\ddot{x}_{cm}; \quad \omega = \sqrt{2k/m}$

Rotation about CM: $\tau = kx_1 \frac{l}{2} - kx_2 \frac{l}{2} = -k\frac{l^2}{4}\theta - k\frac{l^2}{4}\theta = I\ddot{\theta} = \frac{ml^2}{12}\ddot{\theta}; \quad \omega = \sqrt{6k/m}$

3. (30 pts) **Cylindrical hockey player**

Approximate a (not very good) hockey player by a stationary solid cylinder of radius R. A stream of hockey pucks of radius a approach it from the right and scatter elastically. The hockey pucks are uniformly distributed over a length L. The total number of pucks N is very large. A goal net is located a distance d away and has a width w. You can assume that $d>>R$, $d>>w$, $w>>a$.

a) (10 pts) What is a relationship between the impact parameter b and the scattering angle θ of the puck?

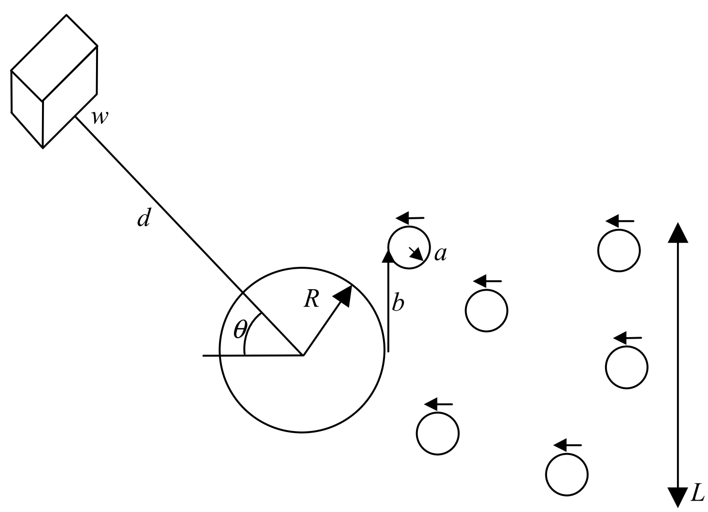
For elastic collisions the angle of incidence α is equal to the angle of reflection

From geometry $\theta = \pi - 2\alpha\,;\sin\alpha = \dfrac{b}{R+a}$

$$\cos\frac{\theta}{2} = \cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha = \frac{b}{R+a}$$

$$b = (R+a)\cos(\theta/2)$$

b) (10 pts) Define a two dimensional scattering cross-section so $(N/L)\sigma(\theta)d\theta$ gives the number of pucks scattered into an angle $d\theta$. What is the differential cross-section $\sigma(\theta)$ in this case?



The number of particles in a range of impact paramenters db that will scatter into angle $d\theta$ is given by:

$$\frac{N}{L}db = \frac{N}{L}\sigma(\theta)d\theta$$

$$\sigma(\theta) = \left|\frac{db}{d\theta}\right| = \frac{(R+a)}{2}\left|\sin\frac{\theta}{2}\right|$$

Note that this result is different from the 3-D equation for the cross-section.

c) (5 pts) Out of the N hockey pucks, how many will end up in the goal net?

$$N_{goal} = \frac{N}{L}\sigma(\theta)d\theta = \frac{N}{L}\frac{w}{\pi d}\frac{(R+a)}{2}\sin\frac{\theta}{2}$$

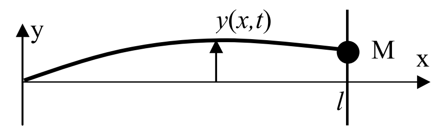
d) (5 pts) Out of the N hockey pucks, now many will strike the player?
(*Note: If you are stuck on parts a or b, you can still answer other parts using simple geometry*)

$$N_{player} = \frac{N}{L}\int_{-\pi}^{\pi}\sigma(\theta)d\theta = \frac{N}{L}\frac{R+a}{2}2\int_{0}^{\pi}\sin\frac{\theta}{2}d\theta = 2\frac{N}{L}(R+a)$$

The same result can be obtained from the fraction of particles with the impact parameters in the range $-(R+a) < b < (R+a)$

4. (30 points) **String and Mass**

A string of mass m and length l with tension τ is attached to a mass M. The shape of the string is described by a function $y(x,t)$. The string is fixed at the origin, $y(0,t)=0$. There is no gravity in this problem.



a) (10 pts) First assume that mass M is held fixed at $y = 0$. Write down the general solution $y(x,t)$ for the standing waves on the string. Express your answer in terms of the given parameters and arbitrary constants.

A standing wave has a general form $y(x,t) = (A\sin kx + B\cos kx)\sin(\omega t + \delta)$

The boundary conditions $y(0,t) = 0$ *and* $y(l,t) = 0$ *impose conditions* $B = 0$ *and* $kl=n\pi$, $n=1,2,3...$

In general for a wave $\omega = kv$ *and for a string* $v = \sqrt{\tau/\rho} = \sqrt{l\tau/m}$

b) (10 pts) Now assume that mass M can slide up and down on a frictionless rod at $x=l$. What is the boundary condition on $y(x,t)$ at $x=l$? You can assume that oscillations are small.

The force needed to accelerate the mass is given by the tension in the string

$$M\frac{\partial^2 y(l,t)}{\partial t^2} = \tau\sin\theta \cong \tau\tan\theta = \tau\left.\frac{\partial y(x,t)}{\partial x}\right|_{x=l}$$

c) (10 pts) Write down an equation for the frequencies of the standing waves on the string when the mass is free to slide. You do not need to solve this equation.

$$M\,\omega^2 A\sin(kl)\sin\omega t = \tau Ak\cos(kl)\sin\omega t$$

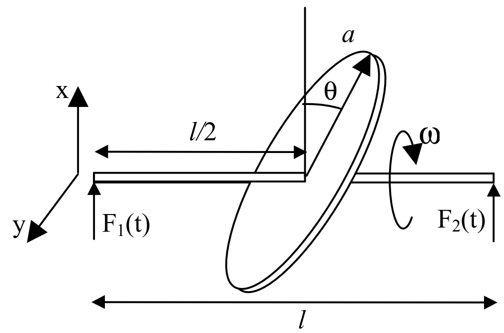
$$Mk^2\frac{l\tau}{m}\sin(kl) = \tau k\cos(kl)$$

$$\cot(kl) = \frac{Mkl}{m}$$

This is an example of a transcendental equation that has to be solved numerically or graphically

5. (35 points) **Rotating Disk**

A disk of mass m and radius a is mounted in the middle of a massless rod of length l. The plane of the disk is tilted by an angle θ away from the normal to the rod. Initially the disk is at rest tilted up, as shown in the figure. At $t = 0$ it begins to rotate around the axis of the rod with an angular velocity ω in the direction shown. The rod is supported at the two ends and forces $F_1(t)$ and $F_2(t)$ are such that it rotates without wobble with a constant angular velocity ω.



a) (5 pts) What are the directions of the principal axes of inertia of the disk?

One axis is perpendicular to the plane of the disk, call it $\hat{e}_3$, the other 2 are orthogonal to each other and lie in the plane of the disk. Take $\hat{e}_1$ to be horizontal out of the page and $\hat{e}_2$ up in the plane of the page before the rotation starts.

b) (10 pts) Calculate the principal moments of inertia of the disk.

$$I_3 = \frac{m}{\pi a^2}\int_0^a 2\pi r^2 r dr = \frac{ma^2}{2};\ I_1 = I_2 = \frac{m}{\pi a^2}\int_0^{2\pi}\int_0^a (r\sin\theta)^2 r dr d\theta = \frac{ma^2}{4\pi}\int_0^{2\pi}\frac{1-\cos 2\theta}{2}d\theta = \frac{ma^2}{4}$$

c) (5 pts) What are the components of the angular velocity ω in the body frame of reference?

$$\omega_3 = \omega\cos\theta, \omega_2 = -\omega\sin\theta,\ \omega_1 = 0$$

d) (10 pts) Determine the torques needed to keep the disk spinning without wobble.

$$N_1 = I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = -\omega^2\sin\theta\cos\theta\frac{ma^2}{4}$$
$$N_2 = I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) = 0$$
$$N_3 = I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) = 0$$

e) (5 pts) Determine the components $F_{1x}(t)$, $F_{1y}(t)$, $F_{2x}(t)$, $F_{2y}(t)$ in the fixed frame of reference that are needed to keep the disk spinning without wobble.

The torques are in the rotating frame and need to be projected back to the lab frame

With the disk oriented as shown: $(F_{2x} - F_{1x})l = N_1; (F_{1y} - F_{2y})l = N_2;$

After rotating by 90 degrees $(F_{1y} - F_{2y})l = N_1; (F_{1x} - F_{2x})l = N_2;$

$$(F_{1x} - F_{2x})l = -N_1\cos\omega t + N_2\sin\omega t = \frac{ma^2\omega^2}{4}\sin\theta\cos\theta\cos\omega t$$

$$(F_{1y} - F_{2y})l = N_2\cos\omega t + N_1\sin\omega t = -\frac{ma^2\omega^2}{4}\sin\theta\cos\theta\sin\omega t$$

$$F_{1x} = \frac{mg}{2} + \frac{ma^2\omega^2}{8l}\sin\theta\cos\theta\cos\omega t, F_{1x} = \frac{mg}{2} - \frac{ma^2\omega^2}{8l}\sin\theta\cos\theta\cos\omega t$$

$$F_{1y} = -\frac{ma^2\omega^2}{8l}\sin\theta\cos\theta\cos\omega t, F_{1y} = \frac{ma^2\omega^2}{8l}\sin\theta\cos\theta\cos\omega t$$