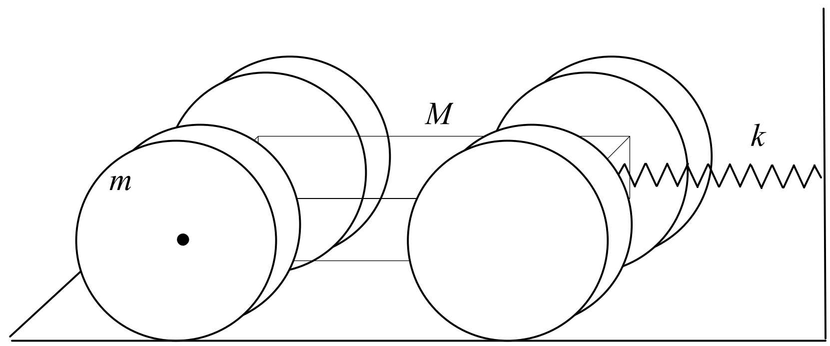
PHYS 203 **Princeton University** **Fall 2005**

Final Exam

1. (30 points) **Toy Car**

A toy car is made from a rectangular block of mass M and four disk wheels of mass m and radius R. The car is attached to a vertical wall by a spring with a spring constant k. The coefficient of static friction between the wheels of the car and the floor is μ. There is no friction in the axles of the wheels.



a) Assuming that the wheels of the car roll without slipping on the floor, calculate the frequency of small oscillations of the car

For no slipping x=Rθ

$$T = \frac{1}{2}(M+4m)\dot{x}^2 + \frac{4I_w}{2R^2}\dot{x}^2 = \frac{1}{2}(M+6m)\dot{x}^2$$

$$U = \frac{1}{2}kx^2$$

$$(M+6m)\ddot{x} + kx = 0, \quad \omega = \sqrt{k/(M+6m)}$$

Now consider the possibility that the wheels can slip on the floor.

b) Write down the Lagrange equations of motion including the equation of constraint.

Keeping x and θ separate now

$$T = \frac{1}{2}(M+4m)\dot{x}^2 + mR^2\dot{\theta}^2$$

$$U = \frac{1}{2}kx^2$$

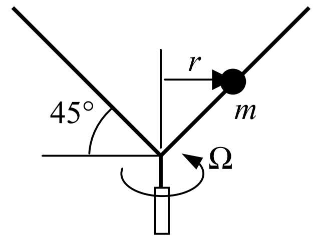
$$f = x - R\theta$$

$$\textit{Lagrange equations}: \begin{cases} -kx-(M+4m)\ddot{x}+\lambda = 0 \\ -2mR^2\ddot{\theta} - R\lambda = 0 \end{cases}$$

c) Find the maximum amplitude of the car oscillation before the wheels begin to slip.

The undetermined multiplier λ *gives the force of friction. From the second equation* $\lambda = -2mR\ddot{\theta} = -2m\ddot{x}$. For harmonic oscillations $x = A\sin\omega t$, $\ddot{x} = -A\omega^2\sin\omega t$. *The maximum* *force of friction is* $F_{\max} = 2Am\omega^2 = \mu N = \mu(M+4m)g$. *Therefore the maximum amplitude is given by* $A = \dfrac{\mu g(M+4m)(M+6m)}{2mk}$

2. (30 points) **Rotating wire and mass**



A Y-shaped massless wire with two arms at 45° is free to rotate around a vertical axis. A small bead of mass *m* is free to slide on one arm of the wire.

a) Write down the Hamiltonian for the system.

Start with 3 degrees of freedom in cylindrical coordinates

$$T = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2, \quad U = mgz$$

Since r = z, can reduce it to two variables:

$$H = m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + mgr$$

b) Identify a conserved momentum

$$\frac{\partial H}{\partial \theta} = -\dot{p}_\theta = 0, \quad p_\theta - const$$

$$p_\theta = \frac{\partial H}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

c) Show that the problem can be reduced to a central force motion problem and determine the effective potential.

$$H = m\dot{r}^2 + \frac{p_\theta^2}{2mr^2} + mgr = \frac{1}{2}\mu\dot{r}^2 + V_{eff}$$

$$V_{eff} = \frac{p_\theta^2}{\mu r^2} + \frac{\mu g r}{2}, \quad \mu = 2m$$

d) Find the equilibrium position of the bead if the wire is rotating with a frequency Ω.

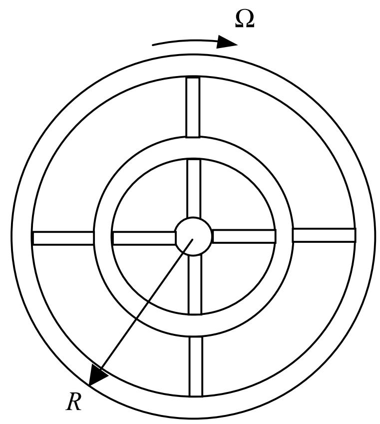
$$\frac{\partial V_{eff}}{\partial r} = -\frac{2p_\theta^2}{\mu r^3} + \frac{\mu g}{2} = 0$$

$$r^3 = \frac{4p_\theta^2}{\mu^2 g} = \frac{r^4\dot{\theta}^2}{g} = \frac{r^4}{g}\Omega^2$$

$$r = \frac{g}{\Omega^2}$$

3. (20 points) **Rotating Space Station**

One commonly proposed solution to alleviate the effects of weightlessness in space is to use a rotating space station.



a) How fast would the space station need to rotate so people on the outer ring of the station with radius R = 100 meters experience normal weight?

For a stationary object on the outer ring of the station the centripetal force is directed radially outward and is given by $F_{cent} = m\Omega^2 R = mg,\ \ \Omega = \sqrt{g/R} = 0.313\,\text{sec}^{-1}$ *corresponding to 1 revolution every 20 seconds*

What unusual effects would you experience if you were

b) Running down the hallway along the outer ring with a velocity of 3 m/sec?

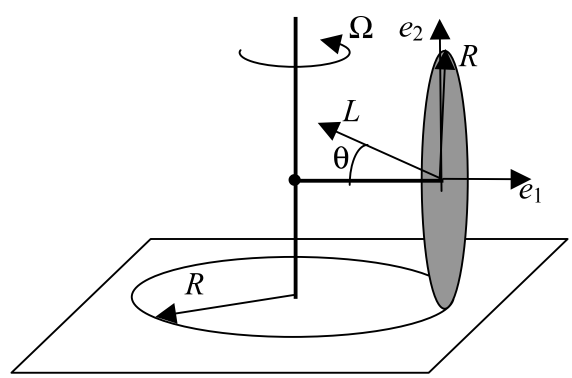
The Coriolis force for someone running along the hallway in the same direction as the spin of the station is directed downward, $F_{cor} = 2mv\Omega = m\ \ 1.9m/s^2$*, thus you would experience an increase in weight by about 20%. The weight would decrease if you run opposite to the rotation of the station.*

c) Riding up in an elevator with a velocity of 1 m/sec?

"Riding up" means toward the center of the station since the fake gravity force is acting radially outward. In this case the Coriolis force is directed sideways, in the direction of the rotation, $F_{cor} = 2mv\Omega = m\ \ 0.63m/s^2$

Calculate the magnitude and direction of the forces.

4. (30 points) **Rolling in circles**



A disk of radius R and mass m is mounted on a massless axle of length R. The other end of the axle is hinged to a stationary post at a height R. The disk rolls around in a circle of radius R without slipping with an angular velocity Ω.

a) What is the direction and magnitude of the angular momentum of the disk as a function of time?

The components of the angular velocity are the same in the lab and rotating frame. Therefore, in the frame of the disk there is a component Ω along the axis of the disk and Ω along the vertical axis in the plane of the disk:

$$\vec{L} = \vec{I} \cdot \vec{\Omega} = \left(\frac{mR^2}{4} \Omega \sin \Omega t, \frac{mR^2}{4} \Omega \cos \Omega t, \frac{mR^2}{2} \Omega \right), |L| = \Omega \frac{mR^2}{4} \sqrt{2^2 + 1} = \Omega \frac{mR^2 \sqrt{5}}{4},$$

$\tan \theta = 1/2$

b) With what force does the disk push down on the floor? You can ignore the effects of gravity.

This can be obtained either in the lab frame or rotating frame.

In the lab frame, the horizontal component of L is rotating with the angular frequency Ω.

$\tau = \frac{d\vec{L}}{dt} = \Omega \frac{\Omega m R^2}{2}$. *The torque is directed in the horizontal plane, perpendicular to the axle and pointing out of the page at the instance shown.*

In the frame of the disk, there is a constant angular velocity Ω along e_3 and also a component Ω rotating in the plane of the disk:

$\omega_1 = \Omega \sin(\Omega t)$

$\omega_2 = \Omega \cos(\Omega t)$

$\omega_3 = \Omega$

The Euler equations then read

$$N_1 = \frac{mR^2}{4} \Omega^2 \cos(\Omega t) + \frac{mR^2}{4} \Omega^2 \cos(\Omega t) = \frac{mR^2}{2} \Omega^2 \cos(\Omega t)$$

$$N_2 = -\frac{mR^2}{4} \Omega^2 \sin(\Omega t) - \frac{mR^2}{4} \Omega^2 \sin(\Omega t) = -\frac{mR^2}{2} \Omega^2 \sin(\Omega t)$$

$$N_3 = 0$$

Going back to the lab frame, the rotating torque components correspond to a torque vector in the horizontal plane, perpendicular to the axle and directed out of the page at the instant shown in the figure, the same as obtained in the lab frame.

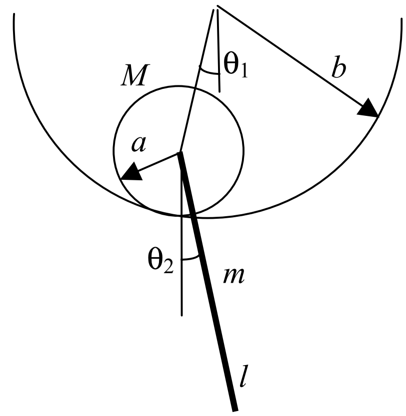
This torque is created by two equal and opposite forces separated by a distance R, one applied up to the disk from the floor and the other down from the central rod to the axle. Therefore, the force with which the disk pushes down on the floor is $F = \frac{mR\Omega^2}{2}$

For your information, the moment of inertia tensor of a disk is given by

$$I = \begin{pmatrix} mR^2/4 & 0 & 0 \\ 0 & mR^2/4 & 0 \\ 0 & 0 & mR^2/2 \end{pmatrix}$$

5. (30 points) Oscillating Contraption

A disk of mass M and radius a rolls without slipping inside a circular rail of radius b. A rod of mass m and length l hangs down from the center of the disk and is free to rotate relative to the disk.



a) Write down the Lagrangian of the system in terms of angles θ_1 and θ_2.

The center of mass of the rod has coordinates:

$$x = (b-a)\sin\theta_1 + \frac{l}{2}\sin\theta_2,$$

$$\dot{x} = (b-a)\cos\theta_1\dot{\theta}_1 + \frac{l}{2}\cos\theta_2\,\dot{\theta}_2 \approx (b-a)\dot{\theta}_1 + \frac{l}{2}\dot{\theta}_2$$

$$y = -(b-a)\cos\theta_1 - \frac{l}{2}\cos\theta_2 \approx -(b-a)\left(1-\frac{\theta_1^2}{2}\right) - \frac{l}{2}\left(1-\frac{\theta_2^2}{2}\right)$$

$$\dot{y} = (b-a)\sin\theta_1\dot{\theta}_1 + \frac{l}{2}\sin\theta_2\,\dot{\theta}_2 \approx 0$$

The disk rolls by an angle $\theta_d = (b/a)\theta_1$,

$$T = \frac{1}{2}M(b-a)^2\dot{\theta}_1^2 + \frac{1}{4}Ma^2\left(\frac{b}{a}\right)^2\dot{\theta}_1^2 + \frac{1}{2}m(b-a)^2\dot{\theta}_1^2 + \frac{1}{8}ml^2\dot{\theta}_2^2 + \frac{1}{2}m(b-a)l\dot{\theta}_1\dot{\theta}_2 + \frac{1}{24}ml^2\dot{\theta}_2^2$$

$$T = \frac{1}{2}(M+m)(b-a)^2\dot{\theta}_1^2 + \frac{1}{4}Mb^2\dot{\theta}_1^2 + \frac{1}{6}ml^2\dot{\theta}_2^2 + \frac{1}{2}m(b-a)l\dot{\theta}_1\dot{\theta}_2$$

$$U = -Mg(b-a)\left(1-\frac{\theta_1^2}{2}\right) - mg(b-a)\left(1-\frac{\theta_1^2}{2}\right) - \frac{mgl}{2}\left(1-\frac{\theta_2^2}{2}\right) = (M+m)g(b-a)\frac{\theta_1^2}{2} + mgl\frac{\theta_2^2}{4} + const$$

b) Determine the matrices A_{ik} and m_{ik} necessary for calculation of the normal frequencies of the system.

We need to identify the terms with the sums $T = \frac{1}{2}\sum_{ik} m_{ik}\dot{\theta}_i\dot{\theta}_k$, $U = \frac{1}{2}\sum_{ik} A_{ik}\theta_i\theta_k$

$$m_{ik} = \begin{pmatrix} (M+m)(b-a)^2 + \frac{1}{2}Mb^2 & \frac{ml(b-a)}{2} \\ \frac{ml(b-a)}{2} & \frac{ml^2}{3} \end{pmatrix}, \quad A_{ik} = \begin{pmatrix} (M+m)g(b-a) & 0 \\ 0 & \frac{mgl}{2} \end{pmatrix}$$

Note that there are two off-diagonal elements of the matrix contributing the same cross term

You do not need to calculate the normal frequencies.