PHY 203: Solutions to Problem Set 2

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1 Laser Beam in Refractive Medium

Here we find the path of a light ray using Fermat's principle. The travel time is

$$T = \int \frac{ds}{v} = \frac{n_0}{c} \int \sqrt{1 + (y')^2}(1 + ky)dx. \tag{1}$$

The first integral ('second form' of the Euler-Lagrange equation) is given by:

$$\mathcal{L} - y' \frac{\partial \mathcal{L}}{\partial y'} \simeq \sqrt{1 + (y')^2}(1 + ky) - y' \frac{y'}{\sqrt{1 + (y')^2}}(1 + ky) = \text{const} = 1, \tag{2}$$

where we have used the boundary conditions to fix the constant. Thus we have the simple differential equation

$$1 + ky = \sqrt{1 + (y')^2}. \tag{3}$$

Separating variables and integrating (using a hyperbolic substitution) we find:

$$y(x) = \frac{2}{k} \sinh^2\left(\frac{kx}{2}\right) = \frac{1}{k}(\cosh(kx) - 1). \tag{4}$$

2 Problem 6.3

Our goal is to show that the shortest distance between two points in three dimensional space is a straight line. The distance between two points that are infinitesimally close is given by $ds = \sqrt{dx^2 + dy^2 + dz^2}$. So the distance between any two points is $S = \int ds = \int \sqrt{dx^2 + dy^2 + dz^2}$. In general a curve in three dimensions is defined by an equation $\vec{r}(t) = (x(t), y(t), z(t))$ where t is some parameter . So if we move along this curve the distance travelled will be $S = \int \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}dt$ where $\dot{x} = \frac{dx}{dt}, \dot{y} = \frac{dy}{dt}, \dot{z} = \frac{dz}{dt}$. Now we have to write down Euler's equation for the three functions x, y, z. We will do it only for x since the equations for y, z are exactly the same with the substitutions $x \leftrightarrow y, x \leftrightarrow z$. For $f = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$ we have

$$\frac{\partial f}{\partial x} = \frac{d}{dt}\frac{\partial f}{\partial \dot{x}}. \tag{5}$$

Since $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}$ the Euler equation is

$$\frac{d}{dt}\frac{dx/dt}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} = 0. \tag{6}$$

If we define a new variable l by $dl = dt\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}$ then our equation becomes

$$\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\frac{d}{dl}\frac{dx}{dl} = 0 \Rightarrow \frac{d^2x}{dl^2} = 0. \tag{7}$$

This equation has a simple solution $x = Al + B$. In exactly the same way we can derive that $y = Cl + D$ and $z = El + F$. These three equations define a line in three dimensional space.

3 Problem 6.14

The surface of the cone given in the problem can be expressed in cylindrical coordinates as $z = 1 - r$. It is the possible to write $dz = -dr$. Therefore the length of a curve in this surface can be written as:

$$L = \int \sqrt{2dr^2 + r^2 d\theta^2} = \int \sqrt{2 + r^2\left(\frac{d\theta}{dr}\right)^2}\,dr = \int \sqrt{r^2 + 2\left(\frac{dr}{d\theta}\right)^2}\,d\theta, \tag{8}$$

depending on which coordinate we use as parameter. Let's take r as the independent variable. Then the Euler equation is:

$$\frac{d}{dr}\left(\frac{r^2\frac{d\theta}{dr}}{\sqrt{2 + r^2\left(\frac{d\theta}{dr}\right)^2}}\right) = 0. \tag{9}$$

This means:

$$\frac{r^2\frac{d\theta}{dr}}{\sqrt{2 + r^2\left(\frac{d\theta}{dr}\right)^2}} = A, \tag{10}$$

where A is an integration constant. This can be rearranged to give:

$$\frac{d\theta}{dr} = \pm\frac{\sqrt{2}A}{r\sqrt{r^2 - A^2}}. \tag{11}$$

We can absorb the sign in the definition of A. After integration we get:

$$\theta = \sqrt{2}\sin^{-1}\left(\frac{A}{r}\right) + B, \tag{12}$$

where B is a new integration constant. Alternatively we can use θ as an independent variable. From (8) we can write the hamiltonian form of Euler equation:

$$\sqrt{r^2 + 2\left(\frac{dr}{d\theta}\right)^2} - \frac{2\left(\frac{dr}{d\theta}\right)^2}{\sqrt{r^2 + 2\left(\frac{dr}{d\theta}\right)^2}} = C. \tag{13}$$

By rearranging we get:

$$\frac{r^4}{C^2} - r^2 = 2\left(\frac{dr}{d\theta}\right)^2. \tag{14}$$

The point where $\frac{dr}{d\theta}$ vanishes is the point of minimal radius, r_{min}. From equations (11) and (14) is easy to see that $A = \sqrt{C} = r_{min}$. We can now substitute our results into (8) and calculate the length of the path:

$$L = \int \sqrt{r^2 + 2\left(\frac{dr}{d\theta}\right)^2} d\theta = \int \frac{r^2}{r_{min}} \left|\frac{d\theta}{dr}\right| dr, \tag{15}$$

where we used (14) and changed variables from θ to r. We know what the jacobian is from (11). Also, we can integrate from r_{min} to 1 and multiply the result by two, as we know the path is symmetric.

$$L = 2\sqrt{2}\int_{r_{min}}^{1} \frac{r}{\sqrt{r^2 - r_{min}^2}} dr. \tag{16}$$

Integrating we get:

$$L = 2\sqrt{2}\sqrt{1 - r_{min}^2}. \tag{17}$$

We just have to find r_{min}. This is done from boundary conditions. If we ask that the curve (12) goes through the points $(1, -\frac{\pi}{2}, 0)$ and $(1, \frac{\pi}{2}, 0)$ we get:

$$B = -\frac{\pi}{\sqrt{2}}, \tag{18}$$

$$A = r_{min} = \cos\frac{\pi}{2\sqrt{2}}. \tag{19}$$

Using this result we find the final answer:

$$L = 2\sqrt{2}\sin\frac{\pi}{2\sqrt{2}}. \tag{20}$$

4 Maximum Area Enclosed by Fixed Perimeter

This problem is worked out in detail in chapter 6 of Thornton and Marion. The solution is of course a circle, with the two integration constants parameterizing the location of its center.