PHY 203: Solutions to Problem Set 1

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1 Firing Shells at Constant Speed

Here we would like to find the set of all points that lie on some trajectory of a shell fired with fixed speed v_0, but at an arbitrary angle α to the horizontal. In particular we would like to find an equation for the surface that can just be reached by such an trajectory (such every point inside the volume bounded by this surface lies on some trajectory, but no point outside this surface does).

First we note that there is rotational symmetry about the z axis. Therefore we have essentially a two-dimensional problem and will choose x as our horizontal coordinate from here on.

The equations of motion are:

$$x(t) = v_0 t \cos\alpha, \tag{1}$$

$$z(t) = v_0 t \sin\alpha - \frac{1}{2} g t^2, \tag{2}$$

Eliminating t we have the trajectory in parametric form:

$$z = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}. \tag{3}$$

The trick now is not to look at one particular trajectory (since we don't a priori know which point on it might touch the required surface), but instead to look at all possible trajectories at once and maximize some appropriate distance over all possible values of α. E.g. we may choose to look at fixed x coordinate and maximize the height z reached at that x:

$$\left.\frac{\partial z}{\partial\alpha}\right|_x = \frac{x}{\cos^2\alpha} - \frac{gx^2}{v_0^2}\frac{\sin\alpha}{\cos^3\alpha} = 0 \qquad \Rightarrow \qquad \tan\alpha = \frac{v_0^2}{gx}. \tag{4}$$

Substituting back this expressions for $\tan\alpha$ we obtain the equation for the "boundary" surface we were looking for:

$$z = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}, \tag{5}$$

which becomes the required expression once we reinstate the y coordinate by replacing $x^2 \to x^2 + y^2$.

Alternatively one might choose to write the surface in polar coordinates as $r(\theta)$ and then maximize r over all α at fixed θ. This gives

$$\alpha = \frac{\pi}{4} + \frac{\theta}{2}, \tag{6}$$

and converting back to Cartesian coordinates leads to the same result.

2 Problem 2.39

The dynamical equation is given by:

$$F = m\dot{v} = -\alpha e^{\beta v}. \tag{7}$$

This can be rearranged and integrated

$$\int_{V_0}^{V} e^{-\beta v} dv = -\int_0^T \frac{\alpha}{m} dt. \tag{8}$$

After integrating,

$$e^{-\beta V} - e^{-\beta V_0} = \frac{\beta\alpha}{m} T. \tag{9}$$

Solving for V yields

$$V = -\frac{1}{\beta} \ln \left[\frac{\beta\alpha}{m} T + e^{-\beta V_0} \right]. \tag{10}$$

In order to find the total time elapsed before the object stops we evaluate the previous expression for $V = 0$. The result is

$$T_F = \frac{m}{\beta\alpha} \left[1 - e^{-\beta V_0} \right]. \tag{11}$$

Now we need to calculate the distance travelled before stoping. This can be done integrating (10):

$$\int_0^{X_F} dx = \int_0^{T_F} -\frac{1}{\beta} \ln \left[\frac{\beta\alpha}{m} T + e^{-\beta V_0} \right] dT. \tag{12}$$

This gives

$$X_F = \frac{m}{\beta^2 \alpha} \left[1 - e^{-\beta V_0} (1 + \beta V_0) \right]. \tag{13}$$

3 Pendulum Hit by a Bullet

The differential equation that governs the motion of a simple pendulum of length l and with mass M is

$$M l^2 \frac{d^2\theta}{dt^2} = -gMl \sin\theta \simeq -gMl\theta, \tag{14}$$

where we have used the small angle approximation. The general solution of this equation can be written

$$\theta(t) = A\sin(\omega t) + B\cos(\omega t), \tag{15}$$

where $\omega \equiv \sqrt{g/l}$. The boundary conditions at the instant the bullet hits are

$$\theta(0) = 0 \qquad \text{and} \qquad \frac{d\theta}{dt}(0) = \frac{mv_0}{Ml}, \tag{16}$$

where the second equality follows from momentum conservation. This forces us to set $B = 0$ and fixes A such that

$$\theta(t) = \frac{mv_0}{Ml\omega}\sin(\omega t). \tag{17}$$

4 Problem 3.7

There are several volumes in this problem. Let's define the ones we will be interested in:

$$\begin{aligned} V &= \text{Volume of fluid displaced at equilibrium} && (18)\\ V_b &= \text{Volume of the body} && (19)\\ V_d &= \text{Volume of fluid displaced at a given time/position} && (20) \end{aligned}$$

We can, then, write the equation of motion for this problem:

$$V_b\rho\ddot{x} = -V_b\rho g + V_d\rho_0 g, \tag{21}$$

where x points upwards and the two terms in the r.h.s. represent gravity and buoyancy. Now, at equilibrium,

$$0 = -V_b\rho g + V\rho_0 g. \tag{22}$$

Then,

$$V_b = \frac{V\rho_0}{\rho}. \tag{23}$$

Also, if we define the origin of coordinates to be at the equilibrium position, we get $V_d = V - xA$. Using this, we can write (21) as:

$$\ddot{x} + \frac{gA}{V}x = 0. \tag{24}$$

This is just the dynamic equation describing a harmonic oscillator. From it we can read the frequency ω to be $\sqrt{\frac{gA}{V}}$. We also know that the period is given by $T = \frac{2\pi}{\omega}$. In the end we get:

$$T = 2\pi\sqrt{\frac{V}{gA}} \simeq 0.18\,\text{s}. \tag{25}$$

5 Suspension Bridge

There are many approaches to this problem. We choose to solve it by writing Newton's equations for a piece of rope with projection dx over the bridge. We define a function $T(x)$ which is the tension along the rope and another $y(x)$ which is just the shape of the rope. Newton's equation in the $\hat{x}$ direction is:

$$-T(x-\frac{dx}{2})\cos\left(\theta(x-\frac{dx}{2})\right)+T(x+\frac{dx}{2})\cos\left(\theta(x+\frac{dx}{2})\right)=0, \tag{26}$$

where $\theta(x)$ is just the angle of the tangent to the rope with respect to the $\hat{x}$ axis at position x. We now expand this in a Taylor series around x to first order and get:

$$\frac{d}{dx}\left(T(x)\cos\theta(x)\right)=0. \tag{27}$$

From this equation we see that $T(x)\cos\theta(x)$ is just a constant we call A. We now need to use Newton's equation for the $\hat{y}$ direction.

$$-\frac{gM}{L}dx-T(x-\frac{dx}{2})\sin\left(\theta(x-\frac{dx}{2})\right)+T(x+\frac{dx}{2})\sin\left(\theta(x+\frac{dx}{2})\right)=0, \tag{28}$$

where $\frac{gM}{L}dx$ is just the weight of the piece of bridge hanging from the piece of rope. M is the total mass of the bridge and L its length. Again, we expand around x, and obtain to first order:

$$\frac{d}{dx}\left(T(x)\sin\theta(x)\right)=\frac{gM}{L}. \tag{29}$$

We can now divide this equation by A. Since it is a constant we can put it inside the derivative in the l.h.s. This yields:

$$\frac{d}{dx}\left(\frac{T(x)\sin\theta(x)}{T(x)\cos\theta(x)}\right)=\frac{gM}{LA}. \tag{30}$$

Now we use that $\frac{dy}{dx}=\tan\theta(x)$. We obtain:

$$\frac{d^2y}{dx^2}=\frac{gM}{LA}. \tag{31}$$

This means that $y(x)=\frac{gM}{2LA}x^2+Bx+C$, where B and C are just constants. B is determined to be zero by fixing the extremes of the rope at, say, $x=-\frac{L}{2}$ and $x=\frac{L}{2}$. C is just related to the origin of coordinates for y, so we can take it to be zero (this amounts to taking the origin at the point where rope lies in the middle of the bridge). Finally, A is determined by knowing the height of the bridge towers with respect to the origin of coordinates. If we take this height to be H,

$$H=\frac{gML}{8A}. \tag{32}$$

The final solution is then

$$y(x)=\frac{4H}{L^2}x^2. \tag{33}$$