# PHY 203: Solutions to Problem Set 10

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## 1 Problem 11.10

This question asks you to consider a body rotating about a *fixed* diameter (i.e. external forces are applied to hold the axis of rotation in place), but with a moment of inertia that changes with time. In particular the inertia tensor consists of an isotropic term representing the sphere and a term arising from the particle that moves on the surface of the sphere

$$\mathbf{I} = \mathbf{I}_s + \mathbf{I}_p. \tag{1}$$

In Cartesian coordinates we have

$$\mathbf{I}_s = \frac{2}{5}MR^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

and

$$\mathbf{I}_p = m \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}, \tag{3}$$

where the position of the particle as a function of time (for $0 < t < T$) is given by

$$x = R\sin\left(\pi\frac{t}{T}\right)\cos(\omega t), \tag{4}$$

$$y = R\sin\left(\pi\frac{t}{T}\right)\sin(\omega t), \tag{5}$$

$$z = R\cos\left(\pi\frac{t}{T}\right). \tag{6}$$

Note that the angular velocity by assumption points in the $z$-direction, but its magnitude varies with time

$$\vec{\omega} = \omega(t)\,\hat{z}. \tag{7}$$

The angular momentum vector precesses about $\vec{\omega}$ and is determined by

$$\vec{L} = \mathbf{I}\cdot\vec{\omega}. \tag{8}$$

The crucial point is that even though $\vec{L}$ varies its $z$-component is a conserved quantity. Substituting for $\mathbf{I}$ and $\vec{\omega}$ from above we have

$$L_z = \left[\frac{2}{5}MR^2 + mR^2 \sin^2\left(\pi \frac{t}{T}\right)\right] \omega(t) = \text{const} = \frac{2}{5}MR^2\omega_0, \tag{9}$$

where in the last equality we have fixed the constant in terms of the initial angular velocity $\omega_0$. Therefore

$$\omega(t) = \frac{\omega_0}{1 + \frac{5m}{2M}\sin^2\left(\pi\frac{t}{T}\right)}. \tag{10}$$

Integrating this expression we find the angle of retardation

$$\alpha = \int_0^T [\omega_0 - \omega(t)]\, dt = \omega_0 T \left(1 - \sqrt{\frac{2M}{2M + 5m}}\right). \tag{11}$$

## 2 Rectangular Plate

This problem is very similar to the previous one. Again the axis of rotation is held fixed and the torque needed to do so is given by $\vec{\tau} = \dot{\vec{L}}$, where $\vec{L}$ is determined by (8). In this case we have

$$\vec{\omega} = \frac{\omega}{\sqrt{a^2 + b^2}} \begin{pmatrix} b \\ a \\ 0 \end{pmatrix}, \tag{12}$$

where $\omega$ is now constant, and

$$\mathbf{I} = \frac{m}{12} \begin{pmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{pmatrix}, \tag{13}$$

where we have aligned the axis with the sides of the rectangle. Then according to (8) we find

$$\vec{L} = \frac{m\omega}{12\sqrt{a^2 + b^2}} \begin{pmatrix} a^2 b \\ b^2 a \\ 0 \end{pmatrix}, \tag{14}$$

and finally the torque is given by

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{\omega} \times \vec{L} = \frac{m\omega^2}{12(a^2 + b^2)}\, ab\,(b^2 - a^2)\,\hat{z}, \tag{15}$$

which points out of the plane of the plate.

## 3 Bicycle Going Around Curve

First let us consider what would happen if the wheels were massless ($m = 0$). In that case the bicycle would still lean into the curve as is easy to see by considering moments about its center of mass. The centripetal force (which equals the frictional force acting on the wheels) has magnitude $Mv^2/R$ and the normal force simply $Mg$. Thus balancing torques requires

$$M\frac{v^2}{R}h\cos\theta = Mgh\sin\theta. \tag{16}$$

Note that if $R$ is the radius at which the wheels touch the ground then what should appear in the denominator on the left hand side is really $R_{eff} = R - h\sin\theta$, but we will assume $R \gg h$ and neglect this correction.

How is this situation modified when $m \neq 0$? The total mass of bike and rider is now $M + 2m$, but more importantly the wheels now have non-zero angular momentum, which rotates as the bicycle rounds the curve and hence there must be a net torque acting on the system. The magnitude of the angular momentum is $L = I\omega = 2ma^2v/a = 2mav$, and it is the radial component $L_r = L\cos\theta$ that rotates with angular frequency $\Omega = v/R$. Thus balancing torques again, we find that (16) is modified to

$$(M+2m)\frac{v^2}{R}h\cos\theta + 2m\frac{av^2}{R}\cos\theta = (M+2m)gh\sin\theta, \tag{17}$$

and thus the bike tilts by an angle

$$\theta = \tan^{-1}\left(\frac{v^2}{Rg}\left(1 + \frac{2m}{M+2m}\frac{a}{h}\right)\right). \tag{18}$$

## 4 String with Density Discontinuity

As so often in this type of question everything boils down to imposing the correct boundary conditions on the solution of the wave equation. A sensible starting point is the ansatz

$$\Psi_1 = A_1\sin(k_1x)\sin(\omega t), \tag{19}$$
$$\Psi_2 = A_2\sin(k_2(L-x))\sin(\omega t), \tag{20}$$

for the left ($0 \leq x < L/2$) and right ($L/2 \leq x \leq L$) half of the string respectively. This is of course not the most general solution of the wave equation. We have picked a single normal mode of frequency $\omega$ and have already imposed some important constraints, namely the Dirichlet boundary conditions at both ends, $\Psi_1(x=0) = 0$ and $\Psi_2(x=L) = 0$.

Note that we know the normal modes of a string with constant density have sinusoidal profile, and our ansatz reflects this. Here, however, we juxtapose two such sine waves with different wavenumbers $k_1$ and $k_2$ for the two regions of different density $\rho_1$ and $\rho_2$, so the combined profile of the normal mode on the interval $[0, L]$ is not a simple sine function. Instead it is made up of two sinusoidal pieces with support on one or the other half of the interval.

Now we simply impose the boundary conditions at the junction. Evidently the string must be continuous $\Psi_1(x = L/2) = \Psi_2(x = L/2)$ and because there is no mass attached at the junction the first derivative must also match $\Psi_1'(x = L/2) = \Psi_2'(x = L/2)$. This leads to the two equations

$$\begin{aligned} A_1 \sin(k_1 L/2) &= A_2 \sin(k_2 L/2), && (21) \\ k_1 A_1 \cos(k_1 L/2) &= -k_2 A_2 \cos(k_2 L/2). && (22) \end{aligned}$$

We can eliminate the amplitudes by dividing the first equation by the second. Noting the the wavenumbers obey

$$\omega = \sqrt{\frac{\tau}{\rho_1}} k_1 = \sqrt{\frac{\tau}{\rho_2}} k_2, \tag{23}$$

we find that the frequencies $\omega$ of the normal modes are given by the solutions of the equation

$$\frac{1}{\sqrt{\rho_1}} \tan\left(\sqrt{\frac{\rho_1}{\tau}} \frac{\omega L}{2}\right) = -\frac{1}{\sqrt{\rho_2}} \tan\left(\sqrt{\frac{\rho_2}{\tau}} \frac{\omega L}{2}\right). \tag{24}$$

As a check on this result note that when $\rho_1 = \rho_2$ we recover the usual set of frequencies $\omega = (n\pi/L)\sqrt{\tau/\rho}$.