PHY 203: Solutions to Problem Set 3

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1 Problem 7.7

Assigning coordinates of the double pendulum in the usual way we have

$$x_1 = l \sin\phi_1 \tag{1}$$

$$y_1 = -l\cos\phi_1 \tag{2}$$

$$x_2 = l(\sin\phi_1 + \sin\phi_2) \tag{3}$$

$$y_2 = -l(\cos\phi_1 + \cos\phi_2). \tag{4}$$

The potential energy is $V = mg(y_1+y_2) = -mgl(2\cos\phi_1 + \cos\phi_2)$. The kinetic energy is $T = \frac{1}{2}m(\dot{x}_1^2+\dot{y}_1^2+\dot{x}_2^2+\dot{y}_2^2)$. Differentiating with respect to time we find that $\dot{x}_1^2+\dot{y}_1^2 = l^2\dot{\phi}_1^2(\cos^2\phi_1+\sin^2\phi_1) = l^2\dot{\phi}_1^2$. Also $\dot{x}_2 = l(\dot{\phi}_1\cos\phi_1 + \dot{\phi}_2\cos\phi_2)$ and $\dot{y}_2 = l(\dot{\phi}_1\sin\phi_1 + \dot{\phi}_2\sin\phi_2)$. After some algebra and using $\cos(a-b) = \cos a\cos b + \sin a \sin b$ we find that $\dot{x}_2^2+\dot{y}_2^2 = l^2(\dot{\phi}_1^2+\dot{\phi}_2^2+2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2)$. Adding up all the pieces our Lagrangian is

$$L = \frac{1}{2}ml^2(2\dot{\phi}_1^2+\dot{\phi}_2^2+2\dot{\phi}_1\dot{\phi}_2\cos(\phi_1-\phi_2)) + mgl(2\cos\phi_1+\cos\phi_2). \tag{5}$$

Now we have to write down the Euler-Lagrange equations for this Lagrangian. First for ϕ_1:

$$\frac{\partial L}{\partial \phi_1} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_1}. \tag{6}$$

$$-ml^2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1-\phi_2) - 2mgl\sin\phi_1 = 2ml^2\ddot{\phi}_1 + ml^2\ddot{\phi}_2\cos(\phi_1-\phi_2) - ml^2\dot{\phi}_2(\dot{\phi}_1-\dot{\phi}_2)\sin(\phi_1-\phi_2). \tag{7}$$

or

$$2\ddot{\phi}_1 + \ddot{\phi}_2\cos(\phi_1-\phi_2) = -2\frac{g}{l}\sin\phi_1 - \dot{\phi}_2^2\sin(\phi_1-\phi_2). \tag{8}$$

The equations for ϕ_2 follow similarly:

$$\frac{\partial L}{\partial \phi_2} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_2}. \tag{9}$$

$$ml^2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1-\phi_2) - mgl\sin\phi_2 = ml^2\ddot{\phi}_2 + ml^2\ddot{\phi}_1 cos(\phi_1-\phi_2) - ml^2\dot{\phi}_1(\dot{\phi}_1-\dot{\phi}_2)\sin(\phi_1-\phi_2). \tag{10}$$

or

$$\ddot{\phi}_2 + \ddot{\phi}_1\cos(\phi_1-\phi_2) = -\frac{g}{l}\sin\phi_2 + \dot{\phi}_1^2\sin(\phi_1-\phi_2). \tag{11}$$

2 Problem 7.9

Let us first choose the origin of coordinates to be on the line where the center of the disc moves. Let's call this line the x-axis and its perpendicular the y-axis. Let's denote the coordinates of the center of the disc as x_d and $y_d = 0$. The coordinates of the pendulum are denoted x_p and y_p. They are related to the ones of the disc by:

$$\begin{aligned} x_p &= x_d + L\sin(\theta+\alpha), \qquad (12)\\ y_p &= -L\cos(\theta+\alpha), \qquad (13)\end{aligned}$$

where θ is the angular degree of freedom of the pendulum measured from the vertical (parallel to the direction of the force of gravity) and α is just the inclination of the plane. L is the length of the rope that holds the pendulum.

The disc rolls down the plane without slipping. That means that the angular velocity of the disc ($\dot{\phi_d}$) is related to $\dot{x_d}$ by $\dot{\phi_d}R = \dot{x_d}$. This is a consequence of the fact that the point on the disc in contact with the plane has zero velocity (no slipping). The only other thing we need to know is that for a disc $I = \frac{1}{2}MR^2$.

With all this information we can write the Lagrangian of the system:

$$\mathcal{L} = \frac{3}{4}M\dot{x_d}^2 + Mgx_d\sin\alpha + \frac{1}{2}m\left(\dot{x_d}^2 + L^2\dot{\theta}^2 + 2\dot{x_d}L\dot{\theta}\cos(\theta+\alpha)\right) + mgx_d\sin\alpha + mgL\cos\theta. \qquad (14)$$

As expected there are only to degrees of freedom: the angle of the pendulum and the x coordinate of the disc. We can now find the equations of motion. For x_d:

$$\left(\frac{3}{2}M + m\right)\ddot{x_d} + mL\ddot{\theta}\cos(\theta+\alpha) - mL\dot{\theta}^2\sin(\theta+\alpha) = (M+m)g\sin\alpha. \qquad (15)$$

For θ we have:

$$mL^2\ddot{\theta} + mL\ddot{x_d}\cos(\theta+\alpha) - mL\dot{x_d}\dot{\theta}\sin(\theta+\alpha) = -mgL\sin\theta - mL\dot{x_d}\dot{\theta}\sin(\theta+\alpha). \qquad (16)$$

3 Problem 7.13

We solve this problem in a similar fashion as the previous one. The Lagrangian for this system is:

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy, \qquad (17)$$

where we picked our origin of coordinates to be some at rest in some inertial frame on the rail.

Since the system is accelerating in x we write:

$$\begin{aligned} x &= \frac{1}{2}at^2 + b\sin\theta, \qquad (18)\\ y &= -b\cos\theta, \qquad (19)\end{aligned}$$

where the origin of coordinates and time was chosen in such a way to make the constant and linear terms in time zero. We denote by a the acceleration of the system in the x axis. We now take derivatives:

$$\begin{aligned} \dot{x} &= at + b\dot{\theta}\cos\theta, & (20) \\ \dot{y} &= b\dot{\theta}\sin\theta. & (21) \end{aligned}$$

Now we plug in our results in (17) and obtain:

$$\mathcal{L} = \frac{1}{2}ma^2t^2 + \frac{1}{2}mb^2\dot{\theta}^2 + mbat\dot{\theta}\cos\theta + mgb\cos\theta. \tag{22}$$

We can drop the first term, which is only function of t. As such it can always be written as the total derivative of some other function of t (its primitive function). Also, we can add to $\mathcal{L}$ another total derivative without affecting the dynamics. If we add $\frac{df}{dt} = \frac{d}{dt}\left(-mbat\sin\theta\right)$, we get:

$$\mathcal{L} = \frac{1}{2}mb^2\dot{\theta}^2 + mgb\cos\theta + mbat\dot{\theta}\cos\theta - mbat\dot{\theta}\cos\theta - mba\sin\theta \tag{23}$$

where the last two terms correspond to $\frac{df}{dt}$. We see that the third and fourth terms cancel out. Also, we can use trigonometric identities to combine the second and fifth term:

$$\mathcal{L} = \frac{1}{2}mb^2\dot{\theta}^2 + mb\sqrt{g^2+a^2}\cos\left(\theta + \arctan\frac{a}{g}\right). \tag{24}$$

From here it is easy to see that this is the Lagrangian of a pendulum with *effective gravity* $g_{eff} = \sqrt{g^2+a^2}$ and equilibrium angle $\theta_e = -\arctan\frac{a}{g}$. As such, the frequency is just $\omega = \sqrt{\frac{\sqrt{g^2+a^2}}{b}}$. Essentially we have added the weight and the inertial force vectorially and treated the result as the *effective gravitational force*.

4 Problem 7.20

The hoop can rotate and move up and down. If we call θ the angle of rotation and z the height we have $z^2 \simeq l^2 - R^2\theta^2$ using the small angle approximation. The Lagrangian is

$$L = \frac{1}{2}MR^2\dot{\theta}^2 + \frac{1}{2}M\dot{z}^2 + Mgz. \tag{25}$$

Substituting for z and using again the small θ approximation we get

$$L = \frac{1}{2}MR^2\dot{\theta}^2 + Mgl - \frac{1}{2}Mg\frac{R^2}{l}\theta. \tag{26}$$

From this we find the equation of motion

$$MR^2\ddot{\theta} = -MR^2\frac{g}{l}\theta, \tag{27}$$

with frequency $\omega = \sqrt{\frac{g}{l}}$.

5 Problem 7.34

The coordinates of mass m are given by

$$x_m = x + r\sin\theta, \qquad y_m = -r\cos\theta, \tag{28}$$

where x measures the displacement of M. The Lagrangian including the constraint $r = R$ is given by

$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x_m}^2 + \dot{y_m}^2) - mgy_m + \lambda(r - R), \tag{29}$$

which implies

$$L = \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + 2\dot{x}\dot{r}\sin\theta + 2\dot{x}r\dot{\theta}\cos\theta) + mgr\cos\theta + \lambda(r - R). \tag{30}$$

The Euler-Lagrange equations that follow from this are

$$(m+M)\ddot{x} + mR\ddot{\theta}\cos\theta - mR\dot{\theta}^2\sin\theta = 0, \tag{31}$$

$$mR^2\ddot{\theta} + mR\ddot{x}\cos\theta = -mgR\sin\theta, \tag{32}$$

$$m\ddot{x}\sin\theta = mR\dot{\theta}^2 + mg\cos\theta + \lambda, \tag{33}$$

for x, θ, r respectively. Note we have substituted $r = R$ *after* varying the Lagrangian.

In order to find the reaction force we have to find $\ddot{x}$ and $\dot{\theta}^2$ in terms of θ. Solving the first two equations for $\ddot{x}$ we get

$$\ddot{x} = \frac{mR^2\dot{\theta}^2\sin\theta + mg\sin\theta\cos\theta}{m + M - m\cos^2\theta}. \tag{34}$$

Thus we only need $\dot{\theta}^2$. We have from conservation of momentum

$$M\dot{x} + m\dot{x_m} = 0 \Rightarrow \dot{x} = -\alpha R\dot{\theta}\cos\theta, \tag{35}$$

where $\alpha = m/(m+M)$. Conservation of energy gives

$$\frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{2}mR^2\dot{\theta}^2 + mR\dot{x}\dot{\theta}\cos\theta - mgR\cos\theta = -mgR\cos\theta_o, \tag{36}$$

Plugging in for $\dot{x}$ we get

$$R\dot{\theta}^2 = \frac{2g(\cos\theta - \cos\theta_o)}{1 - \alpha\cos^2\theta}. \tag{37}$$

Substituting in the expression for λ we find the constraint force

$$\lambda = Mg\alpha\frac{3\cos\theta - 2\cos\theta_o - \alpha\cos^3\theta}{(1 - \alpha\cos^2\theta)^2}. \tag{38}$$