

PHY 203: Solutions to Problem Set 5

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1 Pendulum on Sliding Block

If X is the horizontal displacement of the block the position of the pendulum bob is given by $(x, y) = (X + l \sin \theta, -l \cos \theta)$. The Lagrangian for the system is therefore

$$L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{X}^2 + 2l\dot{X}\dot{\theta}\cos\theta + l^2\dot{\theta}^2) + mgl\cos\theta. \quad (1)$$

The equations of motion that follow from this are

$$M\ddot{X} + m\ddot{X} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = 0, \quad (2)$$

$$ml\ddot{X}\cos\theta + ml^2\ddot{\theta} + mgl\sin\theta = 0. \quad (3)$$

From momentum conservation, or else by integrating the first of these equations we have

$$\dot{X} = -\frac{m}{M+m}l\dot{\theta}\cos\theta. \quad (4)$$

Now substituting for \dot{X} and \ddot{X} in the second equation of motion we find

$$\ddot{\theta}\left(1 - \frac{m}{M+m}\cos^2\theta\right) + \frac{m}{M+m}\dot{\theta}^2\sin\theta\cos\theta + \frac{g}{l}\sin\theta = 0. \quad (5)$$

Using the small angle approximation the second term is negligible and the equation reduces to that of a simple harmonic oscillator with frequency

$$\omega = \sqrt{\frac{g(M+m)}{lM}}. \quad (6)$$

This is slightly larger than it would be if the block were clamped down. This can be understood by considering that the effective length of the pendulum is reduced if the block is allowed to slide back and forth in antiphase with the pendulum.

2 Cosmic String

First we have to find the force on a particle of mass M in the vicinity of the cosmic string. This can be done by integrating over all mass elements λdl of the string

$$F = -GM\lambda \int_{-\infty}^{\infty} \frac{\cos \theta dl}{r^2 + l^2} = -GM\lambda \int_{-\infty}^{\infty} \frac{r dl}{(r^2 + l^2)^{3/2}} = -\frac{2GM\lambda}{r}, \quad (7)$$

or alternatively, because of the high symmetry of the problem one can simply use Gauss' Law to obtain the same result. Note that we essentially have a two-dimensional problem with rotational symmetry in the (x, y) -plane. Thus r above is the radius in *plane* polar coordinates and the component of angular momentum in the z -direction (parallel to the string) is conserved.

The effective potential is simply the (negative) integral of this force plus the usual centripetal term:

$$V_{eff}(r) = 2GM\lambda \ln(r) + \frac{L^2}{2Mr^2}. \quad (8)$$

Note that we have arbitrarily fixed the zero of the logarithmic potential at $r = 1$ in some units. This is not physically significant, since we can always add constant to the potential without affecting the dynamics.

Equating the derivative of the effective potential to zero we find the radius of circular orbits:

$$r_c = \sqrt{\frac{L^2}{2GM^2\lambda}}. \quad (9)$$

To find the frequency of small oscillations we compute the second derivative of the effective potential

$$V''_{eff} = -\frac{2GM\lambda}{r^2} + \frac{3L^2}{Mr^4}, \quad (10)$$

and evaluating it at $r = r_c$ we find the frequency of small oscillations around the circular orbit:

$$\omega_{osc}^2 = \frac{V''_{eff}|_{r_c}}{M} = \frac{8G^2M^2\lambda^2}{L^2} = \frac{4G\lambda}{r_c^2}. \quad (11)$$

3 Problem 8.32

In order to investigate the stability of circular orbits we need to find the sign of the second derivative of the effective potential $V(r)$ at its critical points (where circular orbits are located). If it is positive, then orbits are stable (local minimum of the potential); if it is negative they are unstable (local maximum of the potential).

The effective potential is just:

$$V(r) = U(r) + \frac{L^2}{2\mu r^2}, \quad (12)$$

where $U(r)$ is the potential related to the force through $U'(r) = -F(r)$. Here L is the angular momentum and μ the reduced mass. We find critical points of $V(r)$ wherever $V'(r) = 0$.

$$V'(r) = -F(r) - \frac{L^2}{\mu r^3} = \frac{k}{r^2} e^{-\frac{r}{a}} - \frac{L^2}{\mu r^3} = 0. \quad (13)$$

This means that the condition for a critical point is:

$$k e^{-\frac{r}{a}} = \frac{L^2}{\mu r}. \quad (14)$$

One way to look at this equation is to think that it determines the value of the angular momentum for a circular orbit of radius r . It is clear that the l.h.s. of this equation is always positive, so there is always a solution and we can always find an L for which there is a circular orbit. However, two different values of r could lead to the same L .

Now we calculate the second derivative of $V(r)$:

$$V''(r) = -\frac{k e^{-\frac{r}{a}}}{r^2} \left(\frac{2}{r} + \frac{1}{a} \right) + \frac{3L^2}{\mu r^4}. \quad (15)$$

We substitute (14) into (15) and rearrange to find:

$$V''(r) = \frac{L^2}{\mu r^3} \left(\frac{1}{r} - \frac{1}{a} \right). \quad (16)$$

This means that $V''(r)$ will be positive for $r < a$ and negative for $r > a$. Therefore, orbits are stable in the first case and unstable in the second.

Furthermore we can ask what happens at $r = a$. To answer this questions we have to study higher derivatives. If we calculate $V'''(r)$ and evaluate it at the critical point for $r = a$ we get:

$$V'''(a) = -\frac{L^2}{\mu a^5} < 0. \quad (17)$$

Since the third derivative is different from zero, this means that the critical point in question is neither a maximum nor a minimum. It is a saddle point. Since the third derivative is negative (i.e. the function looks locally like $-r^3$) circular orbits are unstable only towards escaping the gravitational field. They can't move to a lower bounded orbit. For $r > a$ orbits are also unstable but, once perturbed, they can either fall into a lower bounded orbit or escape towards infinity.

4 Problem 8.35

This problem refers us to equation (8.89) in the book. We will, however, solve it from scratch. The basic quantity we want to calculate is the ratio of frequencies of the circular unperturbed motion and the harmonic oscillations around the minimum of the effective potential on top of it. This will tell us how many periods of the perturbation (for small oscillations) can fit inside an orbit.

We first calculate the frequency of oscillations, ω_{osc} . In order to do this, we Taylor expand the effective potential around its minimum:

$$\begin{aligned} V = U(r) + \frac{L^2}{2\mu r^2} &= -\frac{k}{(n-1)r^{n-1}} + \frac{L^2}{2\mu r^2} \simeq V(r_c) + \frac{1}{2} \frac{d^2 V}{dr^2} \Big|_{r_c} (r - r_c)^2 \\ &= V(r_c) + \frac{1}{2} \left(3 \frac{L^2}{\mu} r_c^{-4} - n k r_c^{-n-1} \right) (r - r_c)^2, \end{aligned} \quad (18)$$

where r_c is the radius of the circular orbit. Note that we did not include a first order term in the expansion. This is correct because the first derivative of the potential should vanish at r_c , since it is a critical point. The first order condition that fixes r_c is:

$$k r_c^{3-n} = \frac{L^2}{\mu}. \quad (19)$$

We can plug this result into (18) and drop the constant term $V(r_c)$ as we know that it does not affect the dynamics. We find

$$V(r) \simeq \frac{1}{2} \frac{L^2}{\mu r_c^4} (3-n) (r - r_c)^2. \quad (20)$$

We can now read off the frequency from this result. The potential of a spring with spring constant \tilde{k} is just $\frac{1}{2} \tilde{k} (x - x_0)^2$ and its frequency is $\sqrt{\tilde{k}/m}$, where m is the mass involved in the problem. In our case the mass is just μ and thus

$$\omega_{osc} = \sqrt{\frac{L^2}{\mu^2 r_c^4} (3-n)}. \quad (21)$$

Now we have to find the angular frequency of orbital motion. Since this is uniform circular motion this is not difficult. Angular momentum is conserved and related to the frequency by:

$$L = \mu r_c^2 \omega_{circ}, \quad (22)$$

where ω_{circ} is just the same as $\dot{\theta}$. We are now able to calculate the ratio of periods, i.e. the number of times we can fit a period of the radial oscillations inside the circular orbit:

$$N = \frac{\omega_{osc}}{\omega_{circ}} = \frac{\sqrt{\frac{L^2}{\mu^2 r_c^4} (3-n)}}{\frac{L}{\mu r_c^2}} = \sqrt{3-n}. \quad (23)$$

This allows us to calculate the angle between consecutive maximums of the oscillator as $2\pi/\sqrt{3-n}$. The apsidal angle is the angle between the maximal and minimal distance to the origin, which is

$$\alpha = \frac{\pi}{\sqrt{3-n}}. \quad (24)$$

In order to get closed orbits we need this number to be a rational fraction of 2π . It is easy to see that if we restrict ourselves to $n > -6$, this only happens for $n = -1$ (the harmonic oscillator) and $n = 2$ (Newton's potential).

5 Problem 8.41

To solve this problem we will first calculate the velocities of particles in circular and elliptic orbits. The energy is in both cases:

$$E = \frac{1}{2}mv^2 - \frac{k}{r}. \quad (25)$$

For a circular orbit, r is related to angular momentum by the condition that the particle be at the minimum of the effective potential. This lead to

$$r_c = \frac{L^2}{mk}. \quad (26)$$

We also know that the angular momentum is just $L = mvr_c$. Plugging these results in (25) yields:

$$E_c = -\frac{k}{2r_c}. \quad (27)$$

If we equate this result to (25) we get:

$$-\frac{k}{2r_c} = \frac{1}{2}mv_c^2 - \frac{k}{r_c} \Rightarrow v_c = \sqrt{\frac{k}{mr_c}}. \quad (28)$$

Now let's do the same for the ellipse. In this case we use the fact that the effective potential accounts for all the energy at r_{max} and r_{min} when there is zero velocity in the r -direction. In this case:

$$E_e = -\frac{k}{r_{min}} + \frac{L^2}{2mr_{min}^2} = -\frac{k}{r_{max}} + \frac{L^2}{2mr_{max}^2}. \quad (29)$$

We can treat this equation as a quadratic equation for $\frac{1}{r_{min}}$ and solve. One solution is that $r_{max} = r_{min}$. This represents the circular orbit. The other solution is:

$$\frac{1}{r_{min}} = \frac{2km}{L^2} - \frac{1}{r_{max}}. \quad (30)$$

Now calculate the energy as

$$E_e = \frac{1}{2} \left(-\frac{k}{r_{min}} + \frac{L^2}{2mr_{min}^2} \right) + \frac{1}{2} \left(-\frac{k}{r_{max}} + \frac{L^2}{2mr_{max}^2} \right). \quad (31)$$

We can do this because both terms inside parenthesis are equal, so averaging them gives the same number. Plugging in our result (30) we get after some algebra:

$$E_e = -\frac{k}{r_{max} + r_{min}}. \quad (32)$$

Note that this coincides with the solution for the circle when $r_{max} = r_{min}$. We proceed as before and equate this to (25). For the velocity at r_{min} we find

$$v_{r_{min}} = \sqrt{\frac{2kr_{max}}{mr_{min}(r_{max} + r_{min})}}. \quad (33)$$

At r_{max} the velocity is

$$v_{r_{max}} = \sqrt{\frac{2kr_{min}}{mr_{max}(r_{min} + r_{max})}}. \quad (34)$$

We can now calculate the Δv needed to go from a lower orbit to a higher one. It is just the sum of the two changes in velocity (the first one takes us from the lower circular orbit to the elliptical one and the second one from the elliptical one to the higher circular one). Therefore

$$\Delta v = (v_{er_1} - v_{cr_1}) + (v_{cr_2} - v_{er_2}). \quad (35)$$

From our formulas this is

$$\Delta v = \sqrt{\frac{2kr_2}{mr_1(r_1 + r_2)}} - \sqrt{\frac{2kr_1}{mr_2(r_1 + r_2)}} + \sqrt{\frac{k}{mr_2}} - \sqrt{\frac{k}{mr_1}}. \quad (36)$$

What about the time it takes to switch orbits? The total time is just half the period of the elliptical orbit (as we ride this orbit for half a cycle). We can calculate the period from Kepler's third law:

$$t = \frac{T}{2} = \pi \sqrt{\frac{m}{k}} \left(\frac{r_1 + r_2}{2} \right)^{\frac{3}{2}}. \quad (37)$$

Let's evaluate these quantities for the given problem. We have

$$r_1 = R_{Earth} + 200km = 6.58 \times 10^6 m, \quad (38)$$

$$r_2 = R_{Earth \rightarrow Moon} = 3.84 \times 10^8 m, \quad (39)$$

$$\frac{k}{m} = GM_{Earth} = 3.98 \times 10^{14} \frac{m^3}{s^2}, \quad (40)$$

where R_{Earth} is the radius of the Earth, $R_{Earth \rightarrow Moon}$ is the distance from the Earth to the Moon and M_{Earth} is the mass of the Earth. If we plug these values into the formulas above we find

$$\Delta v = 3.96 \times 10^3 \frac{m}{s}, \quad (41)$$

$$t = 4.97 \text{ days}. \quad (42)$$