PHY 203: Solutions to Problem Set 6

November 22, 2006

1 Problem 9.50

In this problem we need to calculate the differential cross section for a radial force field of magnitude $F = \frac{k}{r^3}$. In order to do this, we first need to obtain the deflection angle as a function of the constants of the motion. We are going to use conservation of energy and angular momentum:

$$L = mr^2 \frac{d\theta}{dt}, \tag{1}$$

$$E = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + \frac{k}{2r^2} + \frac{L^2}{2mr^2}, \tag{2}$$

where the second term in the energy is just the potential associated with the given force and the third term is the usual centrifugal barrier. From the equations above we can isolate dt and equate the results, eliminating time dependence.

$$dt = \frac{mr^2}{L} d\theta = \frac{\sqrt{m}\, dr}{\sqrt{2E - (k + \frac{L^2}{m}) r^{-2}}}. \tag{3}$$

From the reflection symmetry of the orbit around r_{min} we know that the deflection angles is $\alpha = \pi - 2\theta$, where θ is just the angle swept by the motion from r_{min} to $r = \infty$. Therefore:

$$\alpha = \pi - \int d\theta = \pi - \int_{r_{min}}^{\infty} \frac{L\, dr}{r\sqrt{2mEr^2 - (mk + L^2)}}, \tag{4}$$

where we changed variables from θ to r using (3). We can perform the integral and, after evaluating in the limits, we get:

$$\alpha = \pi - \frac{L}{\sqrt{mk + L^2}} \sin^{-1}\left(\frac{\sqrt{mk + L^2}}{r_{min}\sqrt{2mE}}\right). \tag{5}$$

We now have to calculate r_{min}. This is the point where there is no kinetic energy in the r-direction; all energy is potential and centrifugal. That is:

$$E = \frac{k}{2r_{min}^2} + \frac{L^2}{2mr_{min}^2} \rightarrow r_{min} = \sqrt{\frac{mk + L^2}{2mE}}. \tag{6}$$

Plugging this result into (5) yields:

$$\alpha = \pi \left(1 - \frac{L}{\sqrt{mk + L^2}}\right), \tag{7}$$

where we used $\sin^{-1}(1) = \frac{\pi}{2}$. It is possible to write L as a function of the initial velocity v_0 and the impact parameter b as $L = mv_0 b$. Using this result in (7) and solving for b we get:

$$b = \sqrt{\frac{k}{mv_0^2}} \frac{\pi - \alpha}{\sqrt{2\pi - \alpha}\sqrt{\alpha}}. \tag{8}$$

In 3 dimensions and when we have axial symmetry, the differential cross section is given by:

$$\sigma = \frac{b}{\sin(\alpha)} \left|\frac{db}{d\alpha}\right|, \tag{9}$$

(this can be obtained from matching the number of incoming and outcoming particles). We can use (8) in (9) to obtain our final result:

$$\sigma = \frac{k\pi^2(\pi - \alpha)}{mv_0^2\alpha^2(2\pi - \alpha)^2 \sin\alpha}. \tag{10}$$

2 The Hockey Player

This problem is very similar to Example 9.11 in the book. The only real subtlety is that our problem is 2-dimensional, while the one in the book is 3-dimensional. The only effect this has is in the formula for the cross section. By matching the number of outcoming and incoming particles we get:

$$\sigma = \left|\frac{db}{d\theta}\right|, \tag{11}$$

(note that we do not have the usual factor $\frac{b}{\sin\theta}$ in this case). This is only because incoming particles do not come in concentric rings of width db but in usual segments of the same width; the same reasoning applies for outcoming particles). Having said this, the only thing we have to do is to calculate b as a function of θ. This can easily be done by looking at figure 9-23 in the book. The result is geometrical:

$$b = (R + a)\cos\frac{\theta}{2}, \tag{12}$$

where R is the radius of the hockey player and a is the radius of the puck. Using this result we get:

$$\sigma = \frac{R + a}{2}\left|\sin\frac{\theta}{2}\right|. \tag{13}$$

Note that in this case this is not a constant like the 3-D case. To obtain the number of pucks ($\mathcal{N}$) that go inside the goal net we write this as:

$$\mathcal{N} = \frac{N}{L}\sigma d\theta, \tag{14}$$

(this formula is valid for all angles as long as $L > R + a$ so the entire hockey player is *bombarded* with pucks, otherwise $\mathcal{N}$ could be zero for some angles). Since d is much larger that all other distances involved we can approximate $d\theta$ as $d\theta = \frac{w}{d}$ (this is just the definition of radians). This gives the final result:

$$\mathcal{N} = \frac{N}{L}\frac{R+a}{2}\left|\sin\frac{\theta}{2}\right|\frac{w}{d}. \tag{15}$$

3 Problem 12.7

Choosing as coordinates the distances from the equilibrium for the two masses x and y, we find that the kinetic energy is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2. \tag{16}$$

The potential energy is given by

$$V = \frac{1}{2}kx^2 + \frac{1}{2}k(x-y)^2 = \frac{1}{2}(2x^2 - 2xy + y^2). \tag{17}$$

Thus the corresponding matrices $m_{ij} = \frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}$ and $A_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j}$ are

$$\mathbf{m} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \tag{18}$$

and

$$\mathbf{A} = \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix}. \tag{19}$$

The eigenfrequencies are given by

$$\det(\mathbf{m}\omega^2 - \mathbf{A}) = 0 \Rightarrow (m\omega^2)^2 - 3km\omega^2 + k^2 = 0, \tag{20}$$

with solutions

$$\omega^2 = \frac{3 \pm \sqrt{5}}{2}\frac{k}{m}. \tag{21}$$

The eigenvectors are easily found from the equation

$$(\mathbf{m}\omega^2 - \mathbf{A})\vec{u} = 0. \tag{22}$$

This gives the (unnormalized) eigenvectors

$$\vec{u}_{1,2} = \begin{pmatrix} 1 \\ \frac{1 \mp \sqrt{5}}{2} \end{pmatrix}. \tag{23}$$

4 Problem 12.16

The kinetic energy of the hoop is just $T_{hoop} = \frac{1}{2}IR^2\dot{\theta}^2$ with $I = 2MR^2$ from the parallel axis theorem. The kinetic energy of the mass is $T_{mass} = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2)$ where

$$x = R(\sin\theta + \sin\phi), \qquad y = R(\cos\theta + \cos\phi). \tag{24}$$

The sum of the two (expanding around $\theta = 0, \phi = 0$) is thus

$$T = \frac{1}{2}MR^2(3\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2). \tag{25}$$

The potential energy is simply $V = MgR(\frac{1}{2}\phi^2 + \theta^2)$. Therefore the corresponding matrices are

$$\mathbf{m} = \begin{pmatrix} 3MR^2 & MR^2 \\ MR^2 & MR^2 \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} 2MgR & 0 \\ 0 & MgR \end{pmatrix}, \tag{26}$$

and $\det(\mathbf{m}\omega^2 - \mathbf{A}) = 0$ gives

$$2\omega^4 - 5\omega^2\frac{g}{R} + 2\left(\frac{g}{R}\right)^2 = 0 \Rightarrow \omega^2 = \frac{5 \pm 3}{4}\frac{g}{R} \Rightarrow \omega_1^2 = 2\frac{g}{R} \quad \omega_2^2 = \frac{1}{2}\frac{g}{R}. \tag{27}$$

The unnormalized eigenvectors are found to be

$$\vec{u}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \vec{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{28}$$

If one wants to excite one of the normal modes, one can simply set the initial displacement proportional to one of the eigenvectors, with zero initial velocity. So an initial displacement of $\theta = \alpha, \phi = -2\alpha$, for small α excites the first mode.