# PHY 203: Solutions to Problem Set 7

November 30, 2006

## 1 Problem 13.4

In this problem we are asked to solve for the motion of a string with given initial conditions. We start from the wave equation:

$$\rho\frac{\partial^2\psi}{\partial t^2} = T\frac{\partial^2\psi}{\partial x^2}, \tag{1}$$

where $\psi$ is the transverse displacement of the string, $\rho$ its density and $T$ its tension. To solve this equation we use the usual, oscillatory ansatz of the form:

$$\psi(x,t) = \sum_{n=1}^{\infty}(a_n\cos(k_n x) + b_n\sin(k_n x))\cos(\omega_n t + \phi_n), \tag{2}$$

where we still have to determine the coefficients $\omega_n$, $k_n$ and $\phi_n$ and the amplitudes $a_n$ and $b_n$. Plugging each normal mode into equation (1) we get:

$$\rho\omega_n^2\psi = Tk_n^2\psi \quad\rightarrow\quad \omega_n = \sqrt{\frac{T}{\rho}}k_n. \tag{3}$$

Now we use the boundary conditions. Since the string is fixed at $x = 0$ (i.e. $\psi(0,t) = 0$), only solutions which have $\sin(kx)$ are allowed (since $\cos(kx) \neq 0$ at $x = 0$). Thus $a_n = 0$ for all $n$. If we do the same thing for $x = L$ we get the condition $\sin(k_n L) = 0$ for all $n$. Then, $k_n = \frac{n\pi}{L}$. At this point our solution looks like

$$\psi(x,t) = \sum_{n=1}^{\infty} b_n\sin(\frac{n\pi}{L}x)\cos(\sqrt{\frac{T}{\rho}}\frac{n\pi}{L}t + \phi_n). \tag{4}$$

The only thing left for us to do is to fix $b_n$ and $\phi_n$ by initial conditions. In our case the initial conditions are

$$\psi(x,0) = \frac{4x(L-x)}{L^2} = \sum_{n=1}^{\infty} b_n\cos(\phi_n)\sin(\frac{n\pi}{L}x), \tag{5}$$

$$\dot{\psi}(x,0) = 0 = \sum_{n=1}^{\infty} -b_n\omega_n\sin(\phi_n)\sin(\frac{n\pi}{L}x), \tag{6}$$

where we have substituted the result from (4). The only thing left for us to do is to Fourier expand the initial conditions and match coefficients on both sides of (5) and (6). It is easier to start from (6). The Fourier expansion of 0 is just 0 for every coefficient. This means that $\omega_n b_n \sin(\phi_n) = 0$ for all $n$. Since neither $\omega_n$ (these frequencies were already calculated and are positive) nor the $b_n$ can be zero, we have $\sin(\phi_n) = 0 \Rightarrow \phi_n = 0$. Using this result in (5) we find

$$\frac{4x(L-x)}{L^2} = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right). \tag{7}$$

Thus the $b_n$ are just the Fourier coefficients of the initial position of the string.

$$b_n = \frac{2}{L}\int_0^L \frac{4x(L-x)}{L^2} \sin\left(\frac{n\pi}{L}x\right) dx. \tag{8}$$

Performing the integral (by parts) we obtain

$$b_n = \frac{16}{\pi^3 n^3}\left[1-(-1)^n\right], \tag{9}$$

which is the same as $b_n = 0$ for $n$ even and $b_n = \frac{32}{\pi^3 n^3}$ for $n$ odd. These are the amplitudes of the normal modes. Our final solution is therefore

$$\psi(x,t) = \sum_{n=1}^{\infty} \frac{32}{\pi^3(2n-1)^3} \sin\left(\frac{(2n-1)\pi}{L}x\right) \cos\left(\sqrt{\frac{T}{\rho}}\frac{(2n-1)\pi}{L}t\right). \tag{10}$$

## 2 String with Spring

Here we again use the general solution (2) with Dirichlet boundary conditions at $x = 0$ (i.e. $a_n = 0$) and dispersion relation $\omega = \sqrt{T/\lambda}\,k$. The only non-trivial issue is the boundary condition at $x = L$. At a generic point along the string we have

$$\frac{\partial^2\psi}{\partial t^2}\lambda dx = T\left(\left.\frac{\partial\psi}{\partial x}\right|_{x+dx} - \left.\frac{\partial\psi}{\partial x}\right|_{x}\right). \tag{11}$$

At the boundary $x = L$ the first term on the right hand side is zero (since there is no more string to the right of the endpoint) and furthermore there is an extra term proportional to the displacement from the spring:

$$\frac{\partial^2\psi}{\partial t^2}\lambda dx = -T\left.\frac{\partial\psi}{\partial x}\right|_{x=L^-} - k\psi. \tag{12}$$

Now as $dx \to 0$ the right hand side of this equation must vanish at $x = L$. Substituting from (2) this gives

$$\tan(k_n L) = -\frac{T}{k}k_n \qquad \Rightarrow \qquad \tan\left(\omega_n\sqrt{\frac{\lambda}{T}}L\right) = -\frac{\sqrt{\lambda T}\omega_n}{k}. \tag{13}$$

We can find the frequencies $\omega_n$ graphically by plotting $\tan(z)$ and $-Tz/(kL)$ and finding their points of intersection. In particular, in the limit $k \to 0$ the slope of the straight line diverges and the solutions are just the points where $\tan(z)$ diverges, that is

$$\omega_n = (2n-1)\sqrt{\frac{T}{\lambda}}\frac{\pi}{2L}. \tag{14}$$

This coincides with the result for a string with a free end (Neumann boundary condition). Similarly for $k \to \infty$ the solutions are simply the zeroes of tan(z), which implies

$$\omega_n = n\sqrt{\frac{T}{\lambda}}\frac{\pi}{L}. \tag{15}$$

This is of course just a string with Dirichlet boundary conditions at both ends.

# 3 String with Mass in the Middle

Again this is basically just a question of boundary conditions, but before we get to that there's an important point to be understood: the normal modes of a string of length $L$ without a mass attached are either symmetric or antisymmetric about $x = L/2$. For the antisymmetric modes there's a node at $x = L/2$, so if we put a mass there nothing changes. The antisymmetric modes (with even mode numbers) still have sinusoidal $x$-dependence and have frequencies

$$\omega_{2n} = 2n\sqrt{\frac{T}{\lambda}}\frac{\pi}{L}. \tag{16}$$

This is not true for the symmetric modes however. Here the mass at $x = L/2$ sits at a node, and thus the frequencies will be shifted and normal modes will have a kink at this point (they are still piecewise sinusoidal to the left and right of the mass, but they are not sinusoidal on the whole interval $[0, L]$).

To treat these symmetric modes we consider just the left half of the string (the right half is then fixed by symmetry) and treat the mass as a boundary condition. We have

$$\frac{\partial^2\psi}{\partial t^2}(\lambda dx + m) = -2T\frac{\partial\psi}{\partial x}\bigg|_{x=(L/2)^-}, \tag{17}$$

where we take the left derivative (note the profile of the string is not differentiable at $x = L/2$), the factor of two arises because the right half of the string pulls on the mass with the same force by symmetry, and again the $\lambda dx$ term vanishes as $dx \to 0$. Therefore we find for odd mode numbers

$$\cot\left(\omega_{2n-1}\sqrt{\frac{\lambda}{T}}\frac{L}{2}\right) = \frac{m\omega_{2n-1}}{2\sqrt{\lambda T}}. \tag{18}$$

Again this can be solved graphically and the limits $m \to 0$ and $m \to \infty$ are analogous to those discussed in Problem 2.

# 4 Chain of Springs

Consider the equation of motion for a mass $m$ at site $j$, which depends on its displacement $x_j$ and that of its immediate neighbours:

$$m\ddot{x}_j = k(x_{j+1}-x_j)+k(x_{j-1}-x_j) = kd\left(\frac{x_{j+1}-x_j}{d}\right)-kd\left(\frac{x_j-x_{j-1}}{d}\right), \quad (19)$$

where the right hand side should remind you of a difference of two derivatives analogous to eq. (11). Indeed we can rewrite it in the even more suggestive form

$$\frac{m}{d}\ddot{x}_j = kd\left(\frac{x_{j+1}-2x_j+x_{j-1}}{d^2}\right), \quad (20)$$

where now the right hand side is the discrete version of a second derivative. Taking the continuum limit $d \to 0$, $m \to 0$ and $k \to \infty$, such that $\lambda = m/d$ and $T = kd$ are finite, we recover the wave equation

$$\lambda\ddot{x} = Tx''. \quad (21)$$

To find the relationship between Young's modulus and the wave velocity imagine a 3D simple cubic lattice of masses $m$ connected by equal springs, which basically amounts many of the above discrete chains in parallel, with one chain per cross sectional area $d^2$. The three-dimensional density is then $\rho = m/d^3$ and Young modulus is given by

$$E \equiv \frac{\text{stress}}{\text{strain}} = \frac{kx/d^2}{x/d} = \frac{k}{d}. \quad (22)$$

Therefore the wave velocity, i.e. the velocity of longitudinal sound waves $c$, is

$$c^2 = \frac{T}{\lambda} = \frac{kd}{m/d} = \frac{E}{\rho}. \quad (23)$$