PHYS 203 **Princeton University** **Fall 2004**

Midterm Exam

1. (20 points) A boat of mass m is pushed from the shore into a lake with initial velocity v_0. The motion of the boat is resisted by a force $\vec{F} = -mk\vec{v}$.

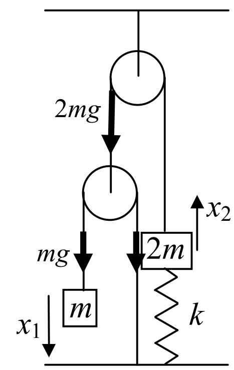
a) (10 points) Find the velocity of the boat as a function of time.

b) (10 points) What is the total distance that the boat will travel?

$$ma = m\dot{v} = m\frac{dv}{dt} = -mkv; \quad \frac{dv}{v} = -kdt; \quad \int_{v_0}^{v} vdv = -k\int_0^t dt; \quad \ln\frac{v}{v_0} = -kt; \quad v(t) = v_0 e^{-kt}$$

$$x_{tot} = \int_0^{\infty} v(t)dt = \frac{v_0}{k}$$

2. (40 points) Consider the setup shown on the right. The pulleys, strings, and the spring are massless and there is no friction. The coordinates x_1 and x_2 are measured from the point of equilibrium.



a) (5 points) What is the extension of the spring in equilibrium?
Adding the tension forces in the pulley system shows that there is no force on the spring and it has zero extension.

b) (10 points) Write down the Lagrangian in terms of coordinates x_1 and x_2.

$$L = \frac{1}{2}m\,\dot{x}_1^2 + \frac{1}{2}2m\,\dot{x}_2^2 - \left(-mgx_1 + 2mgx_2 + \frac{1}{2}kx_2^2\right)$$

b) (10 points) Write down the equation of constraint between x_1 and x_2.
If mass 2m moves up by a distance d, mass m moves down by a distance 2d, therefore $x_1=2x_2$

c) (5 points) Using the constraint, write down the Lagrangian equation of motion for x_1.

$$L = \frac{1}{2}m\,\dot{x}_1^2 + \frac{1}{4}m\,\dot{x}_1^2 - \left(-mgx_1 + mgx_1 + \frac{1}{8}kx_1^2\right)$$

e) (10 points) Find the frequency of small oscillations around the equilibrium.

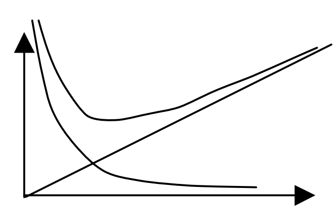
$$\frac{\partial L}{\partial x_1} = -\frac{1}{4}kx_1; \quad \frac{\partial L}{\partial \dot{x}_1} = \frac{3}{2}m\dot{x}_1; \quad \frac{3}{2}m\ddot{x}_1 + \frac{1}{4}kx_1 = 0; \quad \omega = \sqrt{\frac{k}{6m}}$$

3. (40 points) Consider motion of a particle with reduced mass μ in a central force with the potential energy given by $U = F_0 r$,

a) (5 points) Describe the force given by this potential energy.

$F = -\frac{dU}{dr} = -F_0$ *- an inward radial force that does not depend on the distance.*

b) (5 points) Write down and sketch the effective potential.



$$V_{eff} = F_0 r + \frac{l^2}{2\mu r^2}$$

c) (10 points) Find the radius r_0 of a circular orbit for a given angular momentum l.

$$\frac{dV}{dr} = F_0 - \frac{l^2}{\mu r^3} = 0; \quad r_0 = \left(\frac{l^2}{F_0\mu}\right)^{1/3}$$

d) (5 points) Find the period in the circular orbit in terms of F_0, l, μ

$$T=\frac{2\pi r_0}{v};\quad v=\frac{l}{\mu r_0};\ T=\frac{2\pi\mu r_0^2}{l}=2\pi\left(\frac{\mu l}{F_0^2}\right)^{1/3}$$

e) (10 points) Find the frequency of small oscillations around the circular orbit.

$$\frac{d^2V}{dr^2}=\left.\frac{3l^2}{\mu r^4}\right|_{r_0};\ \Omega=\left(\frac{d^2V/dr^2}{\mu}\right)^{1/2}=\sqrt{3}\left(\frac{F_0^2}{\mu l}\right)^{1/3}$$

f) (5 points) By what angle do the orbit apsides precess after one period?

One oscillation around the orbit corresponds to rotation by an angle $\Omega T=2\pi\sqrt{3}$. *Hence, the orbit precesses by an angle of* $2\pi(\sqrt{3}-1)$ *after every period.*