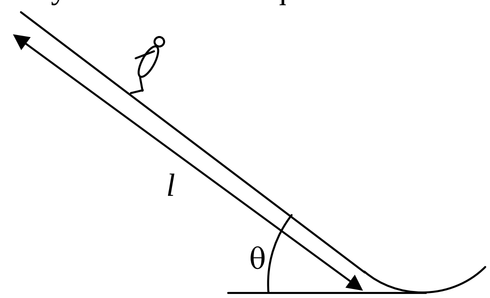
1. (30 points) **Ski jump track**

Consider the motion of a skier of mass *m* down a track with a constant slope θ. The friction of the skies is negligible, but the motion is subject to air resistance. The resistance force is given by $F = -kv^2$. The initial velocity of the skier is equal to zero.



a) What is the velocity of the skier as a function of time?

$$m\frac{dv}{dt} = mg\sin\theta - kv^2$$

$$\frac{dv}{g\sin\theta - kv^2/m} = dt$$

$$\frac{m}{k}\int_0^v \frac{dv}{mg\sin\theta/k - v^2} = t$$

$$\frac{m}{k}\frac{1}{\sqrt{mg\sin\theta/k}}\tanh^{-1}\frac{v}{\sqrt{mg\sin\theta/k}} = t$$

$$v = \sqrt{mg\sin\theta/k}\,\tanh(t\sqrt{kg\sin\theta/m})$$

b) What is the distance *l* traveled by the skier as a function of time? How far does he travel by the time his velocity reaches approximately 90% of its maximum possible value?

$$l = \int_0^t vdt = \sqrt{mg\sin\theta/k}\int_0^t \tanh(t\sqrt{kg\sin\theta/m})dt = \frac{m}{k}\ln\cosh(t\sqrt{kg\sin\theta/m})$$

Tanh(x) reaches 90% of its maximum for $x = \tanh^{-1}(0.9) = 1.47$

$$t_{90}\sqrt{kg\sin\theta/m} = 1.47$$

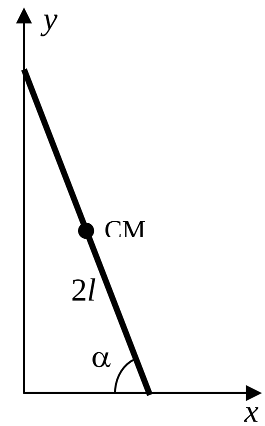
$$l_{90} = \frac{m}{k}\ln\cosh(1.47) = 0.83\frac{m}{k}$$

c) The coefficient of air resistance is approximately given by $k = \rho A/2$, where ρ is the density of air and *A* is the cross-sectional area of the skier. Taking $\rho = 1\ \mathrm{kg/m^3}$, $A = 1\ \mathrm{m^2}$, θ = 45 deg, and *m* = 100 kg, estimate the length *l* needed to achieve 90% of the maximum velocity. How does it compare with the length of a typical ski jump track of about 100 m?

$$l_{90} = \frac{100kg}{1kg/m^3\ 1m^2/2}0.833 = 166\,m$$

So, the track is nearly as long as it takes to reach the maximum velocity. The coefficient of air resistance is probably actually smaller because of the aerodynamic suits.

2. (30 points) **Falling Ladder**



A ladder of length 2*l* is standing up against a vertical wall with initial angle α relative to the horizontal. There is no friction between the ladder and the wall or the floor. The ladder begins to slide down with zero initial velocity. Denote by θ(t) the angle the ladder makes with the horizontal after it starts to slide.

a) Write down Lagrangian equations of motion with two constraints describing the contact of the ladder with the vertical wall and the floor.

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2; \;\; I = \frac{1}{12}m(2l)^2 = \frac{1}{3}ml^2$$

$$U = mgy$$

$$L = T - U$$

$$g_1 = l\cos\theta - x = 0$$

$$g_2 = l\sin\theta - y = 0$$

$$(1) \quad \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \lambda_1\frac{\partial g_1}{\partial x} = -m\ddot{x} - \lambda_1 = 0$$

$$(2) \quad \frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} + \lambda_2\frac{\partial g_2}{\partial y} = -mg - m\ddot{y} - \lambda_2 = 0$$

$$(3) \quad \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda_1\frac{\partial g_1}{\partial \theta} + \lambda_2\frac{\partial g_2}{\partial \theta} = -\frac{1}{3}ml^2\ddot{\theta} - \lambda_1 l\sin\theta + \lambda_2\cos\theta = 0$$

b) Find the expressions for $\dot{\theta}^2$ and $\ddot{\theta}$ as a function of θ

Differentiating the constraint equations and plugging them into the equation for kinetic energy:

$$\dot{x} = -l\sin\theta\,\dot{\theta}$$

$$\dot{y} = l\cos\theta\,\dot{\theta}$$

$$T = \frac{1}{2}ml^2(\sin^2\theta + \cos^2\theta)\dot{\theta}^2 + \frac{1}{6}ml^2\dot{\theta}^2 = \frac{2}{3}ml^2\dot{\theta}^2$$

$$U = mgl\sin\theta$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = -mgl\cos\theta - \frac{4}{3}ml^2\ddot{\theta} = 0$$

$$\ddot{\theta} = -\frac{3g}{4l}\cos\theta$$

From energy conservation $U_{\text{init}} = T + U$

$$mgl\sin\alpha = mgl\sin\theta + \frac{2}{3}ml^2\dot{\theta}^2$$

$$\dot{\theta}^2 = \frac{3g}{2l}(\sin\alpha - \sin\theta)$$

c) Find the angle θ_c when the ladder losses contact with the vertical wall.

The constraint force given by λ_1 is equal to the normal force from the vertical wall. Find when it goes to zero.

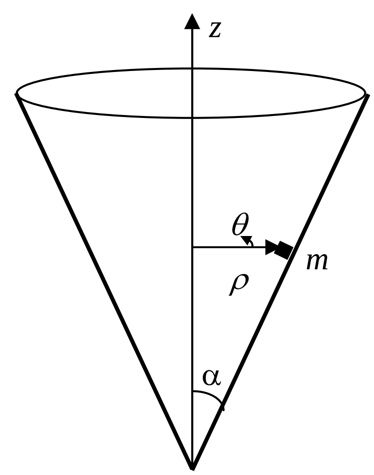
$$\lambda_1 = -m\ddot{x} = ml\sin\theta\;\ddot{\theta} + ml\cos\theta\,\dot{\theta}^2 = 0$$

$$\sin\theta\;\ddot{\theta} + \cos\theta\,\dot{\theta}^2 = -\frac{3g}{4l}\sin\theta\cos\theta + \frac{3g}{2l}\cos\theta(\sin\alpha - \sin\theta) == \frac{3g}{2l}\cos\theta\sin\alpha - \frac{9g}{4l}\cos\theta\sin\theta = 0$$

$$\sin\theta = \frac{2}{3}\sin\alpha$$

3. (40 points) **Conical surface**

A small mass *m* can slide without friction on the inside of a conical surface with opening angle α. Use cylindrical coordinates ρ, θ, and z to describe the position of the mass with $z = 0$ at the apex of the cone.



a) Find the Lagrangian for the mass *m* in terms of coordinates ρ and θ only by eliminating z dependence using the surface constraint.

$$L = \frac{1}{2}m(\dot{\rho}^2 + \dot{z}^2 + \rho^2\dot{\theta}^2) - mgz$$

$$z = \rho\cot\alpha$$

$$L = \frac{1}{2}m(\dot{\rho}^2(1+\cot^2\alpha) + \rho^2\dot{\theta}^2) - mg\rho\cot\alpha$$

b) Is there a conserved momentum in the problem? If so, use it to reduce the problem to only a single dynamical variable.

$$\frac{\partial L}{\partial \theta} = 0; \;\; p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\rho^2\dot{\theta} - const$$

$$\dot{\theta} = \frac{p_\theta}{m\rho^2}$$

$$p_\rho = \frac{\partial L}{\partial \dot{\rho}} = m(1+\cot^2\alpha)\dot{\rho}$$

$$H = \frac{p_\rho{}^2}{2m(1+\cot^2\alpha)} + \frac{p_\theta{}^2}{2m\rho^2} + mg\rho\cot\alpha$$

Since there is no θ dependence and p_θ is a constant, we have only one dynamical variable, ρ

c) Show that the solution can be described by motion of a particle with a certain effective mass in a one-dimensional effective potential. What are the effective potential and the effective mass?

This Hamiltonian corresponds to motion of a single particle with mass $\mu = m(1+\cot^2\alpha) = m/\sin^2\alpha$ in effective potential

$$V_{eff} = \frac{p_\theta{}^2}{2m\rho^2} + mg\rho\cot\alpha$$

Note that the effective potential has to be identified from the Hamiltonian, not Lagrangian, because the Hamiltonian is the function of the momentum and coordinate.

d) Use the effective potential to calculate the radius of a circular orbit. If you are stuck on other parts, use freshman physics techniques to find the radius ρ for a circular orbit.

$$\frac{dV_{eff}}{d\rho} = -\frac{p_\theta{}^2}{m\rho^3} + mg\cot\alpha$$

$$\rho = \sqrt[3]{\frac{p_\theta{}^2\tan\alpha}{m^2 g}}$$