Atomic Physics (Phys 551) Final Exam Solutions

Problem 1. For a Rydberg atom in $n = 50, l = 49$ state estimate within an order of magnitude the numerical value of

a) Decay lifetime

$$A = \frac{1}{\tau} = \frac{4\alpha\omega^3}{3c^2}|D|^2 \tag{1}$$

The dipole matrix element can be rouphly estimated as $\langle r\rangle^2 = \left(\frac{a_0}{2Z}\right)^2 [3n^2 - l(l+1)]^2$. For neutral Rydberg atoms with one electron far outside closed shell $Z = 1$. For $l = n-1$ and $n \gg 1$ we get $|D|^2 \sim \langle r\rangle^2 = a_0^2 n^4$. The same result would be obtained using $\langle r^2\rangle$. The energy difference corresponding to ω is given by $\hbar\omega = Ry\left(\frac{1}{(n-1)^2} - \frac{1}{n^2}\right) \simeq 2Ry/n^3$ for $n \gg 1$ since the stretched state can only decay to the next lowest $n' = n-1$, $l' = n-2$ state. Hence we have

$$A = \frac{4\alpha}{3c^2}\left(\frac{a_0}{Z}\right)^2 n^4 \frac{8Ry^3}{\hbar^3 n^9} = \frac{4\alpha^5 mc^2}{3\hbar n^5} = \frac{4\alpha^4 c}{3n^5 a_0} = 68\,\mathrm{sec}^{-1}, \tau = 15\,\mathrm{msec} \tag{2}$$

Exact calculation with proper angular factors for $|D|^2$ gives $\tau = 30\,\mathrm{msec}$

b) Static polarizability

$$\alpha(0) = \frac{e^2}{\varepsilon_0 m}\sum_k \frac{f_k}{\omega_k^2} = \frac{2e^2}{3\hbar\varepsilon_0}\sum_k \frac{|D|^2}{\omega_k} = \frac{2e^2 a_0^2}{3\hbar\varepsilon_0}\sum_k \frac{n^4}{\omega_k} \tag{3}$$

Taking into account only the closest n' levels, we get

$$\alpha(0) = \frac{e^2 a_0^2}{3\varepsilon_0}\frac{n^7}{Ry} = \frac{4\pi\alpha a_0^2\hbar c}{3}\frac{n^7}{Ry} = \frac{4\pi\alpha^2 a_0^3}{3}\frac{n^7 mc^2}{Ry} = 10^{-12} cm^3 \tag{4}$$

There is an additional complication due to transitions with $n' = n$ (as pointed out by Georgios). The above estimate is correct for atomic states with small l, because the quantum defect $\delta_l \sim 1$ and does not depend on n. For circular states with $l = n-1$ the energy shift is quadratic in E only if the electric field is applied parallel to the direction of l. If the inital state has $l = m = n-1$, then the matrix element $\langle nl'm'|Ez|nlm\rangle$ with $l' = m' = n-2$ is zero because $z \sim Y_{10}$ and is only non-zero for $m' = m$.

Problem 2. Consider a collection of N spin $\frac{1}{2}$ particles initially fully polarized in z direction so that $S_z = N/2$. The spins fill a cubic box of size a. There is no external magnetic field. The spin direction is being measured using optical rotation of a linearly polarized probe laser propagating along x direction. The laser beam has a flux of Φ photons/sec. It operates at frequency in the vicinity of an $\mathrm{S}_{1/2} \rightarrow \mathrm{P}_{1/2}$ transition at frequency ω_0 with oscillator strength f but is sufficiently far detuned from resonance that it does not cause spin relaxation. You can also ignore other sources of spin relaxation.

a) What is the uncertainty in the polarization rotation angle after measurement time t?

$\delta\phi_{pol} = \frac{1}{\sqrt{2N_{ph}}} = \delta\phi_{pol} = \frac{1}{\sqrt{2\Phi t}}$

This can be obtained by considering a balanced polarimeter where one measures the difference between the intensities in the two arms, each getting half of the photons with shot noise level of $\frac{\delta I}{I} = \frac{1}{\sqrt{\Phi/2t}}$

b) What is the uncertainty in S_x after measurement time t?

We need to calculate the rotation of the light polarization produced by atoms. The phase shift caused by light propagation through atomic vapor is given by

$$\phi = \mathrm{Re}[\alpha]\frac{N}{V}\frac{\pi l}{\lambda} = \mathrm{Re}[\alpha]\frac{N}{a^2}\frac{\pi}{\lambda} \tag{5}$$

For $S_{1/2} \rightarrow P_{1/2}$ transition the excitation rate and the polarizability depend on the spin of the $S_{1/2}$ state and the circular polarization of the light. Consider linearly polarized light as superposition of left and right circularly polarized light

$$\varepsilon_y = \frac{1}{2}(\varepsilon_y + i\varepsilon_z) + \frac{1}{2}(\varepsilon_y - i\varepsilon_z) \tag{6}$$

For left-circularly polarized light ($\Delta m = +1$) the transition rate from $S_{1/2}$, $m_S = 1/2$ state to $P_{1/2}$ state is zero while from $m_S = -1/2$ state it is twice the unpolarized rate (this way for unpolarized atoms the excitation rate is the same for any light polarization). Hence $\phi_{LCP} = \mathrm{Re}[\alpha]\frac{N}{a^2}\frac{\pi}{\lambda}2P_{m=-1/2}$, where $P_{m=-1/2}$ is the probability of finding the atom in the $m = -1/2$ state. Note that since the light is propagating in the x direction (and the polarization is in y, z directions) $m = -1/2$ state refers to a quantization axis along x direction. Similarly, $\phi_{RCP} = \mathrm{Re}[\alpha]\frac{N}{a^2}\frac{\pi}{\lambda}2P_{m=+1/2}$. Since $P_{m=+1/2} + P_{m=-1/2} = 1$, and $\langle S_x \rangle = \hbar N(P_{m=+1/2} - P_{m=-1/2})/2$

$$\left\{ \begin{array}{c} \phi_{RCP} \\ \phi_{LCP} \end{array} \right\} = \pm\mathrm{Re}[\alpha]\frac{2\langle S_x\rangle}{\hbar a^2}\frac{\pi}{\lambda} + \mathrm{Re}[\alpha]\frac{N}{a^2}\frac{\pi}{\lambda} \equiv \pm\phi_S + \phi_0 \tag{7}$$

The light polarization at the output will be

$$\begin{aligned} \varepsilon_{out} &= \frac{1}{2}(\varepsilon_y + i\varepsilon_z)e^{i(\phi_S+\phi_0)} + \frac{1}{2}(\varepsilon_y - i\varepsilon_z)e^{-i(\phi_S+\phi_0)} && (8)\\ &= \left[\frac{1}{2}\varepsilon_y(e^{i\phi_S} + e^{-i\phi_S}) + \frac{1}{2}i\varepsilon_z(e^{i\phi_S} - e^{-i\phi_S})\right]e^{i\phi_0} && (9)\\ & \quad (\varepsilon_y\cos\phi_S - \varepsilon_z\sin\phi_S)\,e^{i\phi_0} && (10) \end{aligned}$$

rotated by an angle ϕ_S. Using the expression for polarizability for weak excitation

$$\mathrm{Re}[\alpha] = \frac{fe^2}{2\varepsilon_0 m\omega_0}\frac{(\omega_0-\omega)}{(\omega_0-\omega)^2+\gamma^2} \tag{11}$$

we get

$$\phi_S = \frac{fe^2}{2\varepsilon_0 mc\hbar}\frac{(\omega_0-\omega)}{(\omega_0-\omega)^2+\gamma^2}\frac{\langle S_x\rangle}{a^2} \tag{12}$$

Hence, the uncertainty in $\langle S_x\rangle$ is

$$\delta\langle S_x\rangle = \frac{2\varepsilon_0 mc\hbar}{fe^2}\frac{(\omega_0-\omega)^2+\gamma^2}{(\omega_0-\omega)}\frac{a^2}{\sqrt{2\Phi t}} \tag{13}$$

c) Show that the back-reaction of laser light on the atoms preserves the uncertainty relationship $S_x S_y \sim \hbar S_z/2$

Even though the light is linearly polarized, it will have fluctuations of the degree of circular polarization. Think again of the linearly polarized light as being made of $N_{ph}/2$ of right circularly polarized and $N_{ph}/2$ of left-circularly polarized photons. The difference between then will fluctuate as $\delta(N_{RCP} - N_{LCP}) = \sqrt{N_{ph}}$.

These photons will cause a light shift, which can be calculated using

$$\delta U = -\frac{\varepsilon_0}{4}\alpha E_0^2 \tag{14}$$

Remember that there will be a scalar and vector light shift and the vector light shift is equivalent to a magnetic field along the direction of the laser beam. For $m_S = -1/2$ state (with x quantization axis) only left-circularly polarized photons can cause transitions, and therefore light shift, it is the opposite for the $m_S = 1/2$ state.

$$\delta U_{m=1/2} - \delta U_{m=-1/2} = -\frac{\varepsilon_0}{4}2\alpha(E_{0LCP}^2 - E_{0RCP}^2) = -\frac{\varepsilon_0}{2}\alpha E_0^2/\sqrt{N_{ph}} \tag{15}$$

One can also write the energy of the photons as

$$\frac{\varepsilon_0 c E_0^2}{2} = \frac{\hbar\omega\Phi}{a^2} \tag{16}$$

$$\delta U_{m=1/2} - \delta U_{m=-1/2} = \frac{fe^2}{2\varepsilon_0 m\omega_0}\frac{(\omega_0-\omega)}{(\omega_0-\omega)^2+\gamma^2}\frac{\hbar\omega\Phi}{ca^2}\frac{1}{\sqrt{N_{ph}}} \tag{17}$$

This energy shift will be fluctuating randomly as a white noise. For example, the energy shift averaged for time τ will be given by the above expression with $N_{ph} = \Phi\tau$. The light shift will cause evolution of the spins as a random walk given by $d\rho/dt = -\frac{i}{\hbar}(\delta U_{m=1/2} - \delta U_{m=-1/2})[S_x, \rho]$. It will cause rotation of the S_z spin into y direction by a random angle on the order of $(\delta U_{m=1/2} - \delta U_{m=-1/2})\tau/\hbar$ in each step of length τ. Then after t/τ random walk steps,

$$\delta\langle S_y\rangle = S_z(\delta U_{m=1/2} - \delta U_{m=-1/2})\sqrt{\tau t}/\hbar \tag{18}$$

$$\delta\langle S_y\rangle = \frac{fe^2}{2\varepsilon_0 mc}\frac{(\omega_0-\omega)}{(\omega_0-\omega)^2+\gamma^2}\frac{\sqrt{\Phi t}}{a^2}S_z \tag{19}$$

Hence,

$$\delta\langle S_x\rangle\,\delta\langle S_y\rangle = \frac{\hbar S_z}{\sqrt{2}} \tag{20}$$

independent of any parameters. More accurate consideration of the time evolution gives exactly $\delta\langle S_x\rangle\,\delta\langle S_y\rangle = \hbar S_z/2$.

Problem 3. Consider laser cooling and trapping of two elements, Sr and Cu. The Grotrian diagrams showing their energy levels and oscillator strengths are attached to the exam. Focus on the most abundant isotope of Sr, ^{88}Sr, which has a zero nuclear spin, and the most abundant isotope of Cu, ^{63}Cu, which has $I = 3/2$.

a) What laser wavelength(s) would you need to cool and trap each atom?

For Sr there is only one strong transition at 460.7 nm.

For Cu one has to use 324.7 nm $S_{1/2} \rightarrow P_{3/2}$ transition for cooling because it has to be a cycling transition for circularly polarized light. For $S_{1/2} \rightarrow P_{1/2}$ transition the atoms will be quickly pumped by circularly polarized light into one of the two $S_{1/2}$ states and stop scattering light. This is similar to the alkali metals where D2 transition to $P_{3/2}$ is used for cooling. In copper, however, the atoms can decay from $P_{3/2}$ state to the $D_{3/2}$ and $D_{5/2}$ states with appreciable probability (every thousand photons or so). So, one would need to use a repumping laser at 510.5 nm and perhaps at 570 nm to prevent atoms from being accumulated in metastable $D_{3/2}$ and $D_{5/2}$ states. In addition, ^{63}Cu with nuclear spin 3/2 will have hyperfine states for each electronic level. In the $S_{1/2}$ state there are two hyperfine states and one will need a repumping laser to prevent atoms from accumulating in the hyperfine state that is not being addressed by the cooling laser, similar to alkali metal atoms. In each $D_{3/2}$ and $D_{5/2}$ states there are 4 hyperfine states, and each of them will need to be repumped by a separate laser. That is why Cu atoms have never been cooled, even though they have simple ground state structure and a cooling transition similar to alkali metals.

b) What is the lowest temperature to which each atom can be cooled? (ignoring elaborate techniques like Raman or cavity cooling)

In Sr there is no electron or nuclear spin in the ground state, so the Sisyphus cooling does not work, since there is no optical pumping and polarization-dependent light shifts. The limiting temperture is the Doppler limit

$$T_D = \frac{\hbar A}{2k_B} = \frac{\alpha f}{k_B}\frac{\hbar^2\omega^2}{mc^2} = 2.3\text{mK} \tag{21}$$

More accurate consideration of the angular momentum factors for Sr $S_0 \rightarrow P_1$ transition give a three times smaller value of $A = 2\alpha\hbar\omega^2 f/(3mc^2)$ and $T_D = 0.8\,\text{mK}$. For Cu one can do Sisyphus cooling (with a large number of repumping lasers). The limiting temperature is the recoil limit

$$T_r = \frac{\hbar^2 k^2}{k_B M} = \frac{\hbar^2\omega^2}{k_B M c^2} = 2.9\mu\text{K} \tag{22}$$

c) How strong magnetic fields would be required to hold such cooled atoms in a magnetic trap?

Since Sr has no electron or nuclear magnetic moment in the ground state, it cannot be held in a magnetic dipole trap of any reasonable strength.

For Cu the minimum trap depth is given by

$$\mu_B B \sim k_B T_r,\;\; B \sim 0.04\,\text{G} \tag{23}$$

Also to hold the atoms up against gravity the minimum magnetic field gradient is

$$\mu_B \frac{dB}{dz} = Mg,\;\; \frac{dB}{dz} = 16\,\text{G/cm} \tag{24}$$

Just for strontium atoms, calculate

d) Laser power appropriate for cooling assuming the laser beam diameter of 1 cm.

The required laser power is on the order of the saturation intensity, when the Rabi rate is equal to the spontaneous decay rate

$$A = \frac{4\alpha\omega^3}{3c^2}|D|^2 \sim \Omega = \sqrt{\frac{4\pi\alpha f}{m\omega}I} \tag{25}$$

$$\frac{4\alpha^2\hbar^2\omega^4 f^2}{m^2c^4} = \frac{4\pi\alpha f}{m\omega}I \tag{26}$$

$$I = \frac{\hbar^2\omega^2}{mc^2}\frac{2\pi^2\alpha c f}{\lambda^3} = 190\text{mW/cm}^2 \tag{27}$$

e) The length of Zeeman slower that would be needed to slow a substantial fraction of an atomic beam of Sr emanating from an oven at T=900K?

The acceleration rate of Sr atoms in the Zeeman slower is

$$a = \frac{\hbar k A}{M} \tag{28}$$

To slow the atoms down from initial velocity v_0 the distance needed is

$$d = \frac{v_0^2}{2a} = \frac{c k_B T}{\hbar \omega A} = \frac{k_B T m c^2 \lambda}{4\pi\alpha\hbar^2\omega^2 f} = 1.4cm \tag{29}$$

This distance is so small that one doesn't actually need a Zeeman slower, the beam can be stopped in a magneto-optical trap.