PHYS 551 Homework 3 Solutions

Problem 1

Atomic polarizability is defined as the dipole moment induced in the atom in response to application of electric field, $d = \alpha E$. Near a resonance the dipole moment can be calculated from $d = \langle \Psi | er | \Psi \rangle$, where $\Psi = C_1 |1\rangle e^{-iE_1 t/\hbar} + C_2 |2\rangle e^{-iE_2 t/\hbar}$ and C_1 and C_2 are given by the solution of the two-level problem. This derivation is discussed in section 2.5 of Loudon's text book "Quantum theory of light". The case of collisional broadening is discussed in section 2.9. Collisional broadening results in increase of the Lorentzian linewidth but no change in the spontaneous emission rate. The result for susceptibility χ equal to the polarizability α times the density of atoms (N/V) is given by equation (2.9.8)

$$\chi = \alpha \frac{N}{V} = \frac{2\pi c^3 N}{\omega_0^3 V} \frac{\gamma_{sp}(\omega_0 - \omega + i\gamma)}{(\omega_0 - \omega)^2 + \gamma^2 + (\gamma/2\gamma_{sp})^2 \Omega^2} \tag{1}$$

where $\gamma = \gamma_{sp} + \gamma_{col}$ and Ω is the Rabi rate. The spontaneous emission rate from the excited state $2\gamma_{sp}$ can be related to the square of the dipole matrix element

$$2\gamma_{sp} = \frac{4\alpha\omega_0^3}{3c^2} |D|^2 \tag{2}$$

and to the oscillator strength

$$f = \frac{2m\omega_0}{3\hbar} |D|^2 \tag{3}$$

(Note that there are differences in these definitions depending on whether one assumes that $D = \langle r \rangle$ or $D = \langle er \rangle$, here we assume the former). Hence

$$\chi = \alpha \frac{N}{V} = \frac{Ne^2 f}{2V\varepsilon_0 m\omega_0} \frac{(\omega_0 - \omega + i\gamma)}{(\omega_0 - \omega)^2 + \gamma^2 + (\gamma/2\gamma_{sp})^2 \Omega^2} \tag{4}$$

A similar result can be derived (see section 7.5 of Jackson) assuming a classical dumped oscillator subjected to oscillating field

$$m(\ddot{x} + \gamma' x + \omega_0^2 x) = -eEe^{-i\omega t} \tag{5}$$

One obtains

$$\chi = \frac{Ne^2 f}{V\varepsilon_0 m} \frac{1}{(\omega_0^2 - \omega^2 - i\omega\gamma')} \tag{6}$$

assuming the classical oscillator has "oscillator strength" $= f$. For $\gamma' \ll \omega_0$ one can consider ω only close to ω_0, so that $(\omega + \omega_0) \simeq 2\omega_0$ and $\omega/\omega_0 \simeq 1$ and we obtain

$$\chi = \frac{Ne^2 f}{2V\varepsilon_0 m\omega_0} \frac{1}{(\omega_0 - \omega - i\gamma'/2)}, \tag{7}$$

which is equivalent to Eq. (4) for $\gamma' = 2\gamma$. and small Ω (no saturation). This is consistent with the fact that the decay rate of the excited state is equal to $2\gamma_{sp}$.

The propagation of light $E(x,t) = E_0 e^{-i(kx - \omega t)}$ is derived from the relationship $k = \sqrt{\mu\varepsilon}\omega$ and $\varepsilon = \varepsilon_0(1 + \chi)$. Since the medium is optically thin we can assume that $\chi \ll 1$ (actually it is sufficient that the optical thickness is much less than the wavelength of light) and $k = (1 + \chi/2)\omega/c$. Hence the intensity of the light $I \sim E^2$ will be attenuated as $\exp[Im[\chi]l\omega/c] = \exp[Im[\chi]2\pi l/\lambda]$ where $\lambda = 2\pi c/\omega$ is the wavelength of light in empty space. For optically thin vapor $Im[\chi]2\pi l/\lambda \ll 1$ and the relative attenuation coefficient is equal to

$$\frac{I - I_0}{I_0} = \frac{Ne^2 fl}{2V\varepsilon_0 mc} \frac{\gamma}{(\omega_0 - \omega)^2 + \gamma^2 + (\gamma/2\gamma_{sp})^2\Omega^2} \tag{8}$$

. The phase shift of electric field relative to empty space will be given by $\phi = Re[\chi]\pi l/\lambda$.

Another way to find the attenuation of light is to calculate the absorption cross-section. This is discussed, for example, in Foot's book on "Atomic Physics" in section 7.6. To find the cross-section one can equate the rate of photon absorption $\Phi\sigma$, where Φ is the photon flux (# of photons per unit area), to the rate of actual photon emission from the excited state $\Phi\sigma = 2\gamma_{sp}\rho_{22}$

$$\sigma = \frac{2\gamma_{sp}\rho_{22}}{\Phi} = \frac{\Omega^2}{\Phi} \frac{\gamma/2}{(\omega_0 - \omega)^2 + \gamma^2 + (\gamma/2\gamma_{sp})^2\Omega^2} \tag{9}$$

Now $\Omega^2 = e^2 \left|\langle e|\mathbf{r} \cdot \varepsilon|g\rangle\right|^2 E_0^2/\hbar^2$ and as shown in previous homeworks, when $\left|\langle e|\mathbf{r} \cdot \varepsilon|g\rangle\right|^2$ is averaged over all possible final states one gets a factor of $1/3, \left|\langle e|\mathbf{r} \cdot \varepsilon|g\rangle\right|^2 = |D|^2/3$. Also $\Phi = I/\hbar\omega = \varepsilon_0 c E_0^2/2\hbar\omega$. Hence one gets

$$\sigma = \frac{fe^2}{2\varepsilon_0 mc} \frac{\gamma}{(\omega_0 - \omega)^2 + \gamma^2 + (\gamma/2\gamma_{sp})^2\Omega^2} \tag{10}$$

Light attenuation is then given by $I = I_0 \exp[-\sigma N x/V]$ and for small attenuation

$$\frac{I - I_0}{I_0} = \frac{Ne^2 fl}{2V\varepsilon_0 mc} \frac{\gamma}{(\omega_0 - \omega)^2 + \gamma^2 + (\gamma/2\gamma_{sp})^2\Omega^2}, \tag{11}$$

in agreement with Eq.(8). Incidentally, in the regime of small exciation ($\Omega^2 \ll \gamma_{sp}^2$), the cross-section satisfies a simple sum rule $\int \sigma(\omega)d\omega = \frac{\pi f e^2}{2\varepsilon_0 mc} = 2\pi^2 r_e c f$, where r_e is the "classical electron radius", $r_e = 2.82 \times 10^{-13} cm$. Remembering the sum rule and the fact that the absorption cross-section is a Lorentzian, one can easily obtain all factors in front. The same sum-rule of course applies to Doppler-broadened profiles as well.

Problem 2

The energy shift is given by second-order time-dependent perturbation theory in the basis of S and P states. We can write

$$\Psi = c_s(t)\,|s\rangle + c_p(t)e^{-i\omega_0 t}\,|p\rangle \tag{12}$$

taking the energy of the ground state to be zero. In zeros order $c_s^{(0)}(t) = 1$, $c_p^{(0)}(t) = 1$. The perturbation is $V = -eE_0\mathbf{r} \cdot (\varepsilon e^{-i\omega t} + \varepsilon^* e^{i\omega t})/2$. Plugging it into the Schrodinger equation one gets

$$c_p^{(1)}(t) = -\frac{eE_0}{2i\hbar} \int_0^t e^{i\omega_0 t'} \left\langle p|\mathbf{r} \cdot (\varepsilon e^{-i\omega t'} + \varepsilon^* e^{i\omega t'})|s\right\rangle c_s^0(t')dt' \tag{13}$$

$$= \frac{eE_0}{2\hbar} \left(\frac{\langle p|\mathbf{r} \cdot \varepsilon|s\rangle\,(e^{i(\omega_0-\omega)t} - 1)}{(\omega_0 - \omega)} + \frac{\langle p|\mathbf{r} \cdot \varepsilon^*|s\rangle\,(e^{i(\omega_0+\omega)t} - 1)}{(\omega_0 + \omega)} \right) \tag{14}$$

Now in second order

$$c_s^{(2)}(t) = -\frac{eE_0}{2i\hbar}\int_0^t e^{-i\omega_0 t'}\left\langle s|\mathbf{r}\cdot(\varepsilon e^{-i\omega t'}+\varepsilon^* e^{i\omega t'})|p\right\rangle c_p^{(1)}(t')dt' \tag{15}$$

$$= -\frac{e^2E_0^2}{4i\hbar^2}\left(\frac{\langle p|\mathbf{r}\cdot\varepsilon|s\rangle\langle s|\mathbf{r}\cdot\varepsilon^*|p\rangle}{(\omega_0-\omega)}+\frac{\langle p|\mathbf{r}\cdot\varepsilon^*|s\rangle\langle s|\mathbf{r}\cdot\varepsilon|p\rangle}{(\omega_0+\omega)}\right)t \tag{16}$$

where we only pick terms that do not average to zero. This linear time evolution of $c_s^{(2)}(t)$ can be interpreted as the first term in the expansion $c_s(t) = \exp[-i\delta Et/\hbar] = 1 - i\delta Et/\hbar + ... = c_s^{(0)}(t) + c_s^{(2)}(t) + ...$ Hence the energy shift is given by

$$\delta E = -\frac{e^2E_0^2}{4\hbar}\left(\frac{\langle s|\mathbf{r}\cdot\varepsilon^*|p\rangle\langle p|\mathbf{r}\cdot\varepsilon|s\rangle}{(\omega_0-\omega)}+\frac{\langle s|\mathbf{r}\cdot\varepsilon|p\rangle\langle p|\mathbf{r}\cdot\varepsilon^*|s\rangle}{(\omega_0+\omega)}\right) \tag{17}$$

Now we use spherical expansion $\varepsilon\cdot\mathbf{r} = \sum_{\rho=-1}^{1}(-1)^\rho\varepsilon_\rho r_{-\rho}$ and

$$|\langle l,s,J,m_J|\varepsilon\cdot\mathbf{r}|l',s,J',m'_J\rangle|^2 = |(l,s,J||r||l',s,J')|^2\times \tag{18}$$
$$(-1)^{J+J'-m_J-m'_J+1}\begin{pmatrix} J & 1 & J' \\ -m_J & m_J-m'_J & m'_J\end{pmatrix}^2 \varepsilon_{m'_J-m_J}\varepsilon^*_{m_J-m'_J}$$

and get the following shifts by summing over m'_J

$$\delta E_{m_J=1/2} = -\frac{e^2E_0^2}{4\hbar}|(J||r||J')|^2\left[\left(\frac{\varepsilon_0\varepsilon_0^*-2\varepsilon_{-1}\varepsilon_1^*}{6(\omega_0-\omega)}\right)+\left(\frac{\varepsilon_0\varepsilon_0^*-2\varepsilon_{-1}^*\varepsilon_1}{6(\omega_0+\omega)}\right)\right] \tag{19}$$

$$\delta E_{m_J=-1/2} = -\frac{e^2E_0^2}{4\hbar}|(J||r||J')|^2\left[\left(\frac{\varepsilon_0\varepsilon_0^*-2\varepsilon_1\varepsilon_{-1}^*}{6(\omega_0-\omega)}\right)+\left(\frac{\varepsilon_0\varepsilon_0^*-2\varepsilon_1^*\varepsilon_{-1}}{6(\omega_0+\omega)}\right)\right] \tag{20}$$

Consider the sum of these shifts and their difference

$$\delta E_{m=1/2}+\delta E_{m=-1/2} = -\frac{e^2E_0^2}{4\hbar}|(J||r||J')|^2\left(\frac{\varepsilon_0\varepsilon_0^*-\varepsilon_{-1}\varepsilon_1^*-\varepsilon_1\varepsilon_{-1}^*}{3}\right)\left(\frac{1}{(\omega_0-\omega)}+\frac{1}{(\omega_0+\omega)}\right) \tag{21}$$

$$\delta E_{m=1/2}-\delta E_{m=-1/2} = -\frac{e^2E_0^2}{4\hbar}|(J||r||lJ')|^2\left(\frac{\varepsilon_1\varepsilon_{-1}^*-\varepsilon_{-1}\varepsilon_1^*}{3}\right)\left(\frac{1}{(\omega_0-\omega)}-\frac{1}{(\omega_0+\omega)}\right) \tag{22}$$

Using the definitions of spherical polarization vectors

$$\varepsilon_1 = -\frac{1}{\sqrt{2}}(\varepsilon_x+i\varepsilon_y) \tag{23}$$

$$\varepsilon_0 = \varepsilon_z \tag{24}$$

$$\varepsilon_{-1} = \frac{1}{\sqrt{2}}(\varepsilon_x-i\varepsilon_y) \tag{25}$$

one can show that $\varepsilon_0\varepsilon_0^*-\varepsilon_{-1}\varepsilon_1^*-\varepsilon_1\varepsilon_{-1}^* = \varepsilon_x^2+\varepsilon_y^2+\varepsilon_z^2 = 1$ and $\varepsilon_1\varepsilon_{-1}^*-\varepsilon_{-1}\varepsilon_1^* = \varepsilon_L^2-\varepsilon_R^2$,where ε_L and ε_R specify the components of left and right circular polarization $\varepsilon = \varepsilon_L(-\hat{x}-i\hat{y})/\sqrt{2}+\varepsilon_R(\hat{x}-i\hat{y})/\sqrt{2}$. Hence we can write

$$\delta E(m) = -\frac{e^2E_0^2}{12\hbar}|(l,s,J||r||l',s,J')|^2\left(\frac{\omega_0}{(\omega_0^2-\omega^2)}+\frac{2\omega(\varepsilon_L^2-\varepsilon_R^2)m}{(\omega_0^2-\omega^2)}\right) \tag{26}$$

It follows that the scalar light shift approaches a constant as $\omega \to 0$, while the vector light shift goes to zero at low frequencies. The vector light shift is proportional to the difference in the intensities of left and right circularly polarized light.

One check the scalar shift using the equation for atomic polarizability away from resonance

$$\alpha = \frac{e^2 f}{m} \frac{1}{\omega_0^2 - \omega^2} \tag{27}$$

Note that this equation agrees with Eq. (4) for $\gamma \ll (\omega - \omega_0) \ll \omega_0$, but it gives a result different by a factor of 2 from Eq. (4) for ω close to zero. That is because Eq. (4) is derived using rotating wave approximation and is not valid for $(\omega - \omega_0) \sim \omega_0$. Using

$$f = \frac{2m\omega_0}{3\hbar(2J+1)} |(l, s, J||r||l', s, J')|^2 \tag{28}$$

we get from Eq. (26)

$$\delta E = -\frac{e^2 E_0^2 f}{4m(\omega_0^2 - \omega^2)} \tag{29}$$

which is in agreement with Eq. (27) since $\delta E = -\alpha E_0^2/4$ for an oscillating electric field with amplitude E_0.