We work on the space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = [0,1]$, $\mathcal{F}$ is the Borel σ-algebra, and $\mathbb{P}$ is the uniform measure. In class, we discussed the following family of random variables:

$$X_a(\omega) = \begin{cases} +a & \omega < a^{-2}, \\ 0 & a^{-2} \le \omega \le 1 - a^{-2}, \\ -a & 1 - a^{-2} < \omega. \end{cases}$$

Note that $\mathbb{E}(X_a) = 0$ and $\mathbb{E}(X_a^2) = 2$ for all a, but $\mathbb{E}(|X_a|^3) = 2|a|$. Evidently the mean and the variance do not tell us much here; however, the third moment shows that for a large, the random variable can actually take very large values on sets of sufficiently small probability. Intuitively, the higher moments tell us something about the probability with which the random variable can take very large values.

The point of the example was to give you some intuition for the notion of $\mathcal{L}^p$, but, as was pointed out by one of you in class (thanks!), the example is not really convincing—for fixed a, the random variable X_a is bounded and is hence in $\mathcal{L}^p$ for any $p > 0$! Let us thus build a set of more convincing examples to sharpen your intuition. I hope this will give you a feeling for what class of random variables $\mathcal{L}^p$ really represents.

The idea is simple: rather than considering a fixed size outlier a, as in the random variable X_a, we will make our new examples take every integer value $n \in \mathbb{N}$ with appropriately scaled probabilities. For $p > 1$, define

$$Y_p(\omega) = n + 1 \quad \text{on} \quad \frac{\sum_{k=0}^{n} k^{-p}}{\sum_{k=0}^{\infty} k^{-p}} \le \omega < \frac{\sum_{k=0}^{n+1} k^{-p}}{\sum_{k=0}^{\infty} k^{-p}}, \qquad n = 0, 1, 2, \ldots,$$

and set $Y_p(1) = 0$. Hence $\mathbb{P}(Y_p = n) = C\, n^{-p}$ by construction for $n = 0, 1, 2, \ldots$. This is like all X_a with integer a rolled into one random variable (when $p = 2$). But:

$$\mathbb{E}(|Y_p|^q) = \sum_{n=0}^{\infty} n^q\, \mathbb{P}(Y_p = n) = C \sum_{n=0}^{\infty} n^{q-p}.$$

Hence $\mathbb{E}(|Y_p|^q) < \infty$ for $q < p - 1$, and $\mathbb{E}(|Y_p|^q) = \infty$ for $q \ge p - 1$.

Can we say something more general? Well, recall Chebyshev's inequality:

$$\mathbb{P}(|X| \ge x) = \mathbb{P}(|X|^q \ge x^q) \le \frac{\mathbb{E}(|X|^q)}{x^q};$$

so if $X \in \mathcal{L}^q$, then the probability $\mathbb{P}(|X| \ge x)$ is at least of order $O(x^{-q})$. Note that this is about right for our example above:

$$\mathbb{P}(Y_p \ge n) = C \sum_{k=n}^{\infty} k^{-p} \approx C \int_n^{\infty} x^{-p}\, dx = \frac{C}{p-1} n^{-(p-1)},$$

so $\mathbb{P}(Y_p \ge x)$ is of order $O(x^{-(p-1)})$, and $Y_p \in \mathcal{L}^q$ for $q < p - 1$. So apparently the Chebyshev bound gives a pretty good idea of what it means to be in $\mathcal{L}^q$!

Ultimately, have we gained much intuition? I guess it is a matter of taste. Intuitively, if X is not in $\mathcal{L}^q$ for small q, this means that it can take very large values with non-negligible probability. If X is not in $\mathcal{L}^2$, for example, then X has infinite variance—something you might already have some feeling for. However, in many cases the condition $X \in \mathcal{L}^q$ shows up as a technical condition in theorems, necessary for the proofs to work out. Whether intuitive or not, it is always a good idea to be careful.