Q. 1. Let W_t be an n-dimensional Wiener process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For non-random $x \in \mathbb{R}^n$, we call the process $W_t^x = x + W_t$ a Brownian motion *started at* x. We are going to investigate the behavior of this process in various dimensions.

1. Consider the annulus $D = \{x : r < \|x\| < R\}$ for some $0 < r < R < \infty$, and define the stopping time $\tau_x = \inf\{t : W_t^x \not\in D\}$. For which functions $h : \mathbb{R}^n \to \mathbb{R}$ is $h(W_{t\wedge\tau_x}^x)$ a martingale for all $x \in D$? You may assume that h is C^2 in some neighborhood of D. (Such functions are called *harmonic*).
2. Using the previous part, show that $h(x) = |x|$ is harmonic for $n = 1$, $h(x) = \log \|x\|$ is harmonic for $n = 2$, and $h(x) = \|x\|^{2-n}$ is harmonic for $n \geq 3$.
3. Let us write $\tau_x^R = \inf\{t : \|W_t^x\| \geq R\}$ and $\tau_x^r = \inf\{t : \|W_t^x\| \leq r\}$. What is $\mathbb{P}(\tau_x^r < \tau_x^R)$ for $n = 1, 2, 3, \ldots$? [**Hint**: $\|W_{\tau_x}^x\|$ can only take values r or R.]
4. What is $\mathbb{P}(\tau_x^r < \infty)$? Conclude the Brownian motion is *recurrent* for dimensions 1 and 2, but not for 3 and higher. [**Hint**: $\{\tau_x^r < \infty\} = \bigcup_{R>r}\{\tau_x^r < \tau_x^R\}$.]

Q. 2. We consider a single stock, which, if we were to invest one dollar at time zero, would be worth $S_t = e^{(\mu-\sigma^2/2)t+\sigma W_t}$ dollars by time t; here $\mu > 0$ (the return rate) and $\sigma > 0$ (the volatility) are constants, and W_t is a Wiener process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. We also have a bank account, which, if we were to deposit one dollar at time zero, would contain $R_t = e^{rt}$ dollars at time t, where $r > 0$ (the interest rate) is constant.

If we invest α_0 dollars in stock and β_0 dollars in the bank at time zero, then at time t our *total wealth* is $X_t = \alpha_0 S_t + \beta_0 R_t$ dollars. We can decide to reinvest at time t, so to put α_t dollars in stock and β_t dollars in the bank. However, if our investment is *self-financing*, then we should make sure that $X_t = \alpha_0 S_t + \beta_0 R_t = \alpha_t S_t + \beta_t R_t$ (i.e., the total amount of invested money is the same: we have just transferred some money from stock to the bank or vice versa, without adding in any new money). Note that we will allow α_t and β_t to be negative: you can borrow money or sell short.

1. Show that if we modify our investment at times $t_1, t_2, \ldots$, then

$$X_{t_{n+1}} = \alpha_0 + \beta_0 + \sum_{i=0}^{n} \alpha_{t_i}(S_{t_{i+1}} - S_{t_i}) + \sum_{i=0}^{n} \beta_{t_i}(R_{t_{i+1}} - R_{t_i}),$$

provided our strategy is self-financing. Show that this expression is identical to

$$X_{t_{n+1}} = X_0 + \int_0^{t_{n+1}} (\mu\alpha_s S_s + r\beta_s R_s)\, ds + \int_0^{t_{n+1}} \sigma\alpha_s S_s\, dW_s,$$

where α_t and β_t are the simple integrands that take the values α_{t_i} and β_{t_i} on the interval $[t_i, t_{i+1}]$, respectively. [Assume that α_{t_i} and β_{t_i} are $\mathcal{F}_{t_i}$-measurable (obviously!) and sufficiently integrable.]

The integral expression for X_t still makes sense for continuous time strategies with $\alpha_t S_t$ and $\beta_t R_t$ in $\mathcal{L}^2(\mu_T \times \mathbb{P})$ (which we will always assume). Hence we can *define* a self-financing strategy to be a pair α_t, β_t that satisfies this expression (in addition to $X_t = \alpha_t S_t + \beta_t R_t$, of course). You can see this as a limit of discrete time strategies.

In a sensible model, we should not be able to find a reasonable strategy α_t, β_t that makes money for nothing. Of course, if we put all our money in the bank, then we will always make money for sure just from the interest. It makes more sense to study the normalized market, where all the prices are *discounted* by the interest rate. So we will consider the discounted wealth $\overline{X}_t = X_t/R_t$ and stock price $\overline{S}_t = S_t/R_t$. We want to show that there does not exist a trading strategy with $\overline{X}_0 = a$, $\overline{X}_t \geq a$ a.s., and $\mathbb{P}(\overline{X}_t > a) > 0$. Such a money-for-nothing opportunity is called *arbitrage*.

2. Show that the discounted wealth at time t is given by

$$\overline{X}_t = X_0 + \int_0^t (\mu - r)\alpha_s \overline{S}_s\, ds + \int_0^t \sigma \alpha_s \overline{S}_s\, dW_s.$$

3. Find a new measure $\mathbb{Q}$ such that $\mathbb{Q} \ll \mathbb{P}$, $\mathbb{P} \ll \mathbb{Q}$, and $\overline{X}_t$ is a martingale under $\mathbb{Q}$ (for reasonable α_t). $\mathbb{Q}$ is called the *equivalent martingale measure*.

4. The equivalent martingale measure has a very special property: $\mathbb{E}_{\mathbb{Q}}(\overline{X}_t) = X_0$ (assuming our initial wealth X_0 is non-random), regardless of the trading strategy. Use this to prove that there is no arbitrage in our model.

We are going to do some simple option pricing theory. Consider something called a *European call option*. This is a contract that says the following: at some predetermined time T (the *maturity*), we are allowed to buy one unit of stock at some predetermined price K (the *strike price*). This is a sort of insurance against the stock price going very high: if the stock price goes below K by time T we can still buy stock at the market price, and we only lose the money we paid to take out the option; if the stock price goes above K by time T, then we make money as we can buy the stock below the market price. The total payoff for us is thus $(S_T - K)^+$, minus the option price. The question is what the seller of the option should charge for that service.

5. If we took out the option, we would make $(S_T - K)^+$ dollars (excluding the option price). Argue that we could obtain exactly the same payoff by implementing a particular trading strategy α_t, β_t, a *hedging strategy*, provided that we have sufficient starting capital (i.e., for some X_0, α_t, β_t, we actually have $X_T = (S_T - K)^+$). Moreover, show that there is only one such strategy.

6. Argue that the starting capital required for the hedging strategy is the only fair price for the option. (If a different price is charged, either we or the seller of the option can make money for nothing.)

7. What is the price of the option? [**Hint:** use the equivalent martingale measure.]

Congratulations—you have just developed the famous Black-Scholes model!

Q. 3 (Bonus question: baby steps in the Malliavin calculus). *Very roughly speaking*, whereas the Itô calculus defines integrals $\int \cdots dW_t$ with respect to the Wiener process, the Malliavin calculus defines *derivatives* "$d \cdots / dW_t$" with respect to the Wiener process. This has applications both in stochastic analysis (smoothness of densities, anticipative calculus) and in finance (computation of sensitivities and hedging strategies, variance reduction of Monte Carlo simulation, insider trading models, etc.) This is a much more advanced topic than we are going deal with in this course. As we have the necessary tools to get started, however, I can't resist having you explore some of the simplest ideas (for fun and extra credit—this is not a required problem!).

We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which is defined a Wiener process W_t with its natural filtration $\mathcal{F}_t = \sigma\{W_s : s \le t\}$. We restrict ourselves to a finite time interval $t \in [0, T]$. An $\mathcal{F}_T$-measurable random variable X is called *cylindrical* if it can be written as $X = f(W_{t_1}, \ldots, W_{t_n})$ for a finite number of times $0 < t_1 < \cdots < t_n \le T$ and some function $f \in C_0^\infty$. For such X, the *Malliavin derivative* of X is defined as

$$\mathbf{D}_t X = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(W_{t_1}, \ldots, W_{t_n})\, I_{t \le t_i}.$$

1. For cylindrical X, prove the *Clark-Ocone formula:*

   $$X = \mathbb{E}(X) + \int_0^T \mathbb{E}(\mathbf{D}_t X | \mathcal{F}_t)\, dW_t.$$

   Hint: look at the proofs of lemmas 4.6.5 and 3.1.9 in the notes.

As any $\mathcal{F}_T$-measurable random variable Y in $\mathcal{L}^2(\mathbb{P})$ can be approximated by cylindrical functions, one can now extend the definition of the Malliavin derivative to a much larger class of random variables by taking limits. Not all such Y are Malliavin differentiable, but with a little work one can define a suitable Sobolev space of differentiable random variables. If you want to learn more about this, see the book by Nualart (1995).

Let us take a less general approach (along the lines of Clark's original result), which allows a beautiful alternative development of the Clark-Ocone formula (the idea is due to Haussmann and Bismut, here we follow D. Williams). Let $f : C([0,T]) \to \mathbb{R}$ be a measurable map. We will consider random variables of the form $X = f(W_\cdot)$ (actually, any $\mathcal{F}_T$-measurable random variable can be written in this way.)

2. Let u_t be bounded and $\mathcal{F}_t$-adapted, and let $\varepsilon \in \mathbb{R}$. Prove the *invariance formula*

   $$\mathbb{E}(f(W_\cdot)) = \mathbb{E}\left[f\left(W_\cdot - \varepsilon \int_0^\cdot u_s\, ds \right) e^{\varepsilon \int_0^T u_s\, dW_s - \frac{\varepsilon^2}{2} \int_0^T (u_s)^2 ds} \right].$$

We are now going to impose a (Fréchet) differentiability condition on f. We assume that for any continuous function x and bounded function α on $[0, T]$, we have

$$f\left(x_\cdot + \varepsilon \int_0^\cdot \alpha_s\, ds \right) - f(x_\cdot) = \varepsilon \int_0^T f'(s, x_\cdot)\, \alpha_s\, ds + o(\varepsilon),$$

where $f' : [0,T] \times C([0,T]) \to \mathbb{R}$ is some measurable function. Then for $X = f(W.)$, we define the Malliavin derivative of X as $\mathbf{D}_t X = f'(t, W.)$.

3. Show that this definition of $\mathbf{D}_t X$ coincides with our previous definition for cylindrical random variables X.

4. Let $X = f(W.)$, and assume for simplicity that $f(x.)$ and $f'(t, x.)$ are bounded. By taking the derivative with respect to ε, at $\varepsilon = 0$, of the invariance formula above, prove the *Malliavin integration by parts formula*

$$\mathbb{E}\left[X \int_0^T u_s \, dW_s\right] = \mathbb{E}\left[\int_0^T u_s \, \mathbf{D}_s X \, ds\right]$$

   for any bounded and $\mathcal{F}_t$-adapted process u_t. Show, furthermore, that

$$\mathbb{E}\left[X \int_0^T u_s \, dW_s\right] = \mathbb{E}\left[\int_0^T u_s \, \mathbb{E}(\mathbf{D}_s X | \mathcal{F}_s) \, ds\right].$$

5. Using the Itô representation theorem, prove that there exists a unique $\mathcal{F}_t$-adapted process C_t such that for any bounded and $\mathcal{F}_t$-adapted process u_t

$$\mathbb{E}\left[X \int_0^T u_s \, dW_s\right] = \mathbb{E}\left[\int_0^T u_s \, C_s \, ds\right].$$

   Conclude that the Clark-Ocone formula still holds in this context.