Only one problem this time, as this is the last homework!

Q. 1. A beautiful butterfly is fluttering around a patch of tasty flowers. At a certain time, the butterfly decides that it has got the most out of its current flower patch, and flies off at a rapid rate in search of fresh flowers. We model the position x_t of the butterfly at time t (in one dimension for simplicity, i.e., $x_t \in \mathbb{R}^1$) by the equation

$$dx_t = \gamma\, I_{\tau \le t}\, dt + \sigma\, dB_t, \qquad x_0 = x,$$

where σ determines the vigorousness of the butterfly's fluttering, τ is the time at which it decides to fly away, γ is the speed at which it flies away, and B_t is a Wiener process. We will assume that τ is exponentially distributed, i.e., that $\mathbb{P}(\tau > t) = e^{-\lambda t}$.

Beside the butterfly the forest also features a biologist, who has come equipped with a butterfly net and a Segway. The biologist can move around at will on his Segway by applying some amount of power u_t; his position z_t^u is then given by the equation

$$\frac{dz_t^u}{dt} = \beta\, u_t, \qquad z_0 = z.$$

Mesmerized by the colorful butterfly, the biologist hatches a plan: he will try to intercept the butterfly at a fixed time T, so that he can catch it and bring it back to his laboratory for further study. However, he would like to keep his total energy consumption low, because he knows from experience that if he runs the battery in the Segway dry he will flop over (and miss the butterfly). As such, the biologist wishes to pursue the butterfly using a strategy u that minimizes the cost functional

$$J[u] = \mathbb{E}\left[P \int_0^T (u_t)^2\, dt + Q\,(x_T - z_T^u)^2\right], \qquad P, Q > 0,$$

where the first term quantifies the total energy consumption and the second term quantifies the effectiveness of the pursuit. The entire setup is depicted in figure 1.

Note that this is a partially observed control problem: the control u_t is allowed to be $\mathcal{F}_t^x = \sigma\{x_s : s \le t\}$-adapted, as the biologist can *see* where the butterfly is, but the biologist does not know the time τ at which the butterfly decides to leave.

1. Define the *predicted interception point* $r_t = \mathbb{E}(x_T|\mathcal{F}_t^x)$. Show that

$$r_t = x_t + \gamma \int_t^T \mathbb{P}(\tau \le s|\mathcal{F}_t^x)\, ds.$$

2. Prove that for $s > t$, we have $1 - \mathbb{P}(\tau \le s|\mathcal{F}_t^x) = e^{-\lambda(s-t)}(1 - \mathbb{P}(\tau \le t|\mathcal{F}_t^x))$. Now obtain an explicit expression for r_t in terms of x_t and $\pi_t = \mathbb{P}(\tau \le t|\mathcal{F}_t^x)$.

3. Using Itô's rule and the appropriate filter (which you can find in the lecture notes), find a stochastic differential equation for (r_t, π_t) which is driven by the innovations process $\bar{B}_t$ and in which τ no longer appears explicitly.

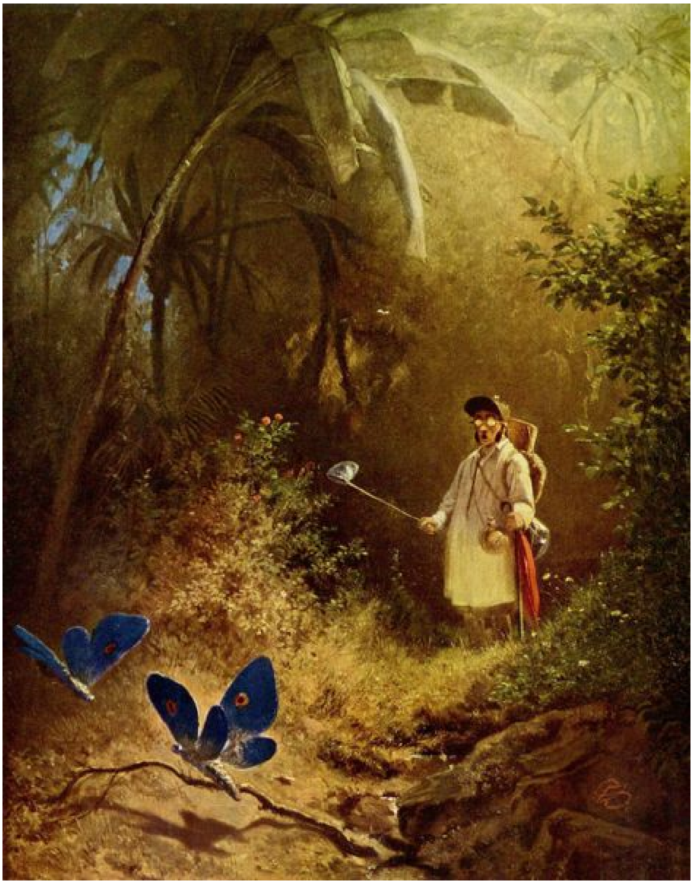

Figure 1: Schematic of problem #1 ("Der Schmetterlingsjäger", Carl Spitzweg, 1840).

4. Define $e_t^u = r_t - z_t^u$. Obtain a stochastic differential equation for (e_t^u, π_t) which is driven by $\bar{B}_t$, and rewrite the cost $J[u]$ in terms of (e_t^u, π_t). You have now converted the partially observed control problem into one with complete observations. What is the corresponding Bellman equation?

5. We are now faced with the difficulty of solving a nonlinear control problem, but nonetheless we will be able to find an analytic solution for the optimal control. To this end, try substituting into the Bellman equation a value function of the form $V(t, e, \pi) = a(t)\, e^2 + b(t, \pi)$, and find equations for $a(t)$ and $b(t, \pi)$. Use this to determine the optimal control strategy. You may assume that the equation you find for $b(t, \pi)$ admits a sufficiently smooth solution (this is in fact the case).

6. Roughly speaking, how could you interpret the optimal strategy? (This is not a deep question, give a one or two line answer.)