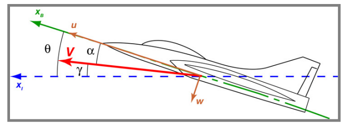
Linearized Longitudinal Equations of Motion

Robert Stengel, Aircraft Flight Dynamics
MAE 331, 2018

Learning Objectives

- 6^{th}-order -> 4^{th}-order -> hybrid equations
- Dynamic stability derivatives
- Long-period (phugoid) mode
- Short-period mode



Reading:
Flight Dynamics
452-464, 482-486

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6^{th}-Order Longitudinal Equations of Motion

Symmetric aircraft
Motions in the vertical plane
Flat earth

Nonlinear Dynamic Equations

$$\dot{u} = X/m - g\sin\theta - qw$$
$$\dot{w} = Z/m + g\cos\theta + qu$$
$$\dot{x}_I = (\cos\theta)u + (\sin\theta)w$$
$$\dot{z}_I = (-\sin\theta)u + (\cos\theta)w$$
$$\dot{q} = M/I_{yy}$$
$$\dot{\theta} = q$$

State Vector, 6 components

$$\begin{bmatrix} u \\ w \\ x \\ z \\ q \\ \theta \end{bmatrix} = \begin{bmatrix} Axial\ Velocity \\ Vertical\ Velocity \\ Range \\ Altitude(-) \\ Pitch\ Rate \\ Pitch\ Angle \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \mathbf{x}_{Lon_6}$$

Range has no dynamic effect
Altitude effect is minimal (air density variation)

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4th-Order Longitudinal Equations of Motion

Nonlinear Dynamic Equations, neglecting range and altitude

$$
\begin{aligned}
\dot{u} &= f_1 = X/m - g\sin\theta - qw \\
\dot{w} &= f_2 = Z/m + g\cos\theta + qu \\
\dot{q} &= f_3 = M/I_{yy} \\
\dot{\theta} &= f_4 = q
\end{aligned}
$$

State Vector, 4 components

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{x}_{Lon_4}
\qquad
\begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} = \begin{bmatrix} \textit{Axial Velocity}, \text{ m/s} \\ \textit{Vertical Velocity}, \text{ m/s} \\ \textit{Pitch Rate}, \text{ rad/s} \\ \textit{Pitch Angle}, \text{ rad} \end{bmatrix}
$$

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Fourth-Order Hybrid Equations of Motion

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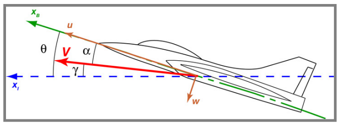

Transform Longitudinal Velocity Components

Replace Cartesian body components of velocity by polar inertial components

$$\dot{u} = f_1 = X/m - g\sin\theta - qw$$
$$\dot{w} = f_2 = Z/m + g\cos\theta + qu$$
$$\dot{q} = f_3 = M/I_{yy}$$
$$\dot{\theta} = f_4 = q$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix} = \begin{bmatrix} \text{Axial Velocity} \\ \text{Vertical Velocity} \\ \text{Pitch Rate} \\ \text{Pitch Angle} \end{bmatrix}$$

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Transform Longitudinal Velocity Components

Replace Cartesian body components of velocity by polar inertial components

$$\dot{V} = f_1 = \left[T\cos(\alpha + i) - D - mg\sin\gamma\right]/m$$
$$\dot{\gamma} = f_2 = \left[T\sin(\alpha + i) + L - mg\cos\gamma\right]/mV$$
$$\dot{q} = f_3 = M/I_{yy}$$
$$\dot{\theta} = f_4 = q$$

Hawker P1127 Kestral

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} V \\ \gamma \\ q \\ \theta \end{bmatrix} = \begin{bmatrix} \text{Velocity} \\ \text{Flight Path Angle} \\ \text{Pitch Rate} \\ \text{Pitch Angle} \end{bmatrix}$$

i = Incidence angle of the thrust vector with respect to the centerline

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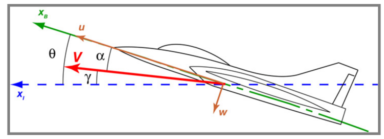
Hybrid Longitudinal Equations of Motion



- **Replace pitch angle by angle of attack**

$$\alpha = \theta - \gamma$$

$$\dot{V} = f_1 = \left[T\cos(\alpha + i) - D - mg\sin\gamma\right]/m$$
$$\dot{\gamma} = f_2 = \left[T\sin(\alpha + i) + L - mg\cos\gamma\right]/mV$$
$$\dot{q} = f_3 = M / I_{yy}$$
$$\dot{\theta} = f_4 = q$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} V \\ \gamma \\ q \\ \theta \end{bmatrix} = \begin{bmatrix} Velocity \\ Flight\ Path\ Angle \\ Pitch\ Rate \\ Pitch\ Angle \end{bmatrix}$$

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Hybrid Longitudinal Equations of Motion

- **Replace pitch angle by angle of attack**

$$\alpha = \theta - \gamma$$

$$\dot{V} = f_1 = \left[T\cos(\alpha + i) - D - mg\sin\gamma\right]/m$$
$$\dot{\gamma} = f_2 = \left[T\sin(\alpha + i) + L - mg\cos\gamma\right]/mV$$
$$\dot{q} = f_3 = M / I_{yy}$$
$$\dot{\alpha} = \dot{\theta} - \dot{\gamma} = f_4 = q - f_2 = q - \frac{1}{mV}\left[T\sin(\alpha + i) + L - mg\cos\gamma\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} V \\ \gamma \\ q \\ \alpha \end{bmatrix} = \begin{bmatrix} Velocity \\ Flight\ Path\ Angle \\ Pitch\ Rate \\ Angle\ of\ Attack \end{bmatrix}$$

$$\theta = \alpha + \gamma$$

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Why Transform Equations and State Vector?

$$\begin{bmatrix} x_1 \\ x_2 \\ \hdashline x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} V \\ \gamma \\ \hdashline q \\ \alpha \end{bmatrix} = \begin{bmatrix} Velocity \\ Flight\ Path\ Angle \\ \hdashline Pitch\ Rate \\ Angle\ of\ Attack \end{bmatrix}$$

Velocity and flight path angle typically have slow variations

Pitch rate and angle of attack typically have quicker variations

Coupling typically small

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Small Perturbations from Steady Path

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \dot{\mathbf{x}}_N(t) + \Delta\dot{\mathbf{x}}(t) \\ &\approx \mathbf{f}[\mathbf{x}_N(t), \mathbf{u}_N(t), \mathbf{w}_N(t), t] \\ &\quad + \mathbf{F}(t)\Delta\mathbf{x}(t) + \mathbf{G}(t)\Delta\mathbf{u}(t) + \mathbf{L}(t)\Delta\mathbf{w}(t) \end{aligned}$$

Steady, Level Flight

$$\dot{\mathbf{x}}_N(t) \equiv \mathbf{0} \approx \mathbf{f}[\mathbf{x}_N(t), \mathbf{u}_N(t), \mathbf{w}_N(t), t]$$
$$\Delta\dot{\mathbf{x}}(t) \approx \mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t) + \mathbf{L}\Delta\mathbf{w}(t)$$

Rates of change are "small"

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Nominal Equations of Motion in Equilibrium (Trimmed Condition)

$$\dot{\mathbf{x}}_N(t) = \mathbf{0} = \mathbf{f}[\mathbf{x}_N(t), \mathbf{u}_N(t), \mathbf{w}_N(t), t]$$

$$\mathbf{x}_N^T = \begin{bmatrix} V_N & \gamma_N & 0 & \alpha_N \end{bmatrix}^T = \text{constant}$$

T, ***D***, ***L***, **and** ***M*** **contain state, control, and disturbance effects**

$$\begin{aligned}
\dot{V}_N &= 0 = f_1 = \left[T\cos(\alpha_N + i) - D - mg\sin\gamma_N\right]/m \\
\dot{\gamma}_N &= 0 = f_2 = \left[T\sin(\alpha_N + i) + L - mg\cos\gamma_N\right]/mV_N \\
\dot{q}_N &= 0 = f_3 = M/I_{yy} \\
\dot{\alpha}_N &= 0 = f_4 = (0) - \frac{1}{mV_N}\left[T\sin(\alpha_N + i) + L - mg\cos\gamma_N\right]
\end{aligned}$$

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Flight Conditions for Steady, Level Flight

Nonlinear longitudinal model

$$\begin{aligned}
\dot{V} &= f_1 = \frac{1}{m}\left[T\cos(\alpha + i) - D - mg\sin\gamma\right] \\
\dot{\gamma} &= f_2 = \frac{1}{mV}\left[T\sin(\alpha + i) + L - mg\cos\gamma\right] \\
\dot{q} &= f_3 = M / I_{yy} \\
\dot{\alpha} &= f_4 = \dot{\theta} - \dot{\gamma} = q - \frac{1}{mV}\left[T\sin(\alpha + i) + L - mg\cos\gamma\right]
\end{aligned}$$

Nonlinear longitudinal model in equilibrium

$$\begin{aligned}
\mathbf{0} &= f_1 = \frac{1}{m}\left[T\cos(\alpha + i) - D - mg\sin\gamma\right] \\
\mathbf{0} &= f_2 = \frac{1}{mV}\left[T\sin(\alpha + i) + L - mg\cos\gamma\right] \\
\mathbf{0} &= f_3 = M / I_{yy} \\
\mathbf{0} &= f_4 = \dot{\theta} - \dot{\gamma} = q - \frac{1}{mV}\left[T\sin(\alpha + i) + L - mg\cos\gamma\right]
\end{aligned}$$

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Numerical Solution for Level Flight Trimmed Condition

- Specify <u>desired</u> altitude and airspeed, h_N and V_N
- Guess starting values for the trim parameters, δT_0, δE_0, and θ_0
- Calculate starting values of f_1, f_2, and f_3

$$f_1 = \dot{V} = \frac{1}{m}\left[T(\delta T, \delta E, \theta, h, V)\cos(\alpha + i) - D(\delta T, \delta E, \theta, h, V)\right]$$

$$f_2 = \dot{\gamma} = \frac{1}{mV_N}\left[T(\delta T, \delta E, \theta, h, V)\sin(\alpha + i) + L(\delta T, \delta E, \theta, h, V) - mg\right]$$

$$f_3 = \dot{q} = M(\delta T, \delta E, \theta, h, V) / I_{yy}$$

- f_1, f_2, and f_3 = 0 in equilibrium, but not for arbitrary δT_0, δE_0, and θ_0
- Define a scalar, positive-definite trim error cost function, e.g.,

$$J(\delta T, \delta E, \theta) = a(f_1^2) + b(f_2^2) + c(f_3^2)$$

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Minimize the Cost Function with Respect to the Trim Parameters

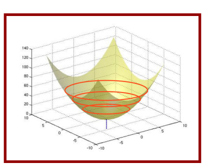
Error cost is "bowl-shaped"

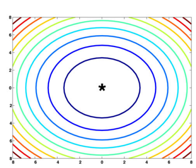
$$J(\delta T, \delta E, \theta) = a(f_1^2) + b(f_2^2) + c(f_3^2)$$

Cost is minimized at bottom of bowl, i.e., when

$$\left[\begin{array}{ccc} \frac{\partial J}{\partial \delta T} & \frac{\partial J}{\partial \delta E} & \frac{\partial J}{\partial \theta} \end{array}\right] = \mathbf{0}$$

Search to find the minimum value of J





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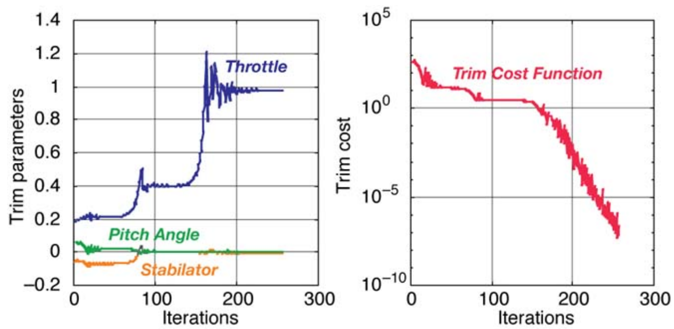
Example of Search for Trimmed Condition (Fig. 3.6-9, *Flight Dynamics*)



In MATLAB, use *fminsearch* or *fsolve* to find trim settings

$$\left(\delta T^*, \delta E^*, \theta^*\right) = \text{fminsearch}\left[J, \left(\delta T, \delta E, \theta\right)\right]$$

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Small Perturbations from Steady Path Approximated by Linear Equations

Linearized Equations of Motion

$$\Delta\dot{\mathbf{x}}_{Lon} = \begin{bmatrix} \Delta\dot{V} \\ \Delta\dot{\gamma} \\ \Delta\dot{q} \\ \Delta\dot{\alpha} \end{bmatrix} = \mathbf{F}_{Lon} \begin{bmatrix} \Delta V \\ \Delta\gamma \\ \Delta q \\ \Delta\alpha \end{bmatrix} + \mathbf{G}_{Lon} \begin{bmatrix} \Delta\delta T \\ \Delta\delta E \\ \cdots \end{bmatrix} + \cdots$$

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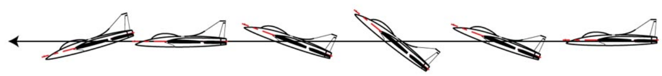
Linearized Equations of Motion

Phugoid (Long-Period) Motion

Short-Period Motion



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Approximate Decoupling of Fast and Slow Modes of Motion

Hybrid linearized equations allow the two modes to be examined separately

Effects of **phugoid** perturbations on **phugoid** motion	Effects of **short-period** perturbations on **phugoid** motion

$$\mathbf{F}_{Lon} = \left[\begin{array}{c|c} \mathbf{F}_{Ph} & \mathbf{F}_{SP}^{Ph} \\ \hline \mathbf{F}_{Ph}^{SP} & \mathbf{F}_{SP} \end{array}\right]$$

Effects of **phugoid** perturbations on **short-period** motion	Effects of **short-period** perturbations on **short-period** motion

$$= \left[\begin{array}{c|c} \mathbf{F}_{Ph} & small \\ \hline small & \mathbf{F}_{SP} \end{array}\right] \approx \left[\begin{array}{c|c} \mathbf{F}_{Ph} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{F}_{SP} \end{array}\right]$$

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Sensitivity Matrices for Longitudinal LTI Model

$$\Delta\dot{\mathbf{x}}_{Lon}(t) = \mathbf{F}_{Lon}\Delta\mathbf{x}_{Lon}(t) + \mathbf{G}_{Lon}\Delta\mathbf{u}_{Lon}(t) + \mathbf{L}_{Lon}\Delta\mathbf{w}_{Lon}(t)$$

$$\mathbf{F}_{Lon} = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix}$$

$$\mathbf{G}_{Lon} = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \\ G_{41} & G_{42} & G_{43} \end{bmatrix} \qquad \mathbf{L}_{Lon} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \\ L_{31} & L_{32} \\ L_{41} & L_{42} \end{bmatrix}$$

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Dimensional Stability and Control Derivatives

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Dimensional Stability-Derivative Notation

- Redefine force and moment symbols as acceleration symbols
- Dimensional stability derivatives portray acceleration sensitivities to state perturbations

$$\frac{Drag}{mass\,(m)} \Rightarrow D \propto \dot{V},\ \mathrm{m/s^2}$$

$$\frac{Lift}{mass} \Rightarrow L \propto V\dot{\gamma},\ \mathrm{m/s^2} \quad \Rightarrow L/V \propto \dot{\gamma},\ \mathrm{rad/s^2}$$

$$\frac{Moment}{moment\ of\ inertia\,(I_{yy})} \Rightarrow M \propto \dot{q},\ \mathrm{rad/s^2}$$

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Dimensional Stability-Derivative Notation

$$F_{11} = \frac{\partial D}{\partial V} = -D_V \triangleq \frac{1}{m}\left[\left(C_{T_V}\cos\alpha_N - C_{D_V}\right)\frac{\rho_N V_N^2}{2}S + \left(C_{T_N}\cos\alpha_N - C_{D_N}\right)\rho_N V_N S\right]$$

Thrust and drag effects are combined and represented by one symbol

$$F_{24} = \frac{\partial L}{\partial \alpha}\Big/V_N = L_\alpha\!\!\not\,/V_N \triangleq \frac{1}{mV_N}\left[\left(C_{T_N}\cos\alpha_N + C_{L_\alpha}\right)\frac{\rho_N V_N^2}{2}S\right]$$

Thrust and lift effects are combined and represented by one symbol

$$F_{34} = \frac{\partial M}{\partial \alpha} = M_\alpha \triangleq \frac{1}{I_{yy}}\left[C_{m_\alpha}\frac{\rho_N V_N^2}{2}S\bar{c}\right]$$

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Longitudinal Stability Matrix

Effects of phugoid perturbations on phugoid motion

Effects of short-period perturbations on phugoid motion

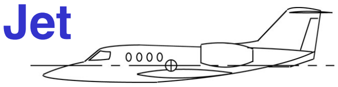
$$\mathbf{F}_{Lon} = \begin{bmatrix} \mathbf{F}_{Ph} & \mathbf{F}_{SP}^{Ph} \\ \mathbf{F}_{Ph}^{SP} & \mathbf{F}_{SP} \end{bmatrix} = \left[\begin{array}{cc|cc} -D_V & -g\cos\gamma_N & -D_q & -D_\alpha \\ L_V/V_N & \dfrac{g}{V_N}\sin\gamma_N & L_q/V_N & L_\alpha/V_N \\ \hline M_V & 0 & M_q & M_\alpha \\ -L_V/V_N & -\dfrac{g}{V_N}\sin\gamma_N & \left(1 - L_q/V_N\right) & -L_\alpha/V_N \end{array}\right]$$

Effects of phugoid perturbations on short-period motion

Effects of short-period perturbations on short-period motion

Comparison of 2nd- and 4th-Order Model Response

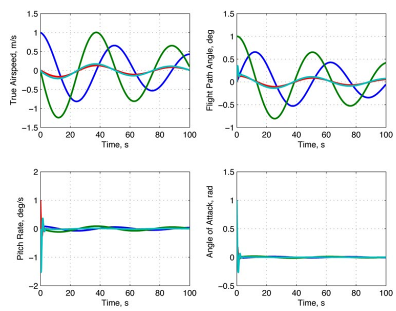
4^{th}-Order Initial-Condition Responses of Business Jet at Two Time Scales



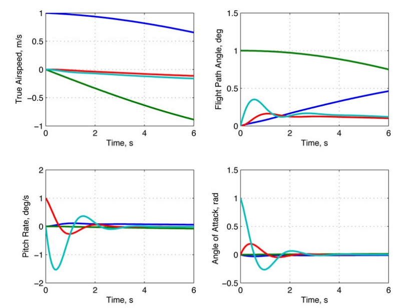
Plotted over different periods of time

4 initial conditions [$V(0)$, $\gamma(0)$, $q(0)$, $\alpha(0)$]

0 - 100 sec
- **Reveals Phugoid Mode**



0 - 6 sec
- **Reveals Short-Period Mode**



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2^{nd}-Order Models of Longitudinal Motion

Assume off-diagonal blocks of (4 x 4) stability matrix are negligible

$$\mathbf{F}_{Lon} = \begin{bmatrix} \mathbf{F}_{Ph} & \sim 0 \\ \sim 0 & \mathbf{F}_{SP} \end{bmatrix}$$

Approximate Phugoid Equation

$$\Delta\dot{\mathbf{x}}_{Ph} =$$

$$\begin{bmatrix} \Delta\dot{V} \\ \Delta\dot{\gamma} \end{bmatrix} \approx \begin{bmatrix} -D_V & -g\cos\gamma_N \\ L_V / V_N & \dfrac{g}{V_N}\sin\gamma_N \end{bmatrix} \begin{bmatrix} \Delta V \\ \Delta\gamma \end{bmatrix}$$

$$+ \begin{bmatrix} T_{\delta T} \\ L_{\delta T} / V_N \end{bmatrix} \Delta\delta T + \begin{bmatrix} -D_V \\ L_V / V_N \end{bmatrix} \Delta V_{wind}$$

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2^{nd}-Order Models of Longitudinal Motion

$$\mathbf{F}_{Lon} = \begin{bmatrix} \mathbf{F}_{Ph} & \sim 0 \\ \sim 0 & \mathbf{F}_{SP} \end{bmatrix}$$

Approximate Short-Period Equation

$$\Delta\dot{\mathbf{x}}_{SP} =$$

$$\begin{bmatrix} \Delta\dot{q} \\ \Delta\dot{\alpha} \end{bmatrix} \approx \begin{bmatrix} M_q & M_\alpha \\ \left(1 - L_q/V_N\right) & -L_\alpha/V_N \end{bmatrix} \begin{bmatrix} \Delta q \\ \Delta\alpha \end{bmatrix}$$

$$+ \begin{bmatrix} M_{\delta E} \\ -L_{\delta E}/V_N \end{bmatrix} \Delta\delta E + \begin{bmatrix} M_\alpha \\ -L_\alpha/V_N \end{bmatrix} \Delta\alpha_{wind}$$

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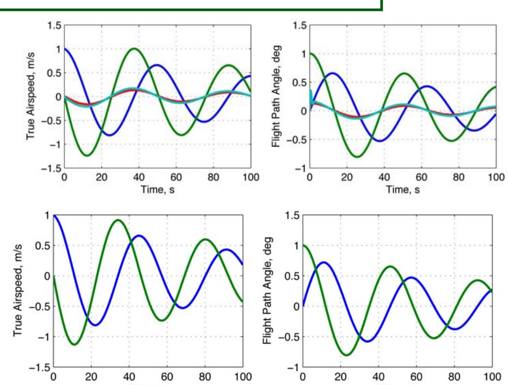
Comparison of Bizjet 4^{th}- and 2^{nd}-Order Model Responses

Phugoid Time Scale, ~100 s

4^{th} Order,
4 initial conditions
$[V(0), \gamma(0), q(0), \alpha(0)]$

2^{nd} Order,
2 initial conditions
$[V(0), \gamma(0)]$



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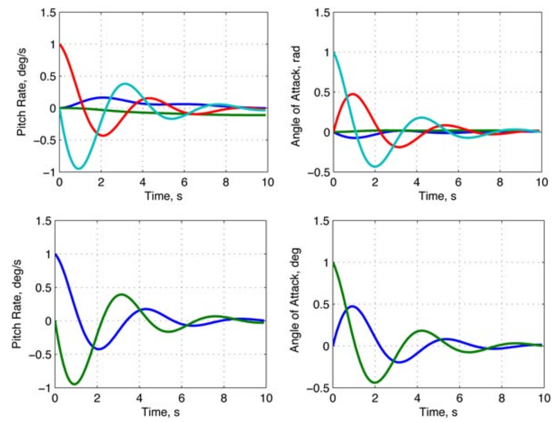
Comparison of Bizjet 4th- and 2nd-Order Model Responses

Short-Period Time Scale, ~10 s

4th Order,
4 initial conditions
[$V(0)$, $\gamma(0)$, $q(0)$, $\alpha(0)$]

2nd Order,
2 initial conditions
[$q(0)$, $\alpha(0)$]



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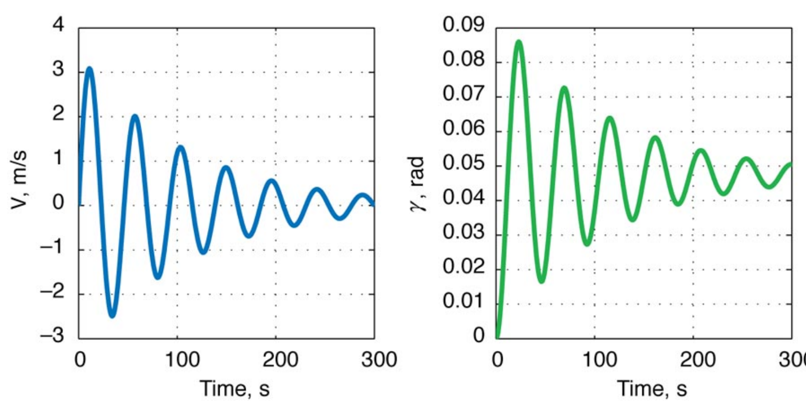
Approximate Phugoid Response to a 10% Thrust Increase



What is the steady-state response?

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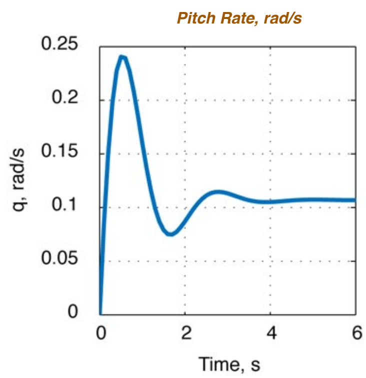
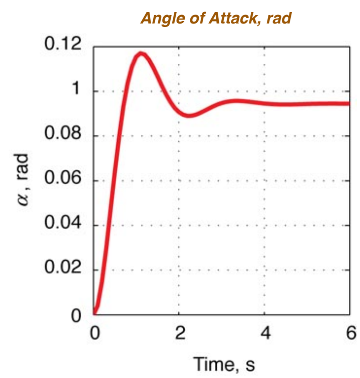
Approximate Short-Period Response to a 0.1-Rad Pitch Control Step Input





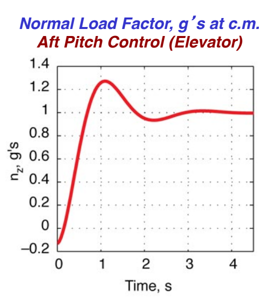
What is the steady-state response?

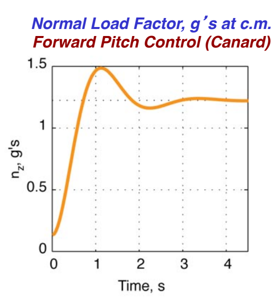
Normal Load Factor Response to a 0.1-Rad Pitch Control Step Input

- **Normal load factor at the center of mass**

$$\Delta n_z = \frac{V_N}{g}\left(\Delta\dot{\alpha} - \Delta q\right) = \frac{V_N}{g}\left(\frac{L_\alpha}{V_N}\Delta\alpha + \frac{L_{\delta E}}{V_N}\Delta\delta E\right)$$

- **Pilot focuses on normal load factor during rapid maneuvering**





Historical Factoids

Flying Cars-1

Curtiss Autocar, 1917

Waterman Arrowbile, 1935

Stout Skycar, 1931

ConsolidatedVultee 111, 1940s

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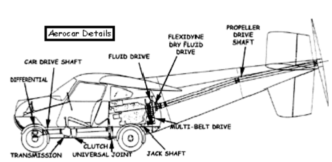
Flying Cars-2

ConvAIRCAR 116 (w/Crosley auto), 1940s

Hallock Road Wing , 1957

Taylor AirCar, 1950s



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Flying Cars-3

"Mitzar" SkyMaster Pinto, 1970s

Lotus Elise Aerocar, concept, 2002

Haynes Skyblazer, concept, 2004

Aeromobil, 2014

Aircraft entered a spin, ballistic parachute was deployed. No fatality.

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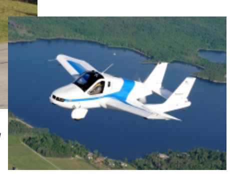
Flying Cars-4

Terrafugia Transition



[2019, $400K (est)]

Terrafugia TF-X, concept

Date? Cost?

… or, now, for the same price

PLUS

plus $120K in savings account

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Next Time: Lateral-Directional Dynamics

Reading:
Flight Dynamics
574-591

Learning Objectives

- 6^{th}-order -> 4^{th}-order -> hybrid equations
- Dynamic stability derivatives
- Dutch roll mode
- Roll and spiral modes

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Supplemental Material

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Elements of the Stability Matrix

Stability derivatives portray acceleration sensitivities to state perturbations

$$\frac{\partial f_1}{\partial V} \equiv -D_V; \quad \frac{\partial f_1}{\partial \gamma} = -g\cos\gamma_N; \quad \frac{\partial f_1}{\partial q} \equiv -D_q; \quad \frac{\partial f_1}{\partial \alpha} \equiv -D_\alpha$$

$$\frac{\partial f_2}{\partial V} \equiv L_V / V_N; \quad \frac{\partial f_2}{\partial \gamma} = \frac{g}{V_N}\sin\gamma_N; \quad \frac{\partial f_2}{\partial q} \equiv L_q / V_N; \quad \frac{\partial f_2}{\partial \alpha} \equiv L_\alpha / V_N$$

$$\frac{\partial f_3}{\partial V} \equiv M_V; \quad \frac{\partial f_3}{\partial \gamma} = 0; \quad \frac{\partial f_3}{\partial q} \equiv M_q; \quad \frac{\partial f_3}{\partial \alpha} \equiv M_\alpha$$

$$\frac{\partial f_4}{\partial V} \equiv -L_V / V_N; \quad \frac{\partial f_4}{\partial \gamma} = -\frac{g}{V_N}\sin\gamma_N; \quad \frac{\partial f_4}{\partial q} \equiv 1 - L_q / V_N; \quad \frac{\partial f_4}{\partial \alpha} \equiv -L_\alpha / V_N$$

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Velocity Dynamics

First row of nonlinear equation

$$\dot{V} = f_1 = \frac{1}{m}\left[T\cos\alpha - D - mg\sin\gamma\right]$$
$$= \frac{1}{m}\left[C_T\cos\alpha\frac{\rho V^2}{2}S - C_D\frac{\rho V^2}{2}S - mg\sin\gamma\right]$$

First row of linearized equation

$$\Delta\dot{V}(t) = \left[F_{11}\Delta V(t) + F_{12}\Delta\gamma(t) + F_{13}\Delta q(t) + F_{14}\Delta\alpha(t)\right]$$
$$+\left[G_{11}\Delta\delta E(t) + G_{12}\Delta\delta T(t) + G_{13}\Delta\delta F(t)\right]$$
$$+\left[L_{11}\Delta V_{wind} + L_{12}\Delta\alpha_{wind}\right]$$

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Sensitivity of Velocity Dynamics to State Perturbations

$$\dot{V} = \left[\left(C_T \cos\alpha - C_D \right) \frac{\rho V^2}{2} S - mg \sin\gamma \right] \Big/ m$$

Coefficients in first row of F

$$F_{11} = \frac{1}{m}\left[\left(C_{T_V} \cos\alpha_N - C_{D_V} \right) \frac{\rho_N V_N^{\ 2}}{2} S + \left(C_{T_N} \cos\alpha_N - C_{D_N} \right) \rho_N V_N S \right]$$

$$F_{12} = -g \cos\gamma_N$$

$$F_{13} = \frac{-1}{m}\left[C_{D_q} \frac{\rho_N V_N^{\ 2}}{2} S \right]$$

$$F_{14} = \frac{-1}{m}\left[\left(C_{T_N} \sin\alpha_N + C_{D_\alpha} \right) \frac{\rho_N V_N^{\ 2}}{2} S \right]$$

$$C_{T_V} \equiv \frac{\partial C_T}{\partial V}$$

$$C_{D_V} \equiv \frac{\partial C_D}{\partial V}$$

$$C_{D_q} \equiv \frac{\partial C_D}{\partial q}$$

$$C_{D_\alpha} \equiv \frac{\partial C_D}{\partial \alpha}$$

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Sensitivity of Velocity Dynamics to Control and Disturbance Perturbations

Coefficients in first rows of G and L

$$G_{11} = \frac{-1}{m}\left[C_{D_{\delta E}} \frac{\rho_N V_N^{\ 2}}{2} S \right]$$

$$G_{12} = \frac{1}{m}\left[C_{T_{\delta T}} \cos\alpha_N \frac{\rho_N V_N^{\ 2}}{2} S \right]$$

$$G_{13} = \frac{-1}{m}\left[C_{D_{\delta F}} \frac{\rho_N V_N^{\ 2}}{2} S \right]$$

$$C_{T_{\delta T}} \equiv \frac{\partial C_T}{\partial \delta T}$$

$$C_{D_{\delta E}} \equiv \frac{\partial C_D}{\partial \delta E}$$

$$C_{D_{\delta F}} \equiv \frac{\partial C_D}{\partial \delta F}$$

$$L_{11} = -\frac{\partial f_1}{\partial V}$$

$$L_{12} = -\frac{\partial f_1}{\partial \alpha}$$

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Flight Path Angle Dynamics

Second row of nonlinear equation

$$\dot{\gamma} = f_2 = \frac{1}{mV}\left[T\sin\alpha + L - mg\cos\gamma\right]$$
$$= \frac{1}{mV}\left[C_T \sin\alpha \frac{\rho V^2}{2} S + C_L \frac{\rho V^2}{2} S - mg\cos\gamma\right]$$

Second row of linearized equation

$$\Delta\dot{\gamma}(t) = \left[F_{21}\Delta V(t) + F_{22}\Delta\gamma(t) + F_{23}\Delta q(t) + F_{24}\Delta\alpha(t)\right]$$
$$+\left[G_{21}\Delta\delta E(t) + G_{22}\Delta\delta T(t) + G_{23}\Delta\delta F(t)\right]$$
$$+\left[L_{21}\Delta V_{wind} + L_{22}\Delta\alpha_{wind}\right]$$

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Sensitivity of Flight Path Angle Dynamics to State Perturbations

$$\dot{\gamma} = \left[\left(C_T \sin\alpha + C_L\right)\frac{\rho V^2}{2} S - mg\cos\gamma\right] \Big/ mV$$

Coefficients in second row of F

$$F_{21} = \frac{1}{mV_N}\left[\left(C_{T_V}\sin\alpha_N + C_{L_V}\right)\frac{\rho_N {V_N}^2}{2} S + \left(C_{T_N}\sin\alpha_N + C_{L_N}\right)\rho_N V_N S\right]$$
$$- \frac{1}{m{V_N}^2}\left[\left(C_{T_N}\sin\alpha_N + C_{L_N}\right)\frac{\rho_N {V_N}^2}{2} S - mg\cos\gamma_N\right]$$

$$F_{22} = g\sin\gamma_N / V_N$$

$$F_{23} = \frac{1}{mV_N}\left[C_{L_q}\frac{\rho_N {V_N}^2}{2} S\right]$$

$$F_{24} = \frac{1}{mV_N}\left[\left(C_{T_N}\cos\alpha_N + C_{L_\alpha}\right)\frac{\rho_N {V_N}^2}{2} S\right]$$

$$C_{T_V} \equiv \frac{\partial C_T}{\partial V}$$
$$C_{L_V} \equiv \frac{\partial C_L}{\partial V}$$
$$C_{L_q} \equiv \frac{\partial C_L}{\partial q}$$
$$C_{L_\alpha} \equiv \frac{\partial C_L}{\partial \alpha}$$

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Pitch Rate Dynamics

Third row of nonlinear equation

$$\dot{q} = f_3 = \frac{M}{I_{yy}} = \frac{C_m\left(\rho V^2/2\right)S\overline{c}}{I_{yy}}$$

Third row of linearized equation

$$\begin{aligned}\Delta\dot{q}(t) = &\left[F_{31}\Delta V(t) + F_{32}\Delta\gamma(t) + F_{33}\Delta q(t) + F_{34}\Delta\alpha(t)\right]\\ &+\left[G_{31}\Delta\delta E(t) + G_{32}\Delta\delta T(t) + G_{33}\Delta\delta F(t)\right]\\ &+\left[L_{31}\Delta V_{wind} + L_{32}\Delta\alpha_{wind}\right]\end{aligned}$$

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Sensitivity of Pitch Rate Dynamics to State Perturbations

$$\dot{q} = C_m\left(\rho V^2/2\right)S\overline{c}/I_{yy}$$

Coefficients in third row of F

$$F_{31} = \frac{1}{I_{yy}}\left[C_{m_V}\frac{\rho_N V_N^{\,2}}{2}S\overline{c} + C_{m_N}\rho_N V_N S\overline{c}\right]$$

$$F_{32} = 0$$

$$F_{33} = \frac{1}{I_{yy}}\left[C_{m_q}\frac{\rho_N V_N^{\,2}}{2}S\overline{c}\right]$$

$$F_{34} = \frac{1}{I_{yy}}\left[C_{m_\alpha}\frac{\rho_N V_N^{\,2}}{2}S\overline{c}\right]$$

$$C_{m_V} \equiv \frac{\partial C_m}{\partial V}$$

$$C_{m_q} \equiv \frac{\partial C_m}{\partial q}$$

$$C_{m_\alpha} \equiv \frac{\partial C_m}{\partial \alpha}$$

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Angle of Attack Dynamics

Fourth row of nonlinear equation

$$\dot{\alpha} = f_4 = \dot{\theta} - \dot{\gamma} = q - \frac{1}{mV}\left[T\sin\alpha + L - mg\cos\gamma\right]$$

Fourth row of linearized equation

$$\begin{aligned}\Delta\dot{\alpha}(t) = &\left[F_{41}\Delta V(t) + F_{42}\Delta\gamma(t) + F_{43}\Delta q(t) + F_{44}\Delta\alpha(t)\right]\\ &+\left[G_{41}\Delta\delta E(t) + G_{42}\Delta\delta T(t) + G_{43}\Delta\delta F(t)\right]\\ &+\left[L_{41}\Delta V_{wind} + L_{42}\Delta\alpha_{wind}\right]\end{aligned}$$

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Sensitivity of Angle of Attack Dynamics to State Perturbations

$$\dot{\alpha} = \dot{\theta} - \dot{\gamma} = q - \dot{\gamma}$$

Coefficients in fourth row of F

$$F_{41} = -F_{21}$$

$$F_{43} = 1 - F_{23}$$

$$F_{42} = -F_{22}$$

$$F_{44} = -F_{24}$$

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Alternative Approach:
Numerical Calculation of the Sensitivity Matrices *("1st Differences")*

$$\frac{\partial f_1}{\partial V}(t) \approx \frac{f_1\begin{bmatrix}(V+\Delta V)\\ \gamma\\ q\\ \alpha\end{bmatrix} - f_1\begin{bmatrix}(V-\Delta V)\\ \gamma\\ q\\ \alpha\end{bmatrix}\Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta V}; \quad \frac{\partial f_1}{\partial \gamma}(t) \approx \frac{f_1\begin{bmatrix}V\\ (\gamma+\Delta\gamma)\\ q\\ \alpha\end{bmatrix} - f_1\begin{bmatrix}V\\ (\gamma-\Delta\gamma)\\ q\\ \alpha\end{bmatrix}\Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta \gamma}$$

$$\frac{\partial f_2}{\partial V}(t) \approx \frac{f_2\begin{bmatrix}(V+\Delta V)\\ \gamma\\ q\\ \alpha\end{bmatrix} - f_2\begin{bmatrix}(V-\Delta V)\\ \gamma\\ q\\ \alpha\end{bmatrix}\Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta V}; \quad \frac{\partial f_2}{\partial \gamma}(t) \approx \frac{f_2\begin{bmatrix}V\\ (\gamma+\Delta\gamma)\\ q\\ \alpha\end{bmatrix} - f_2\begin{bmatrix}V\\ (\gamma-\Delta\gamma)\\ q\\ \alpha\end{bmatrix}\Bigg|_{\substack{\mathbf{x}=\mathbf{x}_N(t)\\ \mathbf{u}=\mathbf{u}_N(t)\\ \mathbf{w}=\mathbf{w}_N(t)}}}{2\Delta \gamma}$$

Remaining elements of F(*t*), G(*t*), and L(*t*) calculated accordingly

Control and Disturbance Sensitivities in Flight Path Angle, Pitch Rate, and Angle-of-Attack Dynamics

$$\frac{\partial f_2}{\partial \delta E} = \frac{1}{mV_N}\left[C_{L_{\delta E}} \frac{\rho V_N^2}{2} S\right]$$
$$\frac{\partial f_2}{\partial \delta T} = \frac{1}{mV_N}\left[C_{T_{\delta T}} \sin\alpha_N \frac{\rho V_N^2}{2} S\right]$$
$$\frac{\partial f_2}{\partial \delta F} = \frac{1}{mV_N}\left[C_{L_{\delta F}} \frac{\rho V_N^2}{2} S\right]$$

$$\frac{\partial f_3}{\partial \delta E} = \frac{1}{I_{yy}}\left[C_{m_{\delta E}} \frac{\rho V_N^2}{2} S\overline{c}\right]$$
$$\frac{\partial f_3}{\partial \delta T} = \frac{1}{I_{yy}}\left[C_{m_{\delta T}} \frac{\rho V_N^2}{2} S\overline{c}\right]$$
$$\frac{\partial f_3}{\partial \delta F} = \frac{1}{I_{yy}}\left[C_{m_{\delta F}} \frac{\rho V_N^2}{2} S\overline{c}\right]$$

$$\frac{\partial f_4}{\partial \delta E} = -\frac{\partial f_2}{\partial \delta E}$$
$$\frac{\partial f_4}{\partial \delta T} = -\frac{\partial f_2}{\partial \delta T}$$
$$\frac{\partial f_4}{\partial \delta F} = -\frac{\partial f_2}{\partial \delta F}$$

$$\frac{\partial f_2}{\partial V_{wind}} = -\frac{\partial f_2}{\partial V} \qquad \frac{\partial f_2}{\partial \alpha_{wind}} = -\frac{\partial f_2}{\partial \alpha}$$

$$\frac{\partial f_3}{\partial V_{wind}} = -\frac{\partial f_3}{\partial V} \qquad \frac{\partial f_3}{\partial \alpha_{wind}} = -\frac{\partial f_3}{\partial \alpha}$$

$$\frac{\partial f_4}{\partial V_{wind}} = \frac{\partial f_2}{\partial V} \qquad \frac{\partial f_3}{\partial \alpha_{wind}} = \frac{\partial f_2}{\partial \alpha}$$

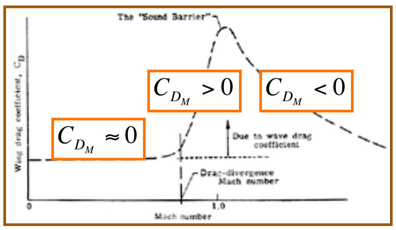
Velocity-Dependent Derivative Definitions

Air compressibility effects are a principal source of velocity dependence

$$C_{D_M} \equiv \frac{\partial C_D}{\partial M} = \frac{\partial C_D}{\partial (V/a)} = a\frac{\partial C_D}{\partial V}$$

a = Speed of Sound

M = Mach number = V/a



$$C_{D_V} \equiv \frac{\partial C_D}{\partial V} = \left(\frac{1}{a}\right) C_{D_M}$$

$$C_{L_V} \equiv \frac{\partial C_L}{\partial V} = \left(\frac{1}{a}\right) C_{L_M}$$

$$C_{m_V} \equiv \frac{\partial C_m}{\partial V} = \left(\frac{1}{a}\right) C_{m_M}$$

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Wing Lift and Moment Coefficient Sensitivity to Pitch Rate

Straight-wing incompressible flow estimate (*Etkin*)

$$C_{L_{\hat{q}_{wing}}} = -2C_{L_{\alpha_{wing}}}\left(h_{cm} - 0.75\right)$$

$$C_{m_{\hat{q}_{wing}}} = -2C_{L_{\alpha_{wing}}}\left(h_{cm} - 0.5\right)^2$$

Straight-wing supersonic flow estimate (*Etkin*)

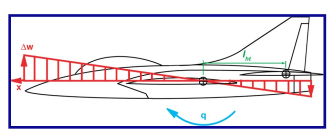
$$C_{L_{\hat{q}_{wing}}} = -2C_{L_{\alpha_{wing}}}\left(h_{cm} - 0.5\right)$$

$$C_{m_{\hat{q}_{wing}}} = -\frac{2}{3\sqrt{M^2-1}} - 2C_{L_{\alpha_{wing}}}\left(h_{cm} - 0.5\right)^2$$

Triangular-wing estimate (*Bryson, Nielsen*)

$$C_{L_{\hat{q}_{wing}}} = -\frac{2\pi}{3}C_{L_{\alpha_{wing}}}$$

$$C_{m_{\hat{q}_{wing}}} = -\frac{\pi}{3AR}$$



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Control- and Disturbance-Effect Matrices

- Control-effect derivatives portray acceleration sensitivities to control input perturbations

$$\mathbf{G}_{Lon} = \begin{bmatrix} -D_{\delta E} & T_{\delta T} & -D_{\delta F} \\ L_{\delta E}/V_N & L_{\delta T}/V_N & L_{\delta F}/V_N \\ M_{\delta E} & M_{\delta T} & M_{\delta F} \\ -L_{\delta E}/V_N & -L_{\delta T}/V_N & -L_{\delta F}/V_N \end{bmatrix}$$

- Disturbance-effect derivatives portray acceleration sensitivities to disturbance input perturbations

$$\mathbf{L}_{Lon} = \begin{bmatrix} -D_{V_{wind}} & -D_{\alpha_{wind}} \\ L_{V_{wind}}/V_N & L_{\alpha_{wind}}/V_N \\ M_{V_{wind}} & M_{\alpha_{wind}} \\ -L_{V_{wind}}/V_N & -L_{\alpha_{wind}}/V_N \end{bmatrix}$$

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Primary Longitudinal Stability Derivatives

$$D_V \triangleq \frac{-1}{m}\left[\left(C_{T_V} - C_{D_V}\right)\frac{\rho V_N^2}{2}S + \left(C_{T_N} - C_{D_N}\right)\rho V_N S\right]$$

$$L_V / V_N \simeq \frac{1}{mV_N}\left[C_{L_V}\frac{\rho V_N^2}{2}S + C_{L_N}\rho V_N S\right] - \frac{1}{mV_N^2}\left[C_{L_N}\frac{\rho V_N^2}{2}S - mg\right]$$

$$M_q = \frac{1}{I_{yy}}\left[C_{m_q}\frac{\rho V_N^2}{2}S\bar{c}\right] \qquad M_\alpha = \frac{1}{I_{yy}}\left[C_{m_\alpha}\frac{\rho V_N^2}{2}S\bar{c}\right]$$

$$L_\alpha / V_N \simeq \frac{1}{mV_N}\left[\left(C_{T_N} + C_{L_\alpha}\right)\frac{\rho V_N^2}{2}S\right]$$

Small angle assumptions

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Primary Phugoid Control Derivatives

$$D_{\delta T} \simeq \frac{-1}{m}\left[C_{T_{\delta T}} \frac{\rho V_N^{\ 2}}{2} S\right]$$

$$L_{\delta F} \Big/ V_N \simeq \frac{1}{mV_N}\left[C_{L_{\delta F}} \frac{\rho V_N^{\ 2}}{2} S\right]$$

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Primary Short-Period Control Derivatives

$$M_{\delta E} = C_{m_{\delta E}} \left(\frac{\rho_N V_N^2}{2I_{yy}}\right) S\overline{c}$$

$$L_{\delta E} \Big/ V = C_{L_{\delta E}} \left(\frac{\rho_N V_N^2}{2m}\right) S$$

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Flight Motions

Dornier Do-128 Short-Period Demonstration
http://www.youtube.com/watch?v=3hdLXE0rc9Q

Dornier Do-128 Phugoid Demonstration
http://www.youtube.com/watch?v=jzxtpQ30nLg&feature=related

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