Transfer Functions and Frequency Response

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MAE 331, 2018

Learning Objectives

- Frequency domain view of initial condition response
- Response of dynamic systems to sinusoidal inputs
- Transfer functions
- Bode plots

Reading:
Flight Dynamics
342-357

1

Fourier and Laplace Transforms

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Fourier Transform of a Scalar Variable

Transformation from "time domain" to "frequency domain"

$$\mathcal{F}\left[\Delta x(t)\right] = \Delta x(j\omega) = \int_{-\infty}^{\infty} \Delta x(t) e^{-j\omega t} dt, \quad \omega = frequency, rad/s$$

$j\omega$: Imaginary frequency operator, rad/s; $\quad j = \sqrt{-1}$

$\Delta x(t)$: **real variable**

$\Delta x(j\omega)$: **complex variable**

$$= a(\omega) + jb(\omega)$$

$$= A(\omega) e^{j\varphi(\omega)}$$

A : **amplitude**

φ : **phase angle**

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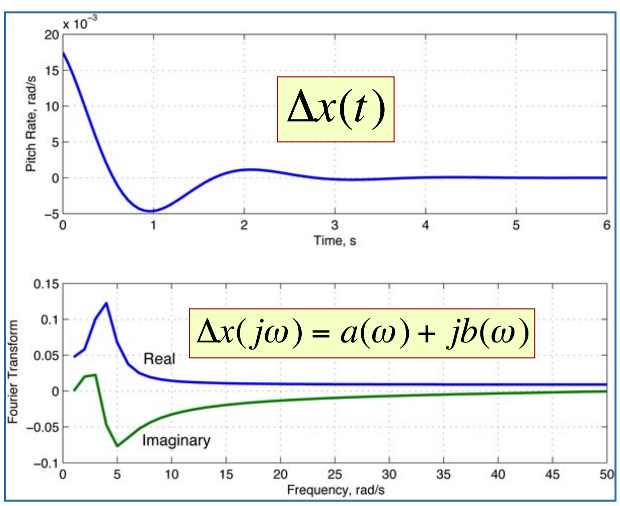
Fourier Transform of a Scalar Variable



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Laplace Transform of a Scalar Variable

Laplace transformation from "time domain" to "frequency domain"

$$\mathcal{L}\left[\Delta x(t)\right] = \Delta x(s) = \int_0^{\infty} \Delta x(t) e^{-st} dt$$

$s = \sigma + j\omega$

$= $ Laplace (complex frequency) operator, rad/s

$\Delta x(t)$: **real variable**

$\Delta x(s)$: **complex variable**

$= a(s) + jb(s)$

$= A(s)e^{j\varphi(s)}$

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Laplace Transformation is a Linear Operation

Sum of Laplace transforms

$$\mathcal{L}\left[\Delta x_1(t) + \Delta x_2(t)\right] = \mathcal{L}\left[\Delta x_1(t)\right] + \mathcal{L}\left[\Delta x_2(t)\right] = \Delta x_1(s) + \Delta x_2(s)$$

Multiplication by a constant

$$\mathcal{L}\left[a\Delta x(t)\right] = a\mathcal{L}\left[\Delta x(t)\right] = a\Delta x(s)$$

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Laplace Transforms of Vectors and Matrices

Laplace transform of a vector variable

$$\mathcal{L}\left[\Delta\mathbf{x}(t)\right] = \Delta\mathbf{x}(s) = \begin{bmatrix} \Delta x_1(s) \\ \Delta x_2(s) \\ \ldots \end{bmatrix}$$

Laplace transform of a matrix variable

$$\mathcal{L}\left[\mathbf{F}(t)\right] = \mathbf{F}(s) = \begin{bmatrix} f_{11}(s) & f_{12}(s) & \ldots \\ f_{21}(s) & f_{22}(s) & \ldots \\ \ldots & \ldots & \ldots \end{bmatrix}$$

Laplace transform of a time-derivative

$$\mathcal{L}\left[\Delta\dot{\mathbf{x}}(t)\right] = s\Delta\mathbf{x}(s) - \Delta\mathbf{x}(0)$$

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Laplace Transform of a Dynamic System

System equation

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{F}\,\Delta\mathbf{x}(t) + \mathbf{G}\,\Delta\mathbf{u}(t) + \mathbf{L}\Delta\mathbf{w}(t)$$

$\dim(\Delta\mathbf{x}) = (n \times 1)$
$\dim(\Delta\mathbf{u}) = (m \times 1)$
$\dim(\Delta\mathbf{w}) = (s \times 1)$

Laplace transform of system equation

$$s\Delta\mathbf{x}(s) - \Delta\mathbf{x}(0) = \mathbf{F}\,\Delta\mathbf{x}(s) + \mathbf{G}\Delta\,\mathbf{u}(s) + \mathbf{L}\Delta\mathbf{w}(s)$$

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Laplace Transform of a Dynamic System

Rearrange Laplace transform of dynamic equation

F to left, *I.C.* to right

$$s\Delta\mathbf{x}(s) - \mathbf{F}\Delta\mathbf{x}(s) = \Delta\mathbf{x}(0) + \mathbf{G}\Delta\mathbf{u}(s) + \mathbf{L}\Delta\mathbf{w}(s)$$

Combine terms

$$\left[s\mathbf{I} - \mathbf{F}\right]\Delta\mathbf{x}(s) = \Delta\mathbf{x}(0) + \mathbf{G}\Delta\mathbf{u}(s) + \mathbf{L}\Delta\mathbf{w}(s)$$

Multiply both sides by inverse of (*s*I – F)

$$\Delta\mathbf{x}(s) = \left[s\mathbf{I} - \mathbf{F}\right]^{-1}\left[\Delta\mathbf{x}(0) + \mathbf{G}\Delta\mathbf{u}(s) + \mathbf{L}\Delta\mathbf{w}(s)\right]$$

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Matrix Inverse

Forward	Inverse
$\mathbf{y} = \mathbf{A}\mathbf{x};$	$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$

$$\dim(\mathbf{x}) = \dim(\mathbf{y}) = (n \times 1)$$
$$\dim(\mathbf{A}) = (n \times n)$$

$$[\mathbf{A}]^{-1} = \frac{\mathrm{Adj}(\mathbf{A})}{|\mathbf{A}|} = \frac{\mathrm{Adj}(\mathbf{A})}{\det \mathbf{A}} \quad \frac{(n \times n)}{(1 \times 1)}$$

$$= \frac{\mathbf{C}^T}{\det \mathbf{A}}; \quad \mathbf{C} = \textit{matrix of cofactors}$$

Cofactors *are signed minors of* **A**

*ij^th^* ***minor*** *of* **A** *is the determinant of* **A** *with the i^th^ row and j^th^ column removed*

Numerator is a square matrix of cofactor transposes
Denominator is a scalar

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Matrix Inverse Examples

$\dim(\mathbf{A}) = (1 \times 1)$

$$\mathbf{A} = a; \quad \mathbf{A}^{-1} = \frac{1}{a}$$

$\dim(\mathbf{A}) = (2 \times 2)$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}; \quad \mathbf{A}^{-1} = \frac{\begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}^T}{a_{11}a_{22} - a_{12}a_{21}} = \frac{\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}{a_{11}a_{22} - a_{12}a_{21}}$$

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Matrix Inverse Examples

$\dim(\mathbf{A}) = (3 \times 3)$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix};$$

$$\mathbf{A}^{-1} = \frac{\begin{bmatrix} (a_{22}a_{33} - a_{23}a_{32}) & -(a_{21}a_{33} - a_{23}a_{31}) & (a_{21}a_{32} - a_{22}a_{31}) \\ -(a_{12}a_{33} - a_{13}a_{32}) & (a_{11}a_{33} - a_{13}a_{31}) & -(a_{11}a_{32} - a_{12}a_{31}) \\ (a_{12}a_{23} - a_{13}a_{22}) & -(a_{11}a_{23} - a_{13}a_{21}) & (a_{11}a_{22} - a_{12}a_{21}) \end{bmatrix}^T}{a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}}$$

$$= \frac{\begin{bmatrix} (a_{22}a_{33} - a_{23}a_{32}) & -(a_{12}a_{33} - a_{13}a_{32}) & (a_{12}a_{23} - a_{13}a_{22}) \\ -(a_{21}a_{33} - a_{23}a_{31}) & (a_{11}a_{33} - a_{13}a_{31}) & -(a_{11}a_{23} - a_{13}a_{21}) \\ (a_{21}a_{32} - a_{22}a_{31}) & -(a_{11}a_{32} - a_{12}a_{31}) & (a_{11}a_{22} - a_{12}a_{21}) \end{bmatrix}}{a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}}$$

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Characteristic Matrix Inverse

Characteristic matrix
(2nd-order example)

$$\left[s\mathbf{I}-\mathbf{F}\right]$$

Inverse of characteristic matrix

$$\left[s\mathbf{I}-\mathbf{F}\right]^{-1}=\frac{Adj\left(s\mathbf{I}-\mathbf{F}\right)}{\left|s\mathbf{I}-\mathbf{F}\right|}=\frac{\mathbf{C}^{T}\left(s\right)}{\Delta(s)}\qquad\frac{(2\times 2)}{(1\times 1)}$$

Denominator is the characteristic polynomial, a scalar

$$\left|s\mathbf{I}-\mathbf{F}\right|\equiv\Delta(s)$$
$$=s^{2}+c_{1}s+c_{0}$$

***Roots* of Δ(*s*) are *eigenvalues* of F**

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Numerator of the Characteristic Matrix Inverse

Numerator is an (*n* x *n*) matrix of polynomials

$$Adj\left(s\mathbf{I}-\mathbf{F}\right)=\begin{bmatrix} n_1^1(s) & n_2^1(s) \\ n_1^2(s) & n_2^2(s) \end{bmatrix}$$

For example,

$$n_1^1\left(s\right)=k\left(s-z\right)$$

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Initial Condition Response of a 2nd-Order System

$$\Delta\mathbf{x}(s) = [s\mathbf{I} - \mathbf{F}]^{-1}\Delta\mathbf{x}(0)$$

Time-domain model

$$\begin{bmatrix} \Delta\dot{x}_1(t) \\ \Delta\dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix}, \quad \begin{bmatrix} \Delta x_1(0) \\ \Delta x_2(0) \end{bmatrix} \textit{given}$$

Frequency-domain model

$$\begin{bmatrix} \Delta x_1(s) \\ \Delta x_2(s) \end{bmatrix} = [s\mathbf{I} - \mathbf{F}]^{-1} \begin{bmatrix} \Delta x_1(0) \\ \Delta x_2(0) \end{bmatrix}$$

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$(s\mathbf{I} - \mathbf{F})^{-1}$ Distributes and Shapes the Effects of Initial Conditions

- **Numerator distributes**
  - each element of the initial condition to
  - each element of the state

Denominator determines the common modes of motion

$$[s\mathbf{I} - \mathbf{F}]^{-1} = \frac{\begin{bmatrix} n_1^1(s) & n_2^1(s) \\ n_1^2(s) & n_2^2(s) \end{bmatrix}}{s^2 + c_1 s + c_0} \quad \frac{(2\times 2)}{(1\times 1)}$$

$$\Delta x_1(s) = \frac{n_1^1(s)\Delta x_1(0) + n_2^1(s)\Delta x_2(0)}{s^2 + c_1 s + c_0}$$

$$\Delta x_2(s) = \frac{n_1^2(s)\Delta x_1(0) + n_2^2(s)\Delta x_2(0)}{s^2 + c_1 s + c_0}$$

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Initial Condition Response of a Single State Element (Frequency Domain)

$$\Delta\mathbf{x}(s) = [s\mathbf{I} - \mathbf{F}]^{-1}\Delta\mathbf{x}(0)$$

$$\begin{bmatrix} \Delta x_1(s) \\ \Delta x_2(s) \\ \cdots \\ \Delta x_n(s) \end{bmatrix} = \frac{\begin{bmatrix} n_{11}(s) & n_{12}(s) & \cdots & n_{1n}(s) \\ n_{21}(s) & n_{22}(s) & \cdots & n_{2n}(s) \\ \cdots & \cdots & \cdots & \cdots \\ n_{n1}(s) & n_{n2}(s) & \cdots & n_{nn}(s) \end{bmatrix}}{\Delta(s)} \begin{bmatrix} \Delta x_1(0) \\ \Delta x_2(0) \\ \cdots \\ \Delta x_n(0) \end{bmatrix}$$

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Initial Condition Response of a Single State Element (Frequency Domain)

Initial condition response of $\Delta x_2(s)$

$$\Delta x_2(s) = \frac{n_{21}(s)}{\Delta(s)}\Delta x_1(0) + \frac{n_{22}(s)}{\Delta(s)}\Delta x_2(0) + \cdots + \frac{n_{2n}(s)}{\Delta(s)}\Delta x_n(0)$$

$$= \frac{n_{21}(s)\Delta x_1(0) + n_{22}(s)\Delta x_2(0) + \cdots + n_{2n}(s)\Delta x_n(0)}{\Delta(s)}$$

$$\triangleq \frac{p_2(s)}{\Delta(s)}$$

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Partial Fraction Expansion of the Initial Condition Response

Scalar frequency response can be expressed with *n* parts, each containing a single mode

$$\Delta x_i(s) = \frac{p_i(s)}{\Delta(s)}$$
$$= \left(\frac{d_1}{(s-\lambda_1)} + \frac{d_2}{(s-\lambda_2)} + \cdots \frac{d_n}{(s-\lambda_n)} \right)_i, \quad i = 1,n$$

For each eigenvalue, λ_i, the coefficients are

$$d_j = (s-\lambda_j)\frac{p_i(s)}{\Delta(s)}\bigg|_{s=\lambda_j} = (s-\lambda_j)\frac{p_i(\lambda_j)}{\Delta(\lambda_j)} \quad j = 1,n$$

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Partial Fraction Expansion of the Initial Condition Response

Time response is the inverse Laplace transform

$$\Delta x_i(t) = \mathcal{L}^{-1}\left[\Delta x_i(s)\right]$$
$$= \mathcal{L}^{-1}\left[\frac{d_1}{(s-\lambda_1)} + \frac{d_2}{(s-\lambda_2)} + \cdots \frac{d_n}{(s-\lambda_n)} \right]_i$$

$$\Delta x_i(t) = \left(d_1 e^{\lambda_1 t} + d_2 e^{\lambda_2 t} + \cdots + d_n e^{\lambda_n t} \right)_i, \quad i = 1,n$$

Each element's time response contains every mode of the system (although some coefficients may be negligible)

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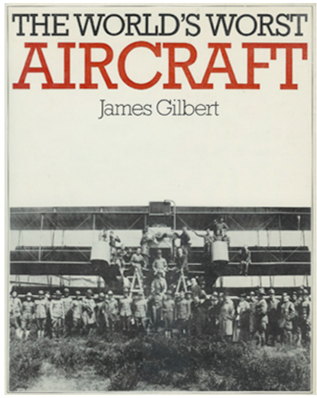
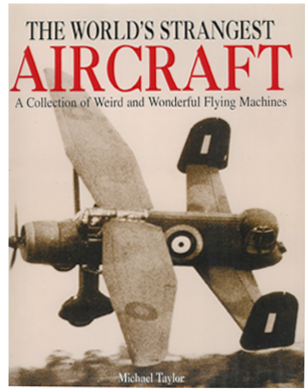
Historical Factoids
Unusual Aircraft





Forssman Tri-Plane (?)

Westland P.12 Lysander

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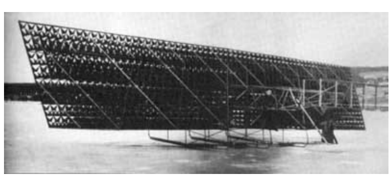
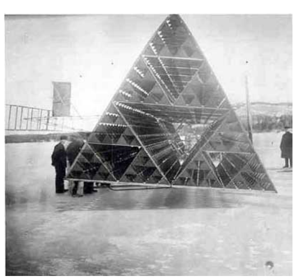
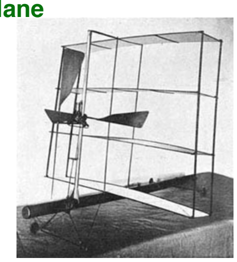
Multiplanes-1

AEA Cygnet II, Alexander Graham Bell, Glenn Curtiss, 1909 (3,393 tetrahedral cells)





Hargrave quadraplane (model), 1889



D'Equevillery, 1908

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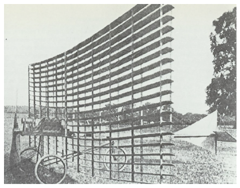
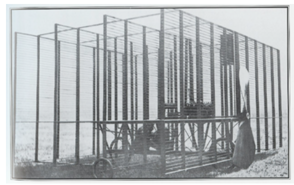
Multiplanes-2

Phillips, 1904



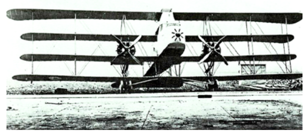
Phillips, 1907



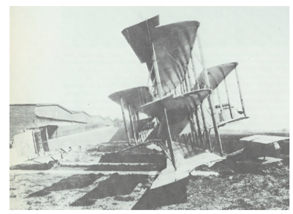
Vedo Villi, 1911

Wight Quadraplane, 1916

Pemberton-Billings Nighthawk, 1916



John Septaplane, 1919



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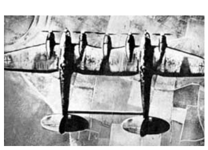
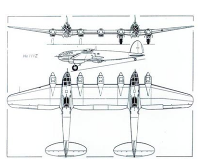
Unusual Engine Layouts

- Farman 3-engine Jabiru

- Tarrant 6-engine Tabor, 1919

- Heinkel 5-engine He111Z





- Farman 4-engine Jabiru, 1923

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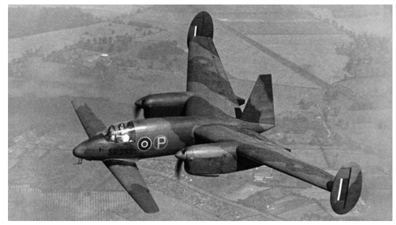
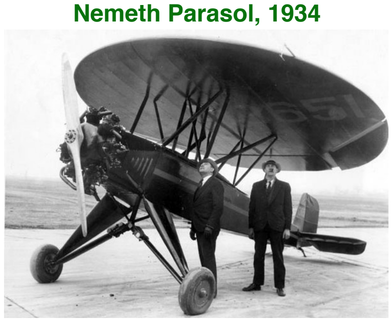
More Unusual Airplanes

Caproni Ca 60, 1920

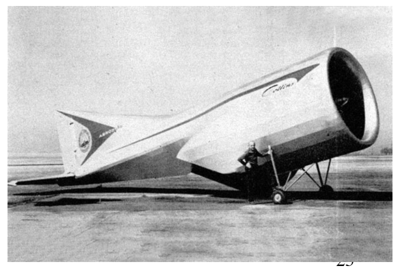
Miles Libellula, 1943



Nemeth Parasol, 1934



Lippisch Aerodyne, 1950



Control Response in Frequency Domain

State Response to Control Input

$$\begin{bmatrix} \Delta x_1(s) \\ \Delta x_2(s) \end{bmatrix} = [s\mathbf{I} - \mathbf{F}]^{-1}\mathbf{G}\begin{bmatrix} \Delta u_1(s) \\ \Delta u_2(s) \end{bmatrix}$$

Output Response *($H_u = 0$)* to Control Input

$$\begin{bmatrix} \Delta y_1(s) \\ \Delta y_2(s) \end{bmatrix} = \mathbf{H_x}[s\mathbf{I} - \mathbf{F}]^{-1}\mathbf{G}\begin{bmatrix} \Delta u_1(s) \\ \Delta u_2(s) \end{bmatrix}$$

$$\triangleq \mathcal{H}(s)\begin{bmatrix} \Delta u_1(s) \\ \Delta u_2(s) \end{bmatrix}$$

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1^{st}-Order Transfer Function

Scalar dynamic system

$$\dot{x}(t) = fx(t) + gu(t)$$
$$y(t) = hx(t)$$

Scalar transfer function (= first-order lag)

$$\frac{y(s)}{u(s)} = \mathscr{H}(s) = h[s-f]^{-1} g = \frac{hg}{(s-f)} \qquad (n = m = r = 1)$$

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2^{nd}-Order Transfer Function

2^{nd}-order system differential equation

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$
$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \cdots$$

2^{nd}-order system transfer function matrix

$$\mathscr{H}(s) = \mathbf{H_x}(s\mathbf{I} - \mathbf{F})^{-1}(s)\mathbf{G}$$
$$= \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \frac{\operatorname{adj}\begin{bmatrix} (s-f_{11}) & -f_{12} \\ -f_{21} & (s-f_{22}) \end{bmatrix}}{\det\begin{pmatrix} (s-f_{11}) & -f_{12} \\ -f_{21} & (s-f_{22}) \end{pmatrix}} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \frac{\begin{bmatrix} n_{u_1}^{y_1}(s) & n_{u_2}^{y_1}(s) \\ n_{u_1}^{y_2}(s) & n_{u_2}^{y_2}(s) \end{bmatrix}}{\Delta(s)}$$

$(r \times n)(n \times n)(n \times m)$
$= (r \times m) = (2 \times 2)$

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Eigenvalues of a Dynamic System

$$\Delta(s) = (s-\lambda_1)(s-\lambda_2)(\ldots)(s-\lambda_n) = 0$$

Eigenvalues are real or complex numbers that can be plotted in the ***s* plane**

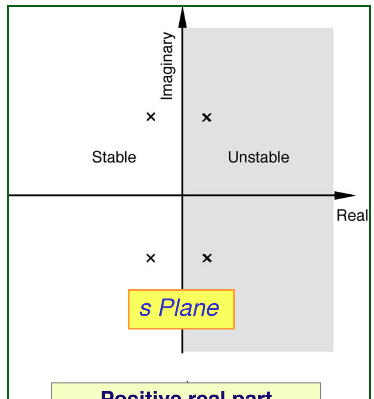
- **Real root**

$$\lambda_i = \sigma_i$$

- **Complex roots occur in conjugate pairs**

$$\lambda_i = \sigma_i + j\omega_i$$

$$\lambda_{i+1} = \lambda^*_i = \sigma_i - j\omega_i$$



Positive real part indicates instability

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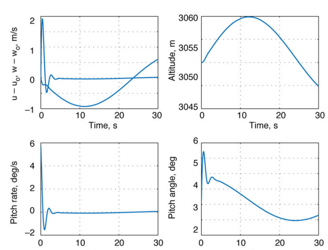
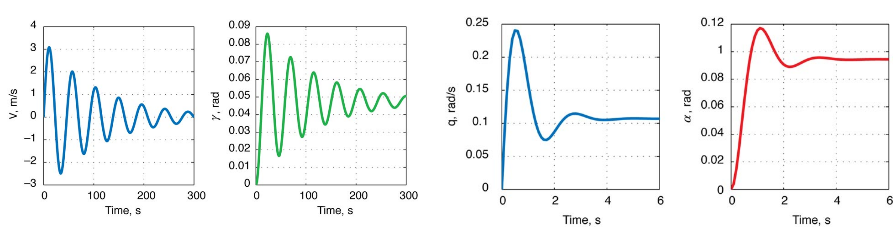


Aircraft Modes of Motion



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Longitudinal and Lateral-Directional Roots of the Characteristic Equation

- 12 roots of the characteristic equation
- Characteristic equation of the system

$$\Delta(s) = s^{12} + c_{11}s^{11} + ... + c_1 s + c_0 = 0$$
$$= (s-\lambda_1)(s-\lambda_2)(...)(s-\lambda_{12}) = 0$$

Up to 12 modes of motion

In steady, level flight, longitudinal and lateral-directional LTI perturbation models are uncoupled

$$\Delta(s) = \Delta_{Long}(s)\Delta_{Lat-Dir}(s) = 0$$
$$= \left[(s-\lambda_1)\cdots(s-\lambda_6)\right]_{Long}\left[(s-\lambda_1)\cdots(s-\lambda_6)\right]_{Lat-Dir}$$

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Longitudinal Modes of Motion in Steady, Level Flight

Longitudinal characteristic equation has 6 roots

$$\Delta_{Lon}(s) = (s-\lambda_R)(s-\lambda_H)\left[(s-\lambda_P)(s-\lambda^*_P)\right]\left[(s-\lambda_{SP})(s-\lambda^*_{SP})\right]$$

Real | Real | Complex | Complex | Complex | Complex

4 modes of motion (typical)

$$\Delta_{Lon}(s) = [s-(0)][s-(\sim 0)](s^2 + 2\zeta_P\omega_{n_P}s + \omega_{n_P}^{\;2})(s^2 + 2\zeta_{SP}\omega_{n_{SP}}s + \omega_{n_{SP}}^2) = 0$$

Range | **Height** | **Phugoid** | **Short Period**

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Each Complex Conjugate Pair Forms One Oscillatory Mode of Motion

Phugoid Roots

$$(s-\lambda_P)(s-\lambda^*_P) = [s-(\sigma_P + j\omega_P)][s-(\sigma_P - j\omega_P)]$$
$$= s^2 - 2\sigma_P s + (\sigma_P{}^2 + \omega_P{}^2) \triangleq (s^2 + 2\zeta_P\omega_{n_P} s + \omega_{n_P}^2)$$

ω_n : **Natural frequency**, rad/s

ζ : **Damping ratio**, -

Short Period Roots

$$(s-\lambda_{SP})(s-\lambda^*_{SP}) = [s-(\sigma_{SP} + j\omega_{SP})][s-(\sigma_{SP} - j\omega_{SP})]$$
$$= s^2 - 2\sigma_{SP} s + (\sigma_{SP}{}^2 + \omega_{SP}{}^2) \triangleq (s^2 + 2\zeta_{SP}\omega_{n_{SP}} s + \omega_{n_{SP}}^2)$$

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Lateral-Directional Modes of Motion in Steady, Level Flight

Lateral-directional characteristic equation has 6 roots

$$\Delta_{LD}(s) = (s-\lambda_{CR})(s-\lambda_{Head})(s-\lambda_S)(s-\lambda_R)\left[(s-\lambda_{DR})(s-\lambda^*_{DR})\right]$$

5 modes of motion (typical)

$$\Delta_{LD}(s) = [s-(0)][s-(0)](s-\lambda_S)(s-\lambda_R)(s^2 + 2\zeta_{DR}\omega_{n_{DR}} s + \omega_{n_{DR}}^2) = 0$$

Crossrange | Heading | Spiral | Roll | Dutch Roll

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Bode Plot
(Frequency Response of a Scalar Transfer Function)

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Scalar (Single-Input/Single-Output) Frequency Response Function

Substitute: *s = jω*

$$\mathscr{H}_{ij}(j\omega) = \mathbf{H}_{x_i{}^{th}{}_{row}} \left[s\mathbf{I} - \mathbf{F} \right]^{-1} \mathbf{G}_{j^{th} column}$$

$$= \frac{k_{ij}\left(j\omega - z_1\right)_{ij}\left(j\omega - z_2\right)_{ij} \ldots \left(j\omega - z_q\right)_{ij}}{\left(j\omega - \lambda_1\right)\left(j\omega - \lambda_2\right)\ldots\left(j\omega - \lambda_n\right)}$$

$$= a(\omega) + jb(\omega) \rightarrow AR(\omega)\, e^{j\phi(\omega)}$$

- **Frequency response is a complex function of input frequency, *ω***
  - **Real and imaginary parts, or**
  - **>>Amplitude ratio and phase angle <<**

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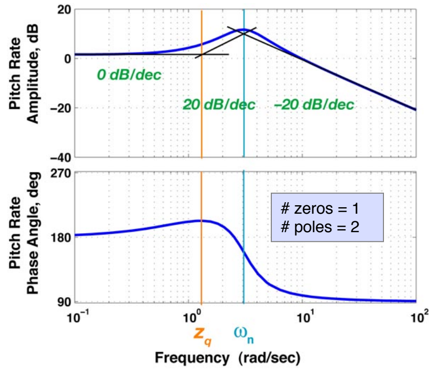
Bode Plot Portrays Response to Sinusoidal Control Input



Express amplitude ratio in decibels

$$AR(dB) = 20\log_{10}\left[AR\left(\text{original units}\right)\right]$$

20 dB = factor of 10

Products in original units are sums in decibels

$$z = xy$$
$$20\log z = 20\log x + 20\log y$$

Asymptotes form "skeleton" of response amplitude ratio

Plot *AR(dB)* vs. $\log_{10}(\omega_{input})$

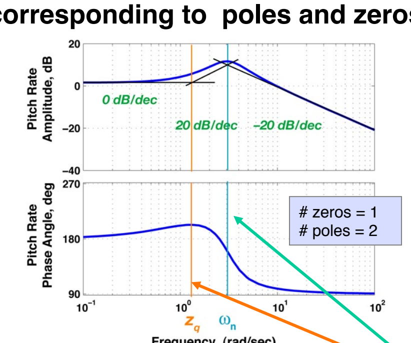
Plot phase angle, ϕ(deg) vs. $\log_{10}(\omega_{input})$

37

Bode Plot Portrays Response to Sinusoidal Control Input

Why Plot Vertical Lines where $\omega = z$ and ω_n?

Asymptotes change at frequencies corresponding to poles and zeros



$$AR_{\delta E}^{q}(\omega)\, e^{j\phi_{\delta E}^{q}(\omega)} = \frac{\Delta q(j\omega)}{\Delta\delta E(j\omega)} = \frac{k_{\delta E}^{q}\left(j\omega - z_{\delta E}^{q}\right)}{-\omega^2 + 2\zeta_{SP}\omega_{n_{SP}} j\omega + \omega_{n_{SP}}^{2}}$$

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Pole and Zero Effects are Additive for *Amplitude Ratio* (dB) and *Phase Angle* (deg)

$$AR^{q}_{\delta E}(\omega)\, e^{j\phi^{q}_{\delta E}(\omega)} = \frac{\Delta q(j\omega)}{\Delta \delta E(j\omega)} = \frac{k^{q}_{\delta E}\left(j\omega - z^{q}_{\delta E}\right)}{-\omega^2 + 2\zeta_{SP}\omega_{n_{SP}} j\omega + \omega_{n_{SP}}{}^2}$$

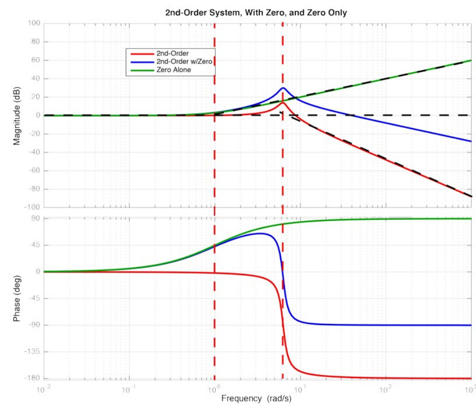
Input frequency, $\omega = -z_q\ (<0)$,

$$k_q\left(j\omega - z_q\right) = k_q z_q\left(-j-1\right) = -k_q z_q\left(j+1\right) = k_q\left|z_q\right| e^{+45°}$$

Input frequency, $\omega = \omega_{n_{SP}}\ (\zeta_{SP} > 0)$

$$-\omega_{n_{SP}}^2 + 2\zeta_{SP} j\omega_{n_{SP}}^2 + \omega_{n_{SP}}^2 = j2\zeta_{SP}\omega_{n_{SP}}^2$$

$$= \frac{1}{j2\zeta_{SP}\omega_{n_{SP}}^2} = \frac{-j}{2\zeta_{SP}\omega_{n_{SP}}^2} = \frac{1}{2\zeta_{SP}\omega_{n_{SP}}^2} e^{-90°}$$



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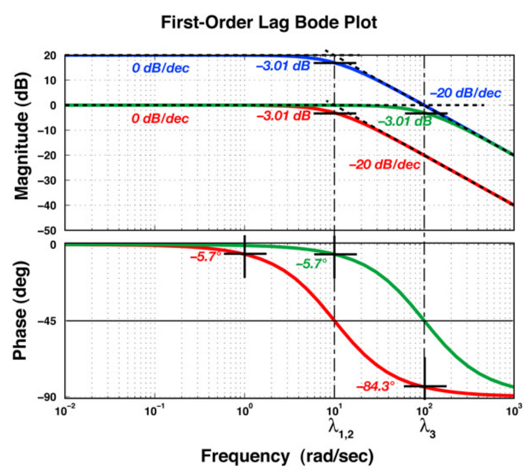
Asymptotes and Departures for 1st-Order Bode Plot

$$\mathscr{H}_{red}(j\omega) = \frac{10}{(j\omega + 10)}$$

$$\mathscr{H}_{blue}(j\omega) = \frac{100}{(j\omega + 10)}$$

$$\mathscr{H}_{green}(j\omega) = \frac{100}{(j\omega + 100)}$$



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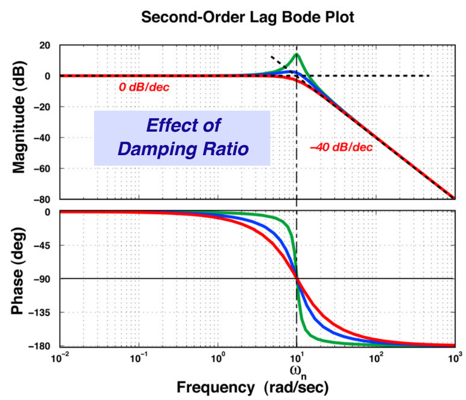
Bode Plots of 2nd-Order System
(No Zeros)

$$\mathcal{H}_{green}(j\omega) = \frac{10^2}{(j\omega)^2 + 2(0.1)(10)(j\omega) + 10^2}$$

$$\mathcal{H}_{blue}(j\omega) = \frac{10^2}{(j\omega)^2 + 2(0.4)(10)(j\omega) + 10^2}$$

$$\mathcal{H}_{red}(j\omega) = \frac{10^2}{(j\omega)^2 + 2(0.707)(10)(j\omega) + 10^2}$$



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Transfer Function Matrix for Short-Period Approximation

$$\left(\mathbf{H_x} = \mathbf{I}_2, \quad \mathbf{H_u} = \mathbf{0}\right)$$

$$\mathcal{H}_{SP}(s) \triangleq \frac{\begin{bmatrix} k_q n_{\delta E}^{q}(s) \\ k_\alpha n_{\delta E}^{\alpha}(s) \end{bmatrix}}{s^2 + 2\zeta_{SP}\omega_{n_{SP}} s + \omega_{n_{SP}}{}^2} = \begin{bmatrix} \dfrac{\Delta q(s)}{\Delta \delta E(s)} \\ \dfrac{\Delta \alpha(s)}{\Delta \delta E(s)} \end{bmatrix}$$

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Components of Approximate Short-Period Transfer Function

$$L_{\delta E}=0$$

Pitch Rate Transfer Function

$$\frac{\Delta q(s)}{\Delta\delta E(s)}=\frac{M_{\delta E}\left[s+\left(L_\alpha/V_N\right)\right]}{s^2+\left(-M_q+L_\alpha/V_N\right)s-\left[M_\alpha\left(1-L_q/V_N\right)+M_q\,L_\alpha/V_N\right]}$$

$$\triangleq\frac{k_q\left(s-z_q\right)}{s^2+2\zeta_{SP}\omega_{n_{SP}}s+\omega_{n_{SP}}^2}$$

Angle of Attack Transfer Function

$$\frac{\Delta\alpha(s)}{\Delta\delta E(s)}=\frac{M_{\delta E}}{s^2+\left(-M_q+L_\alpha/V_N\right)s-\left[M_\alpha\left(1-L_q/V_N\right)+M_q\,L_\alpha/V_N\right]}$$

$$\triangleq\frac{k_\alpha}{s^2+2\zeta_{SP}\omega_{n_{SP}}s+\omega_{n_{SP}}^2}$$

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Short-Period Frequency Response, *Amplitude Ratio and Phase Angle*

$$s\triangleq j\omega$$

Pitch-rate frequency response

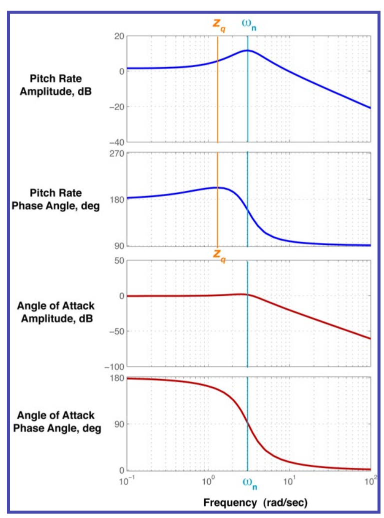
$$\frac{\Delta q(j\omega)}{\Delta\delta E(j\omega)}=\frac{k_q\left(j\omega-z_q\right)}{-\omega^2+2\zeta_{SP}\omega_{n_{SP}}j\omega+\omega_{n_{SP}}^2}$$

$$=AR_q(\omega)\,e^{j\phi_q(\omega)}$$

Angle-of-attack frequency response

$$\frac{\Delta\alpha(j\omega)}{\Delta\delta E(j\omega)}=\frac{k_\alpha}{-\omega^2+2\zeta_{SP}\omega_{n_{SP}}j\omega+\omega_{n_{SP}}^2}$$

$$=AR_\alpha(\omega)\,e^{j\phi_\alpha(\omega)}$$



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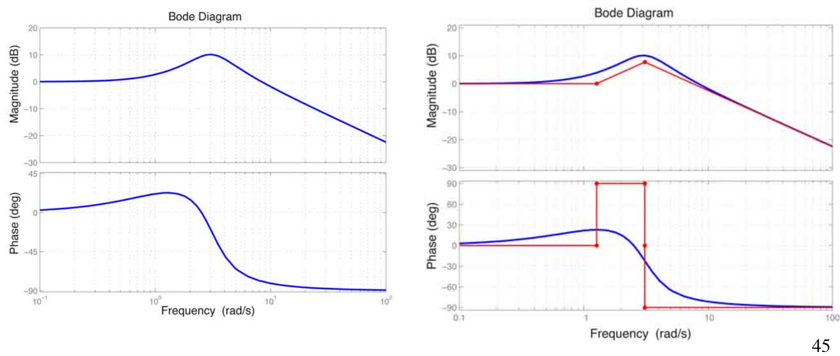
MATLAB Bode Plot with asymp.m

http://www.mathworks.com/matlabcentral/

http://www.mathworks.com/matlabcentral/fileexchange/10183-bode-plot-with-asymptotes

2nd-Order Pitch Rate Frequency Response

bode.m | **asymp.m**



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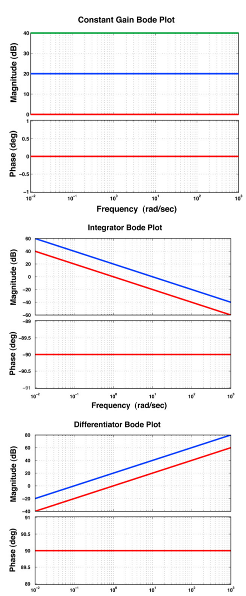
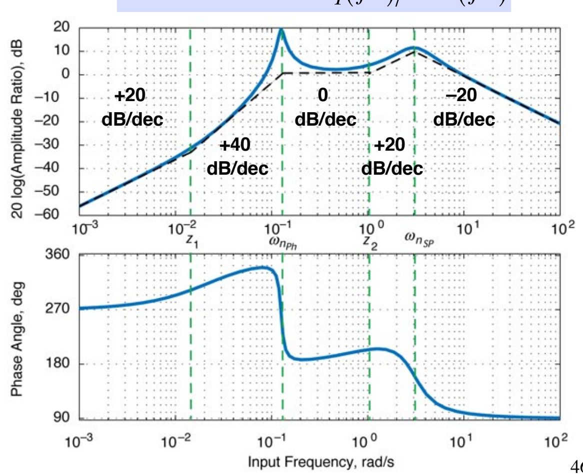
Asymptotes form Complex Bode Plots for Transfer Functions



4th-order model of $\Delta q(j\omega)/\Delta\delta E(j\omega)$



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Next Time: Root Locus Analysis

Reading:
Flight Dynamics
357-361, 465-467, 488-490, 509-514

Learning Objectives

Effects of system parameter variations on modes of motion
Root locus analysis
Evans's rules for construction
Application to longitudinal dynamic models

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Supplementary Material

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Matrix Inverse Examples

$$\mathbf{A} = 5; \quad \mathbf{A}^{-1} = \frac{1}{5} = 0.2$$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}; \quad \mathbf{A}^{-1} = \frac{\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}}{-2} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 7 \\ 8 & 12 & 9 \end{bmatrix}; \quad \mathbf{A}^{-1} = \frac{\begin{bmatrix} -30 & 18 & 4 \\ 20 & -15 & 5 \\ 0 & 4 & -2 \end{bmatrix}}{10} = \begin{bmatrix} -3 & 1.8 & 0.4 \\ 2 & -1.5 & 0.5 \\ 0 & 0.4 & -0.2 \end{bmatrix}$$

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Longitudinal Modes of Motion

Eigenvalues determine the damping and natural frequencies of the linear system's modes of motion

- **6 eigenvalues**
  - **4 eigenvalues normally appear as 2 complex pairs**
  - **Range and height modes usually inconsequential**

$$\lambda_{ran} : range\ mode \approx 0$$
$$\lambda_{hgt} : height\ mode \approx 0$$
$$\left(\zeta_P, \omega_{n_P}\right) : phugoid\ mode$$
$$\left(\zeta_{SP}, \omega_{n_{SP}}\right) : short\text{ - }period\ mode$$

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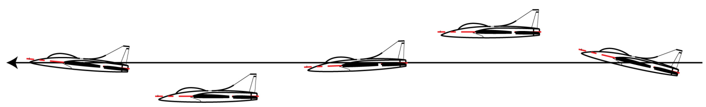
Phugoid (Long-Period) Mode

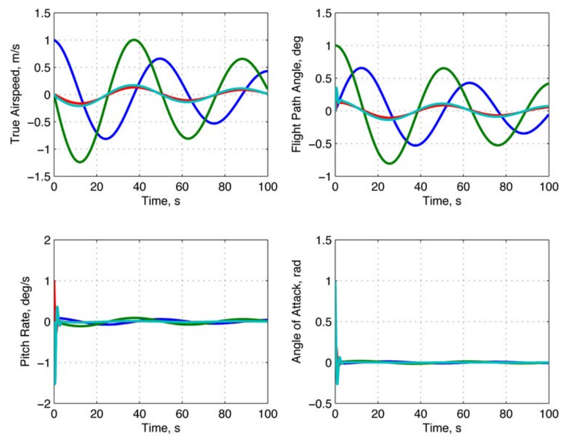


Airspeed

Flight Path Angle

Pitch Rate

Angle of Attack



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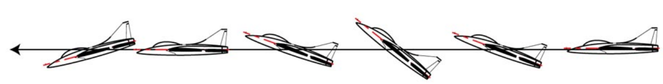
Short-Period Mode



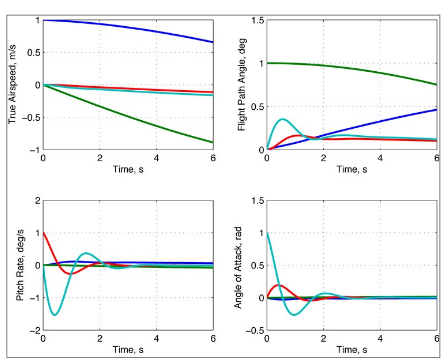
Note change in time scale

Airspeed

Flight Path Angle

Pitch Rate

Angle of Attack



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Lateral-Directional Modes of Motion

- **6 eigenvalues**
  - 2 eigenvalues normally appear as a complex pair
  - Crossrange and heading modes usually inconsequential

$$\lambda_{cr} : crossrange\ mode \approx 0$$
$$\lambda_{head} : heading\ mode \approx 0$$
$$\lambda_S : spiral\ mode$$
$$\lambda_R : roll\ mode$$
$$\left(\zeta_{DR}, \omega_{n_{DR}}\right) : Dutch\ roll\ mode$$

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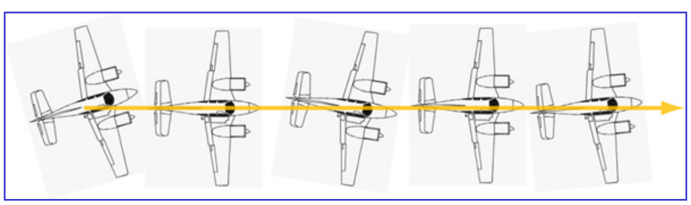
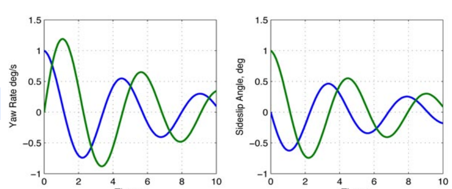
Dutch-Roll Mode



Yaw Rate



Sideslip Angle

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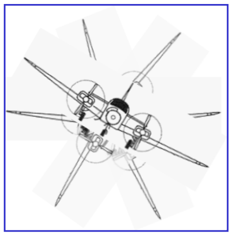
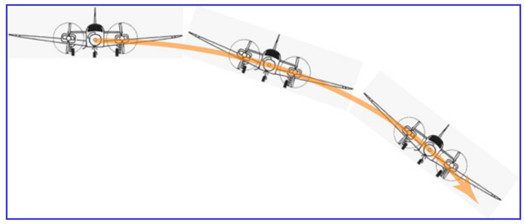
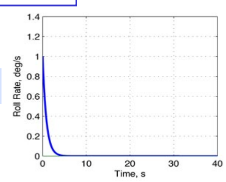
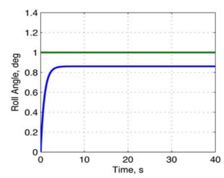
Roll and Spiral Modes





Roll Rate





Roll Angle

Transfer Function Matrix

Frequency-domain effect of all inputs on all outputs *(H_u = 0)*

$$\mathscr{H}(s) = \mathbf{H_x}\left[s\mathbf{I} - \mathbf{F}\right]^{-1}\mathbf{G}$$

$$(r \times n)(n \times n)(n \times m) = (r \times m)$$

Characteristic Polynomial of a LTI Dynamic System

Response to initial condition, control, and disturbance

$$\Delta\mathbf{x}(s) = \left[s\mathbf{I} - \mathbf{F}\right]^{-1}\left[\Delta\mathbf{x}(0) + \mathbf{G}\,\Delta\mathbf{u}(s) + \mathbf{L}\Delta\mathbf{w}(s)\right]$$

Inverse of characteristic matrix

$$\left[s\mathbf{I} - \mathbf{F}\right]^{-1} = \frac{Adj\left(s\mathbf{I} - \mathbf{F}\right)}{\left|s\mathbf{I} - \mathbf{F}\right|} \quad (n\ x\ n)$$

Characteristic polynomial

$$\left|s\mathbf{I} - \mathbf{F}\right| = \det\left(s\mathbf{I} - \mathbf{F}\right) \equiv \Delta(s)$$
$$= s^n + c_{n-1}s^{n-1} + ... + c_1 s + c_0$$

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Eigenvalues (or Roots) of a Dynamic System

Characteristic equation of the system

$$\Delta(s) = \left|s\mathbf{I} - \mathbf{F}\right| = s^n + c_{n-1}s^{n-1} + ... + c_1 s + c_0 = 0$$
$$= \left(s - \lambda_1\right)\left(s - \lambda_2\right)\left(...\right)\left(s - \lambda_n\right) = 0$$

... where λ_i are the eigenvalues of F or the roots of the characteristic polynomial

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Scalar Transfer Function from Δu_j to Δy_i

- Just one element of the matrix, ***H***(*s*)
- Each numerator term is a polynomial with ***q*** zeros, where ***q*** varies from term to term and ≤ ***n*** – 1

$$\mathscr{H}_{ij}(s) = \frac{n_{ij}(s)}{\Delta(s)} = \frac{k_{ij}\left(s^q + b_{q-1}s^{q-1} + \ldots + b_1 s + b_0\right)}{\left(s^n + c_{n-1}s^{n-1} + \ldots + c_1 s + c_0\right)}$$

Denominator polynomial contains *n* roots

$$= k_{ij}\frac{\left(s - z_1\right)_{ij}\left(s - z_2\right)_{ij}\ldots\left(s - z_q\right)_{ij}}{\left(s - \lambda_1\right)\left(s - \lambda_2\right)\ldots\left(s - \lambda_n\right)}$$

\# zeros = *q*
\# poles = *n*

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Control Response of a Single State Element

$$\Delta y_i\left(s\right) = k_{ij}\frac{n_{ij}(s)}{\Delta(s)}\Delta u_j\left(s\right)$$

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Relationship of $(s\mathbf{I} - \mathbf{F})^{-1}$ to State Transition Matrix, $\boldsymbol{\Phi}(t,0)$

Initial condition response

Time Domain	$\Delta\mathbf{x}(t) = \boldsymbol{\Phi}(t,0)\Delta\mathbf{x}(0)$
Frequency Domain	$\Delta\mathbf{x}(s) = \left[s\mathbf{I} - \mathbf{F}\right]^{-1}\Delta\mathbf{x}(0) =$

$\Delta\mathbf{x}(s)$ is the Laplace transform of $\Delta\mathbf{x}(t)$

$$\Delta\mathbf{x}(s) = \mathcal{L}\left[\Delta\mathbf{x}(t)\right] = \mathcal{L}\left[\boldsymbol{\Phi}(t,0)\Delta\mathbf{x}(0)\right] = \mathcal{L}\left[\boldsymbol{\Phi}(t,0)\right]\Delta\mathbf{x}(0)$$

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Relationship of $(s\mathbf{I} - \mathbf{F})^{-1}$ to State Transition Matrix, $\boldsymbol{\Phi}(t,0)$

Therefore,

$$\left[s\mathbf{I} - \mathbf{F}\right]^{-1} = \mathcal{L}\left[\boldsymbol{\Phi}(t,0)\right]$$

$=$ **Laplace transform of the state transition matrix**

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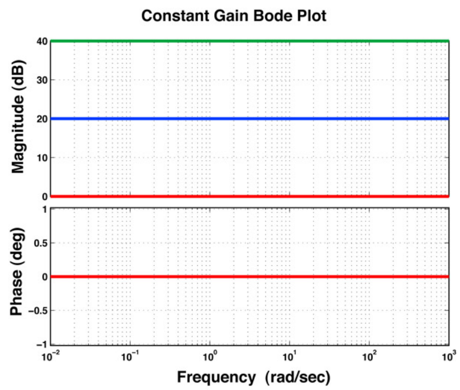
Constant Gain Bode Plot

$y(t) = hu(t)$

$\mathscr{H}(j\omega) = 1$

$\mathscr{H}(j\omega) = 10$

$\mathscr{H}(j\omega) = 100$



Slope = 0 dB/dec, Amplitude Ratio = constant

Phase Angle = 0°

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Integrator Bode Plot

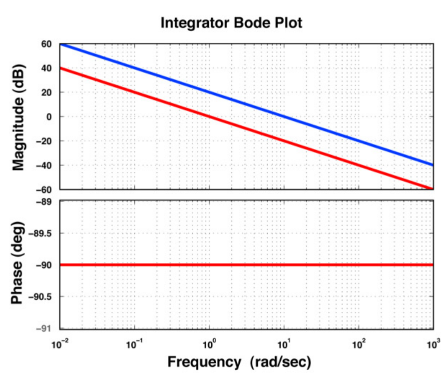
$y(t) = h\int_0^t u(t)\,dt$

$\mathscr{H}(j\omega) = \frac{1}{j\omega}$

$\mathscr{H}(j\omega) = \frac{10}{j\omega}$

Slope = –20 dB/dec

Phase Angle = –90°



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Differentiator Bode Plot

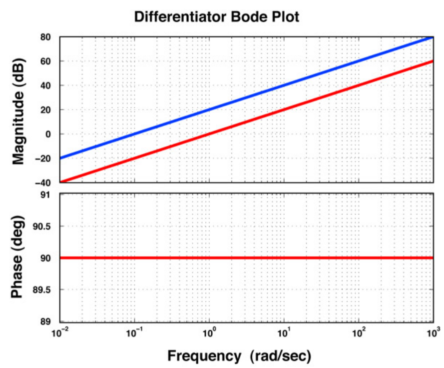
$$y(t) = h\frac{du(t)}{dt}$$

$$\mathscr{H}(j\omega) = j\omega$$

$$\mathscr{H}(j\omega) = 10\,j\omega$$

Slope = +20 dB/dec

Phase Angle = +90°



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Sign Change

Integral

$$y(t) = -h\int_0^t u(t)\,dt$$

$$\mathscr{H}(j\omega) = -\frac{h}{j\omega}$$

Slope = −20 dB/dec

Phase Angle = +90°

Derivative

$$y(t) = -h\frac{du(t)}{dt}$$

$$\mathscr{H}(j\omega) = -j\omega$$

Slope = +20 dB/dec

Phase Angle = −90°

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Multiple Integrators and Differentiators

Double Integral

$$y(t) = h\int_0^t \int_0^t u(t)\,dt^2$$

$$\mathscr{H}(j\omega) = \frac{h}{(j\omega)^2}$$

Slope = −40 dB/dec
Phase Angle = −180°

Double Derivative

$$y(t) = h\frac{d^2u(t)}{dt^2}$$

$$\mathscr{H}(j\omega) = h(j\omega)^2$$

Slope = +40 dB/dec
Phase Angle = +180°

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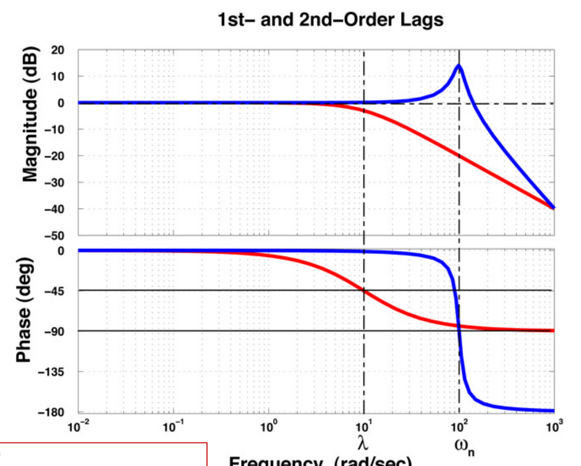
Bode Plots of 1st- and 2nd-Order Lags



$$\mathscr{H}_{red}(j\omega) = \frac{10}{(j\omega+10)}$$

$$\mathscr{H}_{blue}(j\omega) = \frac{100^2}{(j\omega)^2 + 2(0.1)(100)(j\omega) + 100^2}$$

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Numerator and Denominator of 2nd-Order $(s\mathbf{I} - \mathbf{F})^{-1}$

Numerator

$$\mathrm{adj}\begin{bmatrix} (s-f_{11}) & -f_{12} \\ -f_{21} & (s-f_{22}) \end{bmatrix} = \begin{bmatrix} (s-f_{22}) & f_{12} \\ f_{21} & (s-f_{11}) \end{bmatrix}$$

Denominator

$$\det\begin{pmatrix} (s-f_{11}) & -f_{12} \\ -f_{21} & (s-f_{22}) \end{pmatrix} = (s-f_{11})(s-f_{22}) - f_{12}f_{21}$$

$$= s^2 - (f_{11}+f_{22})s + (f_{11}f_{22} - f_{12}f_{21})$$

$$\triangleq s^2 + 2\zeta\omega_n s + \omega_n^2 \triangleq \Delta(s)$$

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2nd-Order Transfer Function, Scalar Input

$$\mathscr{H}(s) = \mathbf{H_x}(s\mathbf{I}-\mathbf{F})^{-1}(s)\mathbf{G} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \frac{\begin{bmatrix} (s-f_{22}) & f_{12} \\ f_{21} & (s-f_{11}) \end{bmatrix}}{s^2 + 2\zeta\omega_n s + \omega_n^2} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

$$\mathscr{H}(s) = \frac{\begin{bmatrix} [h_{11}(s-f_{22}) + h_{12}f_{21}] & [h_{11}f_{12} + h_{12}(s-f_{11})] \\ [h_{21}(s-f_{22}) + h_{22}f_{21}] & [h_{21}f_{12} + h_{22}(s-f_{11})] \end{bmatrix}\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{\begin{bmatrix} [h_{11}(s-f_{22}) + h_{12}f_{21}]g_1 + [h_{11}f_{12} + h_{12}(s-f_{11})]g_2 \\ [h_{21}(s-f_{22}) + h_{22}f_{21}]g_1 + [h_{21}f_{12} + h_{22}(s-f_{11})]g_2 \end{bmatrix}}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

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2nd-Order Transfer Function, Scalar Input

Collect Terms

$$\mathscr{H}(s) = \frac{\begin{bmatrix} \left[h_{11}\left(s-f_{22}\right)+h_{12}f_{21}\right]g_1+\left[h_{11}f_{12}+h_{12}\left(s-f_{11}\right)\right]g_2 \\ \left[h_{21}\left(s-f_{22}\right)+h_{22}f_{21}\right]g_1+\left[h_{21}f_{12}+h_{22}\left(s-f_{11}\right)\right]g_2 \end{bmatrix}}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\triangleq \frac{\begin{bmatrix} k_1\left(s-z_1\right) \\ k_2\left(s-z_2\right) \end{bmatrix}}{s^2+2\zeta\omega_n s+\omega_n^2}$$

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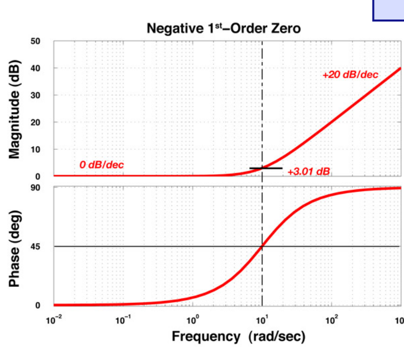
Left-Half-Plane Transfer Function Zero

Zeros are numerator singularities

$$\mathscr{H}(j\omega) = \left(j\omega+10\right)$$

$$\mathscr{H}(j\omega) = \frac{k\left(j\omega-z_1\right)\left(j\omega-z_2\right)\ldots}{\left(j\omega-\lambda_1\right)\left(j\omega-\lambda_2\right)\ldots\left(j\omega-\lambda_n\right)}$$



- Single zero in left half plane
- Introduces a +20 dB/dec slope
- Produces phase lead in vicinity of zero

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Right-Half-Plane Transfer Function Zero

$$\mathscr{H}(j\omega) = -(j\omega - 10)$$

- Single zero in right half plane
- Introduces a +20 dB/dec slope
- Produces phase lag in vicinity of zero

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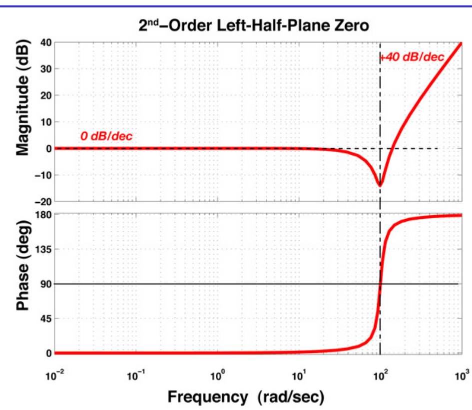
2nd-Order Transfer Function Zero

$$\mathscr{H}(j\omega) = (j\omega - z)(j\omega - z^*) = \left[(j\omega)^2 + 2(0.1)(100)(j\omega) + 100^2\right]$$



- Complex pair of zeros produces an amplitude ratio "notch" at its "natural frequency"

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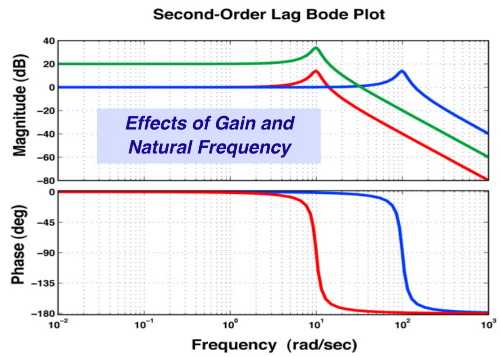
Bode Plots of 2nd-Order Lags (No Zeros)

$$\mathscr{H}_{red}(j\omega) = \frac{10^2}{(j\omega)^2 + 2(0.1)(10)(j\omega) + 10^2}$$

$$\mathscr{H}_{green}(j\omega) = \frac{10^3}{(j\omega)^2 + 2(0.1)(10)(j\omega) + 10^2}$$



$$\mathscr{H}_{blue}(j\omega) = \frac{100^2}{(j\omega)^2 + 2(0.1)(100)(j\omega) + 100^2}$$

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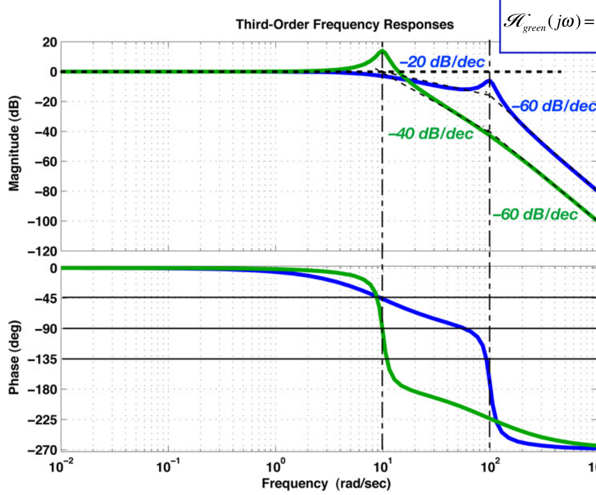
Bode Plots of 3rd-Order Lags

$$\mathscr{H}_{blue}(j\omega) = \left[\frac{10}{(j\omega + 10)}\right]\left[\frac{100^2}{(j\omega)^2 + 2(0.1)(100)(j\omega) + 100^2}\right]$$

$$\mathscr{H}_{green}(j\omega) = \left[\frac{10^2}{(j\omega)^2 + 2(0.1)(10)(j\omega) + 10^2}\right]\left[\frac{100}{(j\omega + 100)}\right]$$



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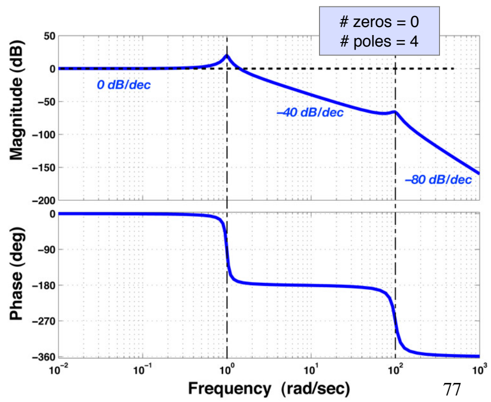
Bode Plot of a 4th-Order System with No Zeros

$$\mathscr{H}(j\omega) = \left[\frac{1^2}{(j\omega)^2 + 2(0.05)(1)(j\omega) + 1^2}\right]\left[\frac{100^2}{(j\omega)^2 + 2(0.1)(100)(j\omega) + 100^2}\right]$$

- Resonant peaks and large phase shifts at each natural frequency
- Additive AR slope shifts at each natural frequency



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Longitudinal Transfer Function Matrix

- With $\mathbf{H_x} = \mathbf{I}$, $\mathbf{H_u} = 0$, and assuming
  - Elevator produces only a pitching moment
  - Throttle affects only the rate of change of velocity
  - Flaps produce only lift

$$\mathscr{H}_{Lon}(s) = \mathbf{H}_{\mathbf{x}_{Lon}}\left[s\mathbf{I} - \mathbf{F}_{Lon}\right]^{-1}\mathbf{G}_{Lon}$$

$$= \frac{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} n_V^V(s) & n_\gamma^V(s) & n_q^V(s) & n_\alpha^V(s) \\ n_V^\gamma(s) & n_\gamma^\gamma(s) & n_q^\gamma(s) & n_\alpha^\gamma(s) \\ n_V^q(s) & n_\gamma^q(s) & n_q^q(s) & n_\alpha^q(s) \\ n_V^\alpha(s) & n_\gamma^\alpha(s) & n_q^\alpha(s) & n_\alpha^\alpha(s) \end{bmatrix}\begin{bmatrix} 0 & T_{\delta T} & 0 \\ 0 & 0 & L_{\delta F}/V_N \\ M_{\delta E} & 0 & 0 \\ 0 & 0 & -L_{\delta F}/V_N \end{bmatrix}}{\Delta_{Lon}(s)}$$

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Longitudinal Transfer Function Matrix

- There are 4 outputs and 3 inputs

$$\mathcal{H}_{Lon}(s) = \frac{\begin{bmatrix} n_{\delta E}^{V}(s) & n_{\delta T}^{V}(s) & n_{\delta F}^{V}(s) \\ n_{\delta E}^{\gamma}(s) & n_{\delta T}^{\gamma}(s) & n_{\delta F}^{\gamma}(s) \\ n_{\delta E}^{q}(s) & n_{\delta T}^{q}(s) & n_{\delta F}^{q}(s) \\ n_{\delta E}^{\alpha}(s) & n_{\delta T}^{\alpha}(s) & n_{\delta F}^{\alpha}(s) \end{bmatrix}}{\left(s^2 + 2\zeta_P \omega_{nP} s + \omega_{n_P}{}^2\right)\left(s^2 + 2\zeta_{SP} \omega_{n_{SP}} s + \omega_{n_{SP}}{}^2\right)}$$

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Longitudinal Transfer Function Matrix Relates All Inputs to All Outputs

- Input-output relationship

$$\begin{bmatrix} \Delta V(s) \\ \Delta\gamma(s) \\ \Delta q(s) \\ \Delta\alpha(s) \end{bmatrix} = \mathcal{H}_{Lon}(s) \begin{bmatrix} \Delta\delta E(s) \\ \Delta\delta T(s) \\ \Delta\delta F(s) \end{bmatrix}$$

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Transfer Function Matrix for 2nd-Order Short-Period Approximation

Dynamic Equation

$$\Delta\dot{\mathbf{x}}_{SP}(t)=\begin{bmatrix}\Delta\dot{q}(t)\\ \Delta\dot{\alpha}(t)\end{bmatrix}\approx\begin{bmatrix}M_q & M_\alpha\\ \left(1-L_q/V_N\right) & -L_\alpha/V_N\end{bmatrix}\begin{bmatrix}\Delta q(t)\\ \Delta\alpha(t)\end{bmatrix}+\begin{bmatrix}M_{\delta E}\\ -L_{\delta E}/V_N\end{bmatrix}\Delta\delta E(t)$$

Transfer Function Matrix (with $H_x = I$, $H_u = 0$)

$$\mathcal{H}_{SP}(s)=\mathbf{I}_2\left(s\mathbf{I}-\mathbf{F}\right)_{SP}^{-1}(s)\mathbf{G}_{SP}=\begin{bmatrix}\left(s-M_q\right) & -M_\alpha\\ -\left(1-L_q/V_N\right) & \left(s+L_\alpha/V_N\right)\end{bmatrix}^{-1}\begin{bmatrix}M_{\delta E}\\ -L_{\delta E}/V_N\end{bmatrix}$$

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Transfer Function Matrix for Short-Period Approximation

Transfer Function Matrix (with $H_x = I$, $H_u = 0$)

Expand Inverse

$$\mathcal{H}_{SP}(s)=\left[s\mathbf{I}-\mathbf{F}_{Lon}\right]^{-1}\mathbf{G}_{SP}$$

$$=\frac{\begin{bmatrix}\left(s+L_\alpha/V_N\right) & M_\alpha\\ \left(1-L_q/V_N\right) & \left(s-M_q\right)\end{bmatrix}\begin{bmatrix}M_{\delta E}\\ -L_{\delta E}/V_N\end{bmatrix}}{\left(s-M_q\right)\left(s+L_\alpha/V_N\right)-M_\alpha\left(1-L_q/V_N\right)}$$

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Transfer Function Matrix for Short-Period Approximation

Expand Products

$$\mathscr{H}_{SP}(s)=\frac{\begin{bmatrix}\left[M_{\delta E}\left(s+\frac{L_\alpha}{V_N}\right)-\frac{L_{\delta E}M_\alpha}{V_N}\right]\\ \left[M_{\delta E}\left(1-\frac{L_q}{V_N}\right)-\left(\frac{L_{\delta E}}{V_N}\right)\left(s-M_q\right)\right]\end{bmatrix}}{s^2+\left(-M_q+\frac{L_\alpha}{V_N}\right)s-\left[M_\alpha\left(1-\frac{L_q}{V_N}\right)+M_q\frac{L_\alpha}{V_N}\right]}$$

Collect

$$=\frac{\begin{bmatrix}M_{\delta E}\left[s+\left(\frac{L_\alpha}{V_N}-\frac{L_{\delta E}M_\alpha}{V_N M_{\delta E}}\right)\right]\\ -\left(\frac{L_{\delta E}}{V_N}\right)\left\{s+\left[\frac{V_N M_{\delta E}}{L_{\delta E}}\left(1-\frac{L_q}{V_N}\right)-M_q\right]\right\}\end{bmatrix}}{\Delta_{SP}(s)}$$

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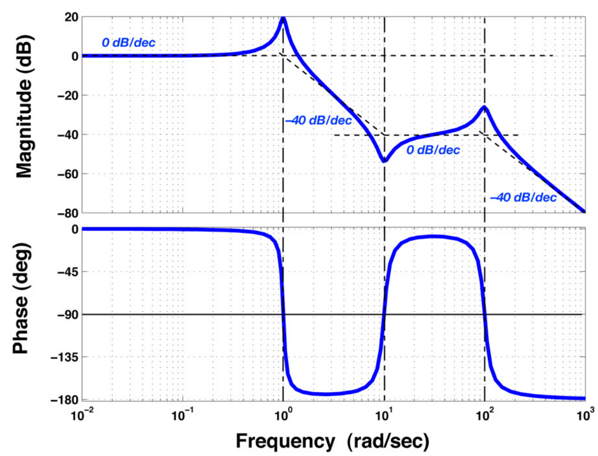
4^{th}-Order Transfer Function with 2^{nd}-Order Zero

$$\mathscr{H}(j\omega)=\frac{\left[(j\omega)^2+2(0.1)(10)(j\omega)+10^2\right]}{\left[(j\omega)^2+2(0.05)(1)(j\omega)+1^2\right]\left[(j\omega)^2+2(0.1)(100)(j\omega)+100^2\right]}$$



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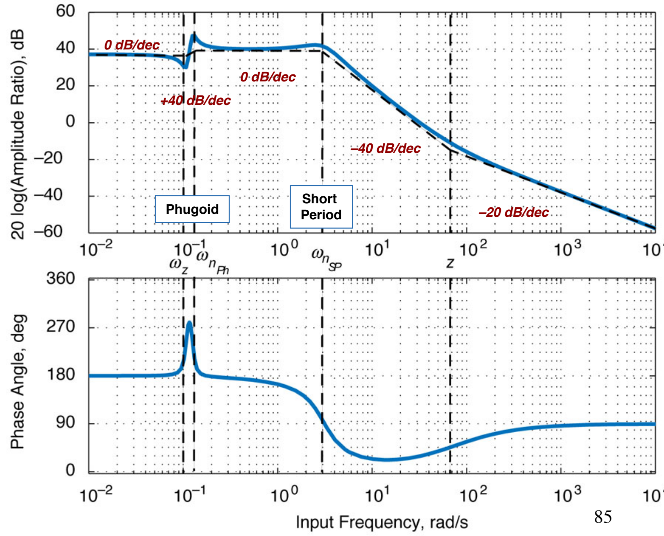
Elevator-to-Normal-Velocity Frequency Response

$$\frac{\Delta w(s)}{\Delta\delta E(s)} = \frac{n^{w}_{\delta E}(s)}{\Delta_{Lon}(s)} \approx \frac{M_{\delta E}\left(s^2+2\zeta\omega_n s+\omega_n^2\right)_{Approx\ Ph}\left(s-z_3\right)}{\left(s^2+2\zeta\omega_n s+\omega_n^2\right)_{Ph}\left(s^2+2\zeta\omega_n s+\omega_n^2\right)_{SP}}$$

- **$(n-q)=1$**
- **Complex zero almost (but not quite) cancels phugoid response**



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Response to a Control Input

Neglect initial condition and disturbance

State response to control

$$s\Delta\mathbf{x}(s) = \mathbf{F}\Delta\mathbf{x}(s) + \mathbf{G}\Delta\mathbf{u}(s) + \Delta\mathbf{x}(0), \quad \Delta\mathbf{x}(0) \triangleq \mathbf{0}$$

$$\Delta\mathbf{x}(s) = \left[s\mathbf{I}-\mathbf{F}\right]^{-1}\mathbf{G}\,\Delta\mathbf{u}(s)$$

Output response to control

$$\begin{aligned}\Delta\mathbf{y}(s) &= \mathbf{H_x}\Delta\mathbf{x}(s) + \mathbf{H_u}\Delta\mathbf{u}(s) \\ &= \mathbf{H_x}\left[s\mathbf{I}-\mathbf{F}\right]^{-1}\mathbf{G}\,\Delta\mathbf{u}(s) + \mathbf{H_u}\Delta\mathbf{u}(s) \\ &= \left\{\mathbf{H_x}\left[s\mathbf{I}-\mathbf{F}\right]^{-1}\mathbf{G} + \mathbf{H_u}\right\}\Delta\mathbf{u}(s)\end{aligned}$$

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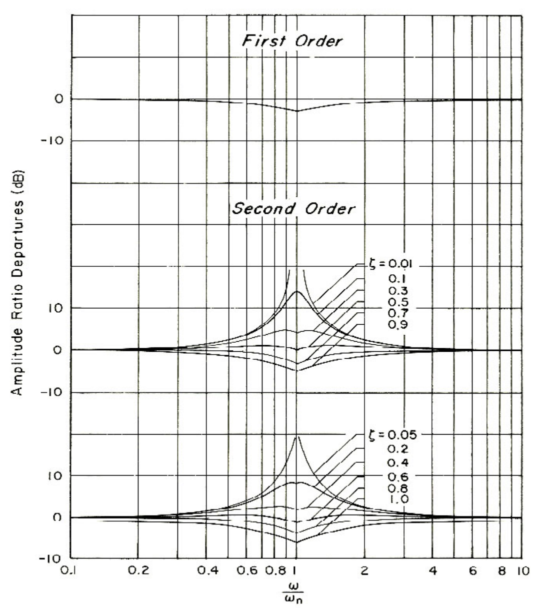
First- and Second-Order Departures from Amplitude Ratio Asymptotes



Frequency Response AR Departures in the Vicinity of Poles

- Difference between actual amplitude ratio (*dB*) and asymptote = departure (dB)
- Results for multiple roots are additive
- Zero departures have opposite sign

McRuer, Ashkenas, and Graham, *Aircraft Dynamics and Automatic Control*, Princeton University Press, 1973

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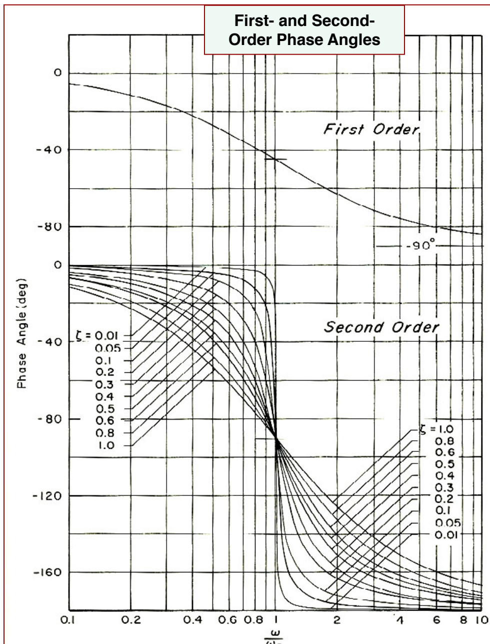
First- and Second-Order Phase Angles



Phase Angle Variations in the Vicinity of Poles

- Results for multiple roots are additive
- LHP zero variations have opposite sign
- RHP zeros have same sign

McRuer, Ashkenas, and Graham, *Aircraft Dynamics and Automatic Control*

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