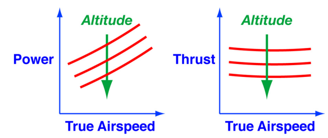
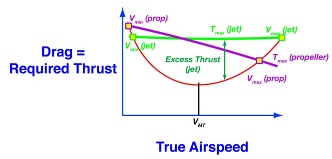
Power and Thrust for Cruising Flight

Robert Stengel, Aircraft Flight Dynamics, MAE 331, 2018

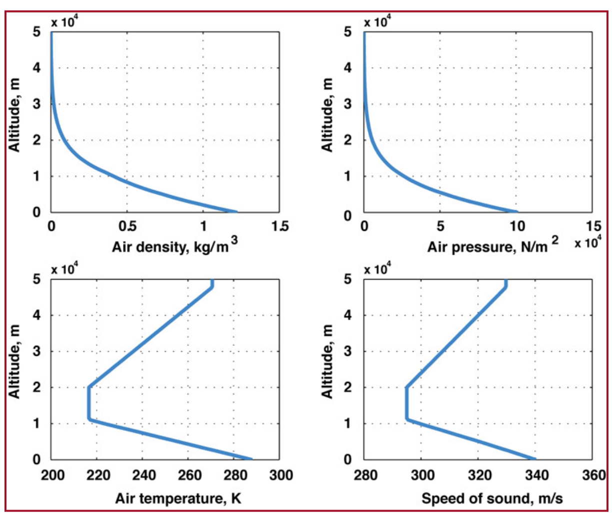




http://www.princeton.edu/~stengel/MAE331.html
http://www.princeton.edu/~stengel/FlightDynamics.html

1

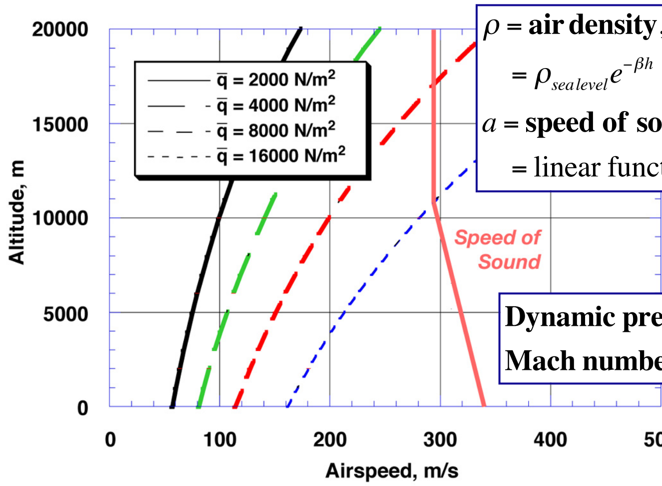
U.S. Standard Atmosphere, 1976



http://en.wikipedia.org/wiki/U.S._Standard_Atmosphere

2

Dynamic Pressure and Mach Number



$\rho =$ **air density**, function of height

$= \rho_{sealevel} e^{-\beta h}$

$a =$ **speed of sound**

$=$ linear function of height

Dynamic pressure $= \bar{q} \triangleq \rho V^2/2$

Mach number $= V/a$

3

Definitions of Airspeed

Airspeed is speed of aircraft measured with respect to air mass

Airspeed = Inertial speed if wind speed = 0

- **Indicated Airspeed (IAS)**

$$IAS = \sqrt{2\left(p_{stagnation} - p_{ambient}\right)/\rho_{SL}} = \sqrt{\frac{2\left(p_{total} - p_{static}\right)}{\rho_{SL}}}$$

$$\triangleq \sqrt{\frac{2q_c}{\rho_{SL}}}, \text{ with } q_c \triangleq \textbf{impact pressure}$$

- **Calibrated Airspeed (CAS)***

$$CAS = IAS \text{ corrected for instrument and position errors}$$

$$= \sqrt{\frac{2\left(q_c\right)_{corr\#1}}{\rho_{SL}}}$$

4

*** Kayton & Fried, 1969; NASA TN-D-822, 1961***

Definitions of Airspeed

Airspeed is speed of aircraft measured with respect to air mass

Airspeed = Inertial speed if wind speed = 0

Equivalent Airspeed (EAS)*

$$EAS = CAS\ corrected\ \ for\ compressibility\ effects = \sqrt{\frac{2(q_c)_{corr\#2}}{\rho_{SL}}}$$

True Airspeed (TAS)*

$$V \triangleq TAS = EAS\sqrt{\frac{\rho_{SL}}{\rho(z)}} = IAS_{corrected}\sqrt{\frac{\rho_{SL}}{\rho(z)}}$$

Mach number

$$M = \frac{TAS}{a}$$

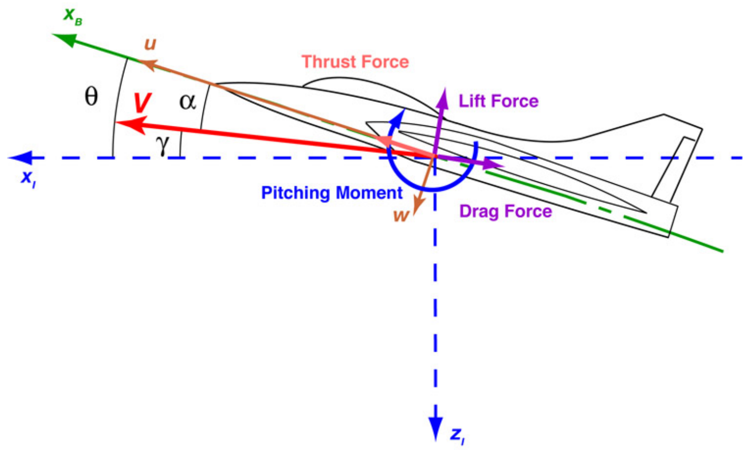
*** Kayton & Fried, 1969; NASA TN-D-822, 1961***

5

Flight in the Vertical Plane

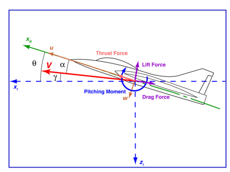
6

Longitudinal Variables



7

Longitudinal Point-Mass Equations of Motion



- **Assume thrust is aligned with the velocity vector (small-angle approximation for α)**
- **Mass = constant**

$$\dot{V} = \frac{\left(C_T \cos\alpha - C_D\right)\frac{1}{2}\rho V^2 S - mg\sin\gamma}{m} \approx \frac{\left(C_T - C_D\right)\frac{1}{2}\rho V^2 S - mg\sin\gamma}{m}$$

$$\dot{\gamma} = \frac{\left(C_T \sin\alpha + C_L\right)\frac{1}{2}\rho V^2 S - mg\cos\gamma}{mV} \approx \frac{C_L\frac{1}{2}\rho V^2 S - mg\cos\gamma}{mV}$$

$$\dot{h} = -\dot{z} = -v_z = V\sin\gamma$$

$$\dot{r} = \dot{x} = v_x = V\cos\gamma$$

$V = velocity =$ Earth-relative airspeed
= True airspeed with zero wind
$\gamma = flight\ path\ angle$
$h = height\ (altitude)$
$r = range$

8

Conditions for Steady, Level Flight

- Flight path angle = 0
- Altitude = constant
- Airspeed = constant
- Dynamic pressure = constant

$$0 = \frac{(C_T - C_D)\frac{1}{2}\rho V^2 S}{m}$$

- Thrust = Drag

$$0 = \frac{C_L \frac{1}{2}\rho V^2 S - mg}{mV}$$

- Lift = Weight

$$\dot{h} = 0$$

$$\dot{r} = V$$

Power and Thrust

Propeller

$$Power = P = T \times V = C_T \frac{1}{2}\rho V^3 S \approx \text{independent of airspeed}$$

Turbojet

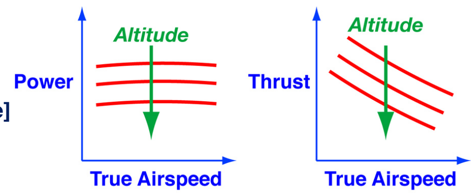
$$Thrust = T = C_T \frac{1}{2}\rho V^2 S \approx \text{independent of airspeed}$$

Throttle Effect

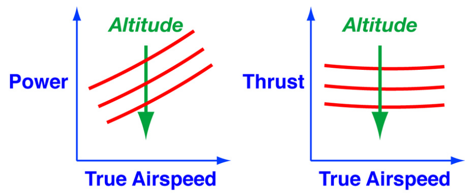
$$T = T_{\max}\delta T = \left[C_{T_{\max}} \bar{q} S\right]\delta T, \quad 0 \le \delta T \le 1$$

Typical Effects of Altitude and Velocity on Power and Thrust

- **Propeller**
  [Air-breathing engine]



- **Turbofan** — ***[In between]***
- **Turbojet**



- **Battery** — ***[Independent of altitude and airspeed]***

11

Models for Altitude Effect on Turbofan Thrust

From *Flight Dynamics*, pp.117-118

$$
\begin{aligned}
Thrust &= C_T\left(V,\delta T\right)\frac{1}{2}\rho\left(h\right)V^2 S \\
&= \left[\left(k_o + k_1 V^n\right)\frac{1}{2}\rho\left(h\right)V^2 S\right]\delta T\text{, N}
\end{aligned}
$$

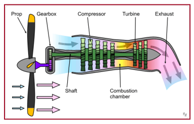
$k_o =$ Static thrust coefficient at sea level

$k_1 =$ Velocity sensitivity of thrust coefficient

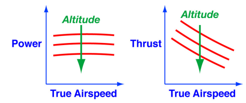
$n =$ Exponent of velocity sensitivity [$= -2$ for turbojet]

$$\rho\left(h\right) = \rho_{SL} e^{-\beta h}, \quad \rho_{SL} = 1.225\, kg/m^3, \quad \beta = \left(1/9{,}042\right) m^{-1}$$

12



Thrust of a Propeller-Driven Aircraft



With constant *rpm*, variable-pitch propeller

$$T = \eta_P \eta_I \frac{P_{engine}}{V} = \eta_{net} \frac{P_{engine}}{V}$$

$$\begin{aligned} \eta_P &= propeller\ efficiency \\ \eta_I &= ideal\ propulsive\ efficiency \\ &= TV/T\left(V + \Delta V_{inflow}\right) = V/\left(V + \Delta V_{freestream}/2\right) \\ \eta_{net_{\max}} &\approx 0.85 - 0.9 \end{aligned}$$

Efficiencies decrease with airspeed

Engine power decreases with altitude

Proportional to air density, w/o supercharger

13

Reciprocating-Engine Power and Specific Fuel Consumption (SFC)

$$\frac{P(h)}{P_{SL}} = 1.132\frac{\rho(h)}{\rho_{SL}} - 0.132$$

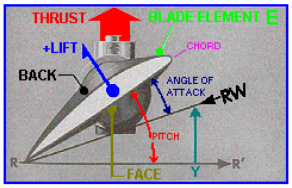
$$SFC \propto \text{ Independent of Altitude}$$

- **Engine power decreases with altitude**
  - **Proportional to air density, w/o supercharger**
  - **Supercharger increases inlet manifold pressure, increasing power and extending maximum altitude**

Anderson (Torenbeek)

14

Propeller Efficiency, η_P, and Advance Ratio, J



Advance Ratio

$$J = \frac{V}{nD}$$

where
$V = airspeed, m/s$
$n = rotation\ rate, revolutions/s$
$D = propeller\ diameter, m$

Effect of propeller-blade pitch angle

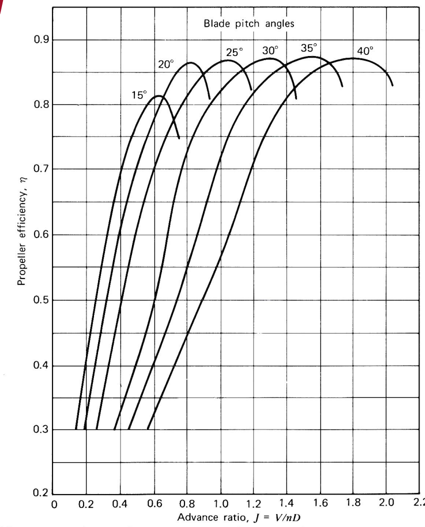
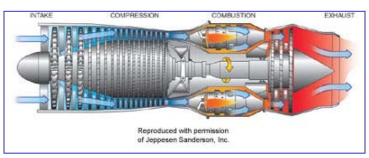
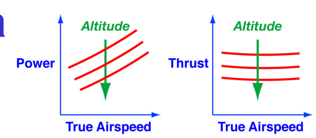


Figure 6.19 Estimated propeller efficiency for the Piper Cherokee Arrow PA-28R.

from McCormick



Thrust of a Turbojet Engine



$$T = \dot{m}V\left\{\left[\left(\frac{\theta_o}{\theta_o - 1}\right)\left(\frac{\theta_t}{\theta_t - 1}\right)(\tau_c - 1) + \frac{\theta_t}{\theta_o \tau_c}\right]^{1/2} - 1\right\}$$

$$\dot{m} = \dot{m}_{air} + \dot{m}_{fuel}$$

$$\theta_o = \left(p_{stag}/p_{ambient}\right)^{(\gamma-1)/\gamma};\quad \gamma = ratio\ of\ specific\ heats \approx 1.4$$

$$\theta_t = \left(turbine\ inlet\ temp./freestream\ ambient\ temp.\right)$$

$$\tau_c = \left(compressor\ outlet\ temp./compressor\ inlet\ temp.\right)$$

from Kerrebrock

Little change in thrust with airspeed below M_{crit}
Decrease with increasing altitude

Electric Propulsion

Available Energy	*Power Harvesting*	*Usable Power*	*Available Thrust*
Energy Storage	**Energy Conversion**	**Power Conditioning**	**Thrust Generation**
Batteries Fuel Sun	Solar Cells Fuel Cells Turbine/Reciprocating Engines	Transmission Inverters Regulators Generators Gearing	Motors Propellers Ducted Fans
	Conversion Efficiency < 1	Conditioning Efficiency < 1	Propulsive Efficiency < 1

Specific Energy and Energy Density of Fuel and Batteries *(typical)*

- Specific energy = energy/unit mass
- Energy density = energy/unit volume

Energy Storage Material	Specific Energy, MJ/kg	Energy Density, MJ/L
Lithium-Ion Battery	0.4-0.9	0.9-2.6
Jet Engine Fuel (Kerosene)	43	37
Gasoline	46	34
Methane (Liquified)	56	22
Hydrogen (Liquified)	142	9

Fuel cell energy conversion efficiency: 40-60%

Solar cell power conversion efficiency: 30-45%
Solar irradiance: 1 kW/m^2

Engine/Motor Power, Thrust, and Efficiency *(typical)*

Engine/Motor Type	Power/Mass, kW/kg	Thermal Efficiency	Propulsive Efficiency
Supercharged Radial Engine	1.8	25-50%	~Propeller Efficiency
Turboshaft Engine	5	40-60%	~Propeller Efficiency
Brushless DC Motor	1-2	-	~Propeller Efficiency
	Thrust/Weight, -		
Turbojet Engine	10	40-60%	$\sim 1 - \lvert V_{exhaust} - V \rvert / V$
Turbofan Engine	4-5	40-60%	$\sim 1 - \lvert V_{exhaust} - V \rvert / V$

Zunum ZA-10 Hybrid-Electric Aircraft

- **12-passenger commuter aircraft (2023)**
- **Safran Ardiden 3Z turbine engine, 500kW (~ 650 shp)**
- **Lithium-ion batteries (TBD)**
- **Boeing and Jet Blue funding**
- **Goal: 610-nm (700-sm) range**
- **Turbo Commander test aircraft (2019)**

Performance Parameters

Lift-to-Drag Ratio $L/D = C_L/C_D$

Load Factor $n = L/W = L/mg$, "g"s

Thrust-to-Weight Ratio $T/W = T/mg$, "g"s

Wing Loading W/S, N/m^2 or lb/ft^2

21

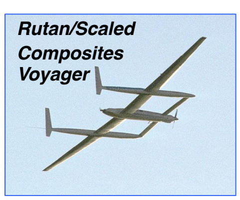
Historical Factoid

- **Aircraft Flight Distance Records**

http://en.wikipedia.org/wiki/Flight_distance_record

- **Aircraft Flight Endurance Records**

http://en.wikipedia.org/wiki/Flight_endurance_record



22

Steady, Level Flight

Trimmed Lift Coefficient, C_L

- **Trimmed lift coefficient, C_L**
  - Proportional to weight and wing loading factor
  - Decreases with V^2
  - At constant true airspeed, increases with altitude

$$\boxed{W = C_{L_{trim}}\left(\frac{1}{2}\rho V^2\right)S = C_{L_{trim}}\bar{q}S}$$

$$\boxed{C_{L_{trim}} = \frac{1}{\bar{q}}(W/S) = \frac{2}{\rho V^2}(W/S) = \left(\frac{2\,e^{\beta h}}{\rho_0 V^2}\right)(W/S)}$$

$\beta = 1/9{,}042$ m, inverse scale height of air density

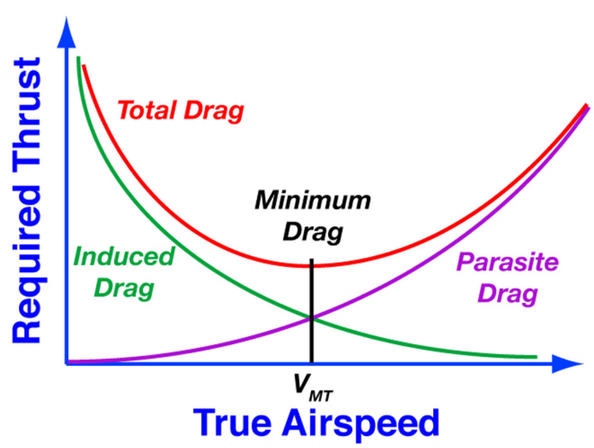
Trimmed Angle of Attack, α

- Trimmed angle of attack, α
  - Constant if dynamic pressure and weight are constant
  - If dynamic pressure decreases, angle of attack must increase

$$\boxed{\alpha_{trim} = \frac{2W/\rho V^2 S - C_{L_o}}{C_{L_\alpha}} = \frac{\frac{1}{\bar{q}}(W/S) - C_{L_o}}{C_{L_\alpha}}}$$

25

Thrust Required for Steady, Level Flight



26

Thrust Required for Steady, Level Flight

Trimmed thrust

Parasitic Drag *Induced Drag*

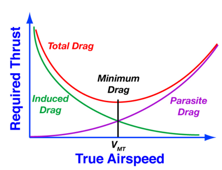
$$T_{trim} = D_{cruise} = C_{D_o}\left(\frac{1}{2}\rho V^2 S\right) + \varepsilon \frac{2W^2}{\rho V^2 S}$$

Minimum required thrust conditions

$$\frac{\partial T_{trim}}{\partial V} = C_{D_o}\left(\rho V S\right) - \frac{4\varepsilon W^2}{\rho V^3 S} = 0$$

Necessary Condition: Slope = 0

27



Necessary and Sufficient Conditions for Minimum Required Thrust

Necessary Condition = Zero Slope

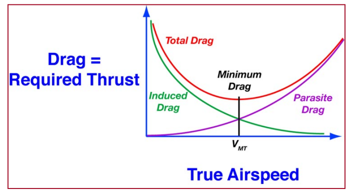
$$C_{D_o}\left(\rho V S\right) = \frac{4\varepsilon W^2}{\rho V^3 S}$$

Sufficient Condition for a Minimum = Positive Curvature when slope = 0

$$\frac{\partial^2 T_{trim}}{\partial V^2} = \underbrace{C_{D_o}\left(\rho S\right)}_{(+)} + \underbrace{\frac{12\varepsilon W^2}{\rho V^4 S}}_{(+)} > 0$$

28

Airspeed for Minimum Thrust in Steady, Level Flight



Satisfy necessary condition

$$V^4 = \left(\frac{4\varepsilon}{C_{D_o}\rho^2}\right)(W/S)^2$$

Fourth-order equation for velocity

Choose the positive root

$$V_{MT} = \sqrt{\frac{2}{\rho}\left(\frac{W}{S}\right)\sqrt{\frac{\varepsilon}{C_{D_o}}}}$$

29

Lift, Drag, and Thrust Coefficients in Minimum-Thrust Cruising Flight

Lift coefficient

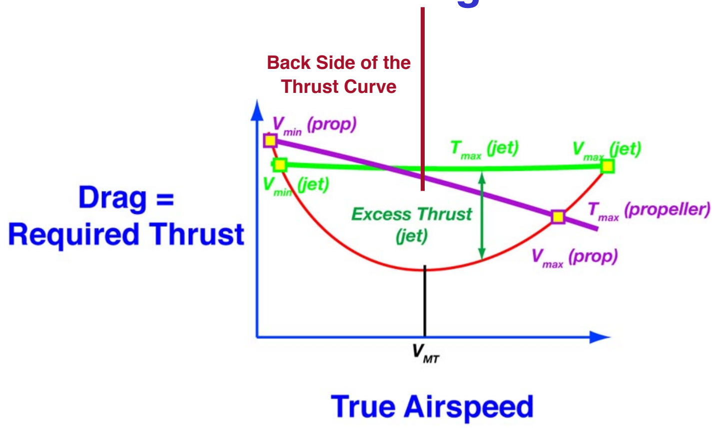
$$C_{L_{MT}} = \frac{2}{\rho V_{MT}^2}\left(\frac{W}{S}\right) = \sqrt{\frac{C_{D_o}}{\varepsilon}} = (C_L)_{(L/D)_{\max}}$$

Drag and thrust coefficients

$$C_{D_{MT}} = C_{D_o} + \varepsilon C_{L_{MT}}^2 = C_{D_o} + \varepsilon\frac{C_{D_o}}{\varepsilon} = 2C_{D_o} \equiv C_{T_{MT}}$$

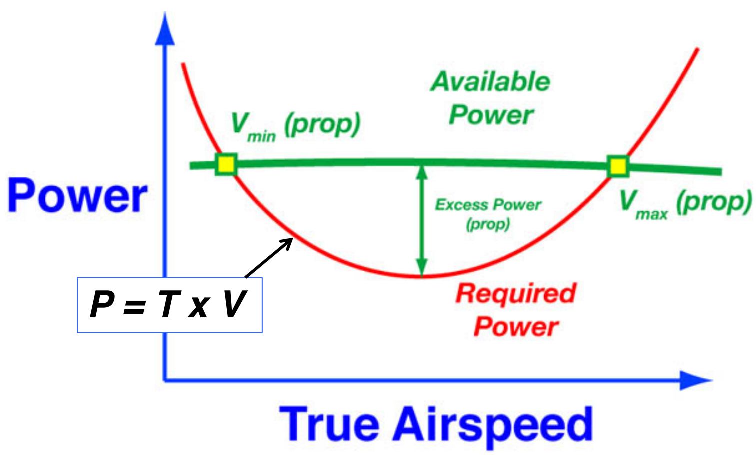
30

Achievable Airspeeds in Constant-Altitude Flight



- Two equilibrium airspeeds for a given thrust or power setting
  - Low speed, high C_L, high α
  - High speed, low C_L, low α
- Achievable airspeeds between minimum and maximum values with maximum thrust or power

Power Required for Steady, Level Flight



Power Required for Steady, Level Flight

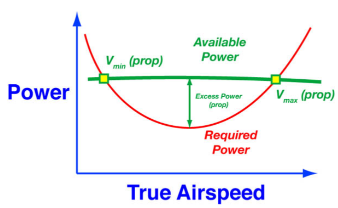
Trimmed power

Parasitic Drag *Induced Drag*

$$P_{trim} = T_{trim}V = D_{cruise}V = \left[C_{D_o}\left(\frac{1}{2}\rho V^2 S\right) + \frac{2\varepsilon W^2}{\rho V^2 S}\right]V$$

Minimum required power conditions

$$\frac{\partial P_{trim}}{\partial V} = C_{D_o}\frac{3}{2}\left(\rho V^2 S\right) - \frac{2\varepsilon W^2}{\rho V^2 S} = 0$$



Airspeed for Minimum Power in Steady, Level Flight

- **Satisfy necessary condition**

$$C_{D_o}\frac{3}{2}\left(\rho V^2 S\right) = \frac{2\varepsilon W^2}{\rho V^2 S}$$

- **Fourth-order equation for velocity**
  - Choose the positive root

$$V_{MP} = \sqrt{\frac{2}{\rho}\left(\frac{W}{S}\right)}\sqrt{\frac{\varepsilon}{3C_{D_o}}}$$

- **Corresponding lift and drag coefficients**

$$C_{L_{MP}} = \sqrt{\frac{3C_{D_o}}{\varepsilon}}$$

$$C_{D_{MP}} = 4C_{D_o}$$

Achievable Airspeeds for Jet in Cruising Flight

Thrust = constant

$$\boxed{T_{avail} = C_D \bar{q} S = C_{D_o}\left(\frac{1}{2}\rho V^2 S\right) + \frac{2\varepsilon W^2}{\rho V^2 S}}$$

$$\boxed{C_{D_o}\left(\frac{1}{2}\rho V^4 S\right) - T_{avail}V^2 + \frac{2\varepsilon W^2}{\rho S} = 0}$$

$$\boxed{V^4 - \frac{2T_{avail}}{C_{D_o}\rho S}V^2 + \frac{4\varepsilon W^2}{C_{D_o}(\rho S)^2} = 0}$$

4^{th}-order algebraic equation for V

Achievable Airspeeds for Jet in Cruising Flight

Solutions for V^2 can be put in quadratic form and solved easily

$$\boxed{V^2 \triangleq x; \quad V = \pm\sqrt{x}}$$

$$V^4 - \frac{2T_{avail}}{C_{D_o}\rho S}V^2 + \frac{4\varepsilon W^2}{C_{D_o}(\rho S)^2} = 0$$

$$x^2 + bx + c = 0$$

$$\boxed{x = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c} = V^2}$$

Thrust Required and Thrust Available for a Typical Bizjet

Available thrust *decreases* with altitude

Stall limitation at low speed

Mach number effect on lift and drag *increases* thrust required at high speed

Typical Simplified Jet Thrust Model

$$T_{\max}(h) = T_{\max}(SL)\left[\frac{\rho(SL)e^{-\beta h}}{\rho(SL)}\right]^x$$

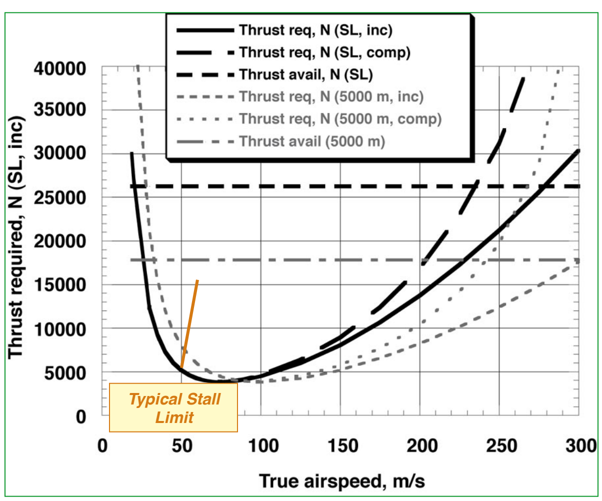
$$= T_{\max}(SL)\left[e^{-\beta h}\right]^x = T_{\max}(SL)e^{-x\beta h}$$

Empirical correction to force thrust to zero at a given altitude, h_{max}. c is a convergence factor.

$$T_{\max}(h) = T_{\max}(SL)e^{-x\beta h}\left[1 - e^{-(h-h_{\max})/c}\right]$$

37

Thrust Required and Thrust Available for a Typical Bizjet



38

Next Time: Cruising Flight Envelope

39

Supplemental Material

40

Models for Altitude Effect on Turbofan Thrust

From *AeroModelMach.m* in *FLIGHT.m*, *Flight Dynamics,*
http://www.princeton.edu/~stengel/AeroModelMach.m

```
[airDens,airPres,temp,soundSpeed] = Atmos(-x(6));
Thrust =  u(4) * StaticThrust * (airDens / 1.225)^0.7 * (1 - exp((-x(6) – 17000)/2000));
```

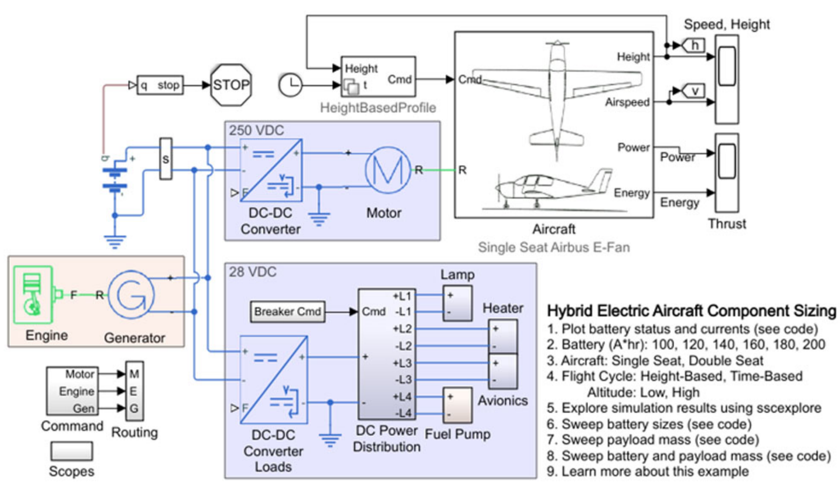
$$\text{Atmos(-x(6))}: \text{ 1976 U.S. Standard Atmosphere function}$$
$$\text{-x(6)} = h = \text{ Altitude, m}$$
$$\text{airDens} = \rho = \text{ Air density at altitude } h\text{, kg/m}^3$$
$$\text{u(4)} = \delta T = \text{ Throttle setting, } (0,1)$$

Empirical fit to match known characteristics of powerplant for generic business jet

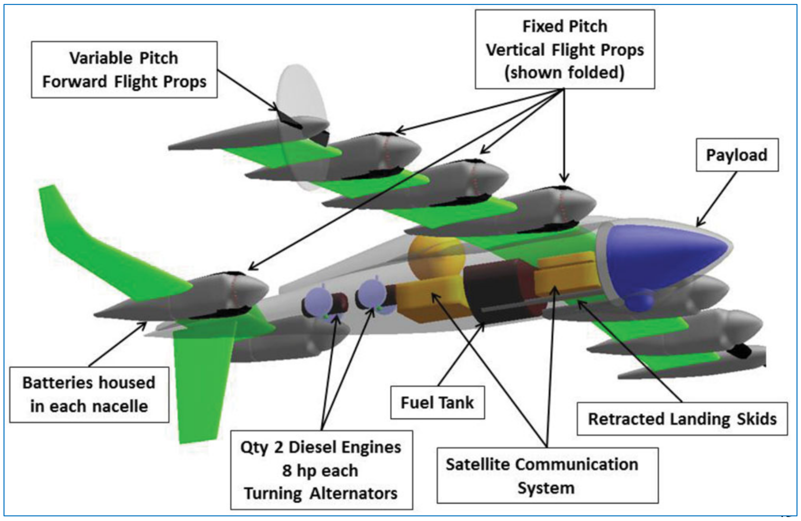
(airDens / 1.225)^0.7 * (1 - exp((-x(6) – 17000)/2000))

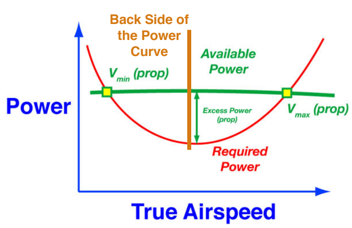
Hybrid-Electric Power System

Simulink Design Example



NASA Hybrid-Electric V/STOL UAV Concept





Achievable Airspeeds in Propeller-Driven Cruising Flight

Power = constant

$$\boxed{\begin{gathered} P_{avail} = T_{avail} V \\ V^4 - \frac{P_{avail} V}{C_{D_o} \rho S} + \frac{4 \varepsilon W^2}{C_{D_o} (\rho S)^2} = 0 \end{gathered}}$$

Solutions for *V* cannot be put in quadratic form; solution is more difficult, e.g., Ferrari's method

$$\boxed{aV^4 + (0)V^3 + (0)V^2 + dV + e = 0}$$

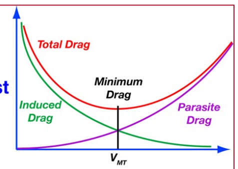
<u>Best bet</u>: *roots* in MATLAB

44

P-51 Mustang Minimum-Thrust Example

$$Wing\ Span = 37\ ft\ (9.83\ m)$$
$$Wing\ Area = 235\ ft^2\ (21.83\ m^2)$$
$$Loaded\ Weight = 9{,}200\ lb\ (3{,}465\ kg)$$
$$C_{D_o} = 0.0163$$
$$\varepsilon = 0.0576$$
$$W/S = 39.3\ lb/ft^2\ (1555.7\ N/m^2)$$

Drag = Required Thrust



True Airspeed

Airspeed for minimum thrust

$$V_{MT} = \sqrt{\frac{2}{\rho}\left(\frac{W}{S}\right)}\sqrt{\frac{\varepsilon}{C_{D_o}}} = \sqrt{\frac{2}{\rho}(1555.7)}\sqrt{\frac{0.947}{0.0163}} = \frac{76.49}{\sqrt{\rho}}\ m/s$$

Altitude, m	Air Density, kg/m^3	VMT, m/s
0	1.23	69.11
2,500	0.96	78.20
5,000	0.74	89.15
10,000	0.41	118.87

45

P-51 Mustang Maximum L/D Example

$$Wing\ Span = 37\ ft\ (9.83\ m)$$
$$Wing\ Area = 235\ ft\ (21.83\ m^2)$$
$$Loaded\ Weight = 9{,}200\ lb\ (3{,}465\ kg)$$
$$C_{D_o} = 0.0163$$
$$\varepsilon = 0.0576$$
$$W/S = 1555.7\ N/m^2$$

$$\left(C_D\right)_{L/D_{max}} = 2C_{D_o} = 0.0326$$

$$\left(C_L\right)_{L/D_{max}} = \sqrt{\frac{C_{D_o}}{\varepsilon}} = C_{L_{MT}} = 0.531$$

$$\left(L/D\right)_{max} = \frac{1}{2\sqrt{\varepsilon C_{D_o}}} = 16.31$$

$$V_{L/D_{max}} = V_{MT} = \frac{76.49}{\sqrt{\rho}}\ m/s$$

Altitude, m	Air Density, kg/m^3	VMT, m/s
0	1.23	69.11
2,500	0.96	78.20
5,000	0.74	89.15
10,000	0.41	118.87

46