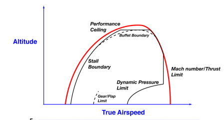
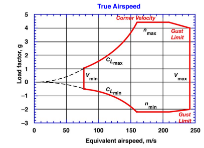
Gliding, Climbing, and Turning Flight Performance

Robert Stengel, Aircraft Flight Dynamics, MAE 331, 2018

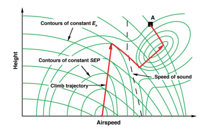


Learning Objectives

- Conditions for gliding flight
- Parameters for maximizing climb angle and rate
- Review the *V-n* diagram
- Energy height and specific excess power
- Alternative expressions for steady turning flight
- The *Herbst maneuver*



Reading:
Flight Dynamics
Aerodynamic Coefficients, 130-141



http://www.princeton.edu/~stengel/MAE331.html
http://www.princeton.edu/~stengel/FlightDynamics.html

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Review Questions

- *How does air density decrease with altitude?*
- *What are the different definitions of airspeed?*
- *What is a "lift-drag polar"?*
- *Power and thrust: How do they vary with altitude?*
- *What factors define the "flight envelope"?*
- *What were some features of the first commercial transport aircraft?*
- *What are the important parameters of the "Breguet Range Equation"?*
- *What is a "step climb", and why is it important?*

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Gliding Flight

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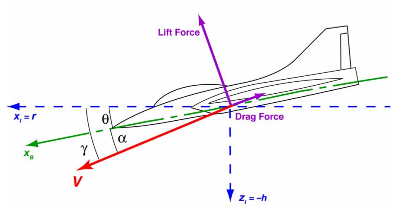
Equilibrium Gliding Flight



$$C_D \frac{1}{2}\rho V^2 S = -W \sin\gamma$$

$$C_L \frac{1}{2}\rho V^2 S = W \cos\gamma$$

$$\dot{h} = V \sin\gamma$$

$$\dot{r} = V \cos\gamma$$

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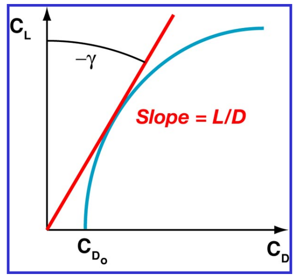
Gliding Flight

- Thrust = 0
- Flight path angle < 0 in gliding flight
- Altitude is decreasing
- Airspeed ~ constant
- Air density ~ constant



Gliding flight path angle

$$\tan\gamma = -\frac{D}{L} = -\frac{C_D}{C_L} = \frac{\dot{h}}{\dot{r}} = \frac{dh}{dr}; \quad \gamma = -\tan^{-1}\left(\frac{D}{L}\right) = -\cot^{-1}\left(\frac{L}{D}\right)$$

Corresponding airspeed

$$V_{glide} = \sqrt{\frac{2W}{\rho S\sqrt{C_D^2 + C_L^2}}}$$

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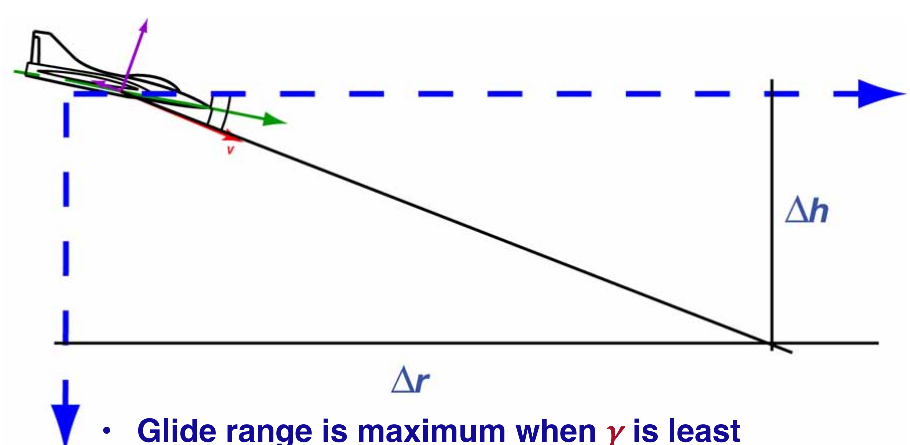
Maximum Steady Gliding Range



- Glide range is maximum when γ is least negative, i.e., most positive
- This occurs at $(L/D)_{max}$

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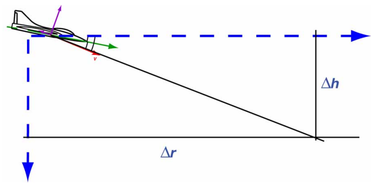

Maximum Steady Gliding Range

- Glide range is maximum when γ is least negative, i.e., most positive
- This occurs at $(L/D)_{max}$

$$\gamma_{max} = -\tan^{-1}\left(\frac{D}{L}\right)_{min} = -\cot^{-1}\left(\frac{L}{D}\right)_{max}$$

$$\tan\gamma = \frac{\dot{h}}{\dot{r}} = negative\ constant = \frac{(h - h_o)}{(r - r_o)}$$

$$\Delta r = \frac{\Delta h}{\tan\gamma} = \frac{-\Delta h}{-\tan\gamma} = maximum\ when\ \frac{L}{D} = maximum$$

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Sink Rate, m/s

Lift and drag define γ and V in gliding equilibrium

$$D = C_D \frac{1}{2}\rho V^2 S = -W \sin\gamma$$

$$\sin\gamma = -\frac{D}{W}$$

$$L = C_L \frac{1}{2}\rho V^2 S = W \cos\gamma$$

$$V = \sqrt{\frac{2W\cos\gamma}{C_L \rho S}}$$

Sink rate = altitude rate, *dh/dt* (negative)

$$\dot{h} = V\sin\gamma$$

$$= -\sqrt{\frac{2W\cos\gamma}{C_L\rho S}}\left(\frac{D}{W}\right) = -\sqrt{\frac{2W\cos\gamma}{C_L\rho S}}\left(\frac{L}{W}\right)\left(\frac{D}{L}\right)$$

$$= -\sqrt{\frac{2W\cos\gamma}{C_L\rho S}}\cos\gamma\left(\frac{1}{L/D}\right)$$

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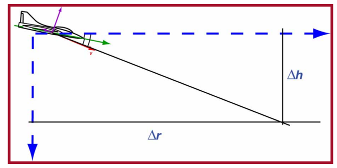
Conditions for Minimum Steady Sink Rate



- Minimum sink rate provides **maximum endurance**
- Minimize sink rate by setting $\partial(dh/dt)/\partial C_L = 0$ ($\cos\gamma \sim 1$)

$$\dot{h} = -\sqrt{\frac{2W\cos\gamma}{C_L\rho S}}\cos\gamma\left(\frac{C_D}{C_L}\right)$$

$$= -\sqrt{\frac{2W\cos^3\gamma}{\rho S}}\left(\frac{C_D}{C_L^{3/2}}\right) \approx -\sqrt{\frac{2}{\rho}\left(\frac{W}{S}\right)}\left(\frac{C_D}{C_L^{3/2}}\right)$$

$$C_{L_{ME}} = \sqrt{\frac{3C_{D_o}}{\varepsilon}} \quad and \quad C_{D_{ME}} = 4C_{D_o}$$

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L/D and V_{ME} for Minimum Sink Rate

$$\left(L/D\right)_{ME} = \frac{1}{4}\sqrt{\frac{3}{\varepsilon C_{D_o}}} = \frac{\sqrt{3}}{2}\left(L/D\right)_{\max} \approx 0.86\left(L/D\right)_{\max}$$

$$V_{ME} = \sqrt{\frac{2W}{\rho S\sqrt{C_{D_{ME}}^2 + C_{L_{ME}}^2}}} \approx \sqrt{\frac{2(W/S)}{\rho}}\sqrt{\frac{\varepsilon}{3C_{D_o}}} \approx 0.76V_{L/D_{\max}}$$

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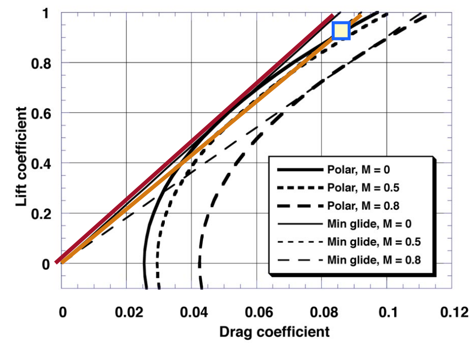
L/D for Minimum Sink Rate

- For *L/D* < *L/D*$_{max}$, there are two solutions
- Which one produces smaller sink rate?

$$\left(L/D\right)_{ME} \approx 0.86\left(L/D\right)_{\max}$$
$$V_{ME} \approx 0.76 V_{L/D_{\max}}$$



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Historical Factoids

Lifting-Body Reentry Vehicles

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Climbing Flight

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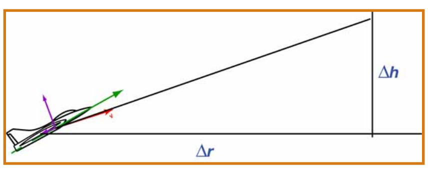

Climbing Flight

- **Flight path angle**

$$\dot{V} = 0 = \frac{(T - D - W\sin\gamma)}{m}$$

$$\sin\gamma = \frac{(T-D)}{W}; \quad \gamma = \sin^{-1}\frac{(T-D)}{W}$$

- **Required lift**

$$\dot{\gamma} = 0 = \frac{(L - W\cos\gamma)}{mV}$$

$$L = W\cos\gamma$$

Rate of climb, *dh/dt* = Specific Excess Power

$$\dot{h} = V\sin\gamma = V\frac{(T-D)}{W} = \frac{(P_{thrust} - P_{drag})}{W}$$

$$\textit{Specific Excess Power (SEP)} = \frac{\textit{Excess Power}}{\textit{Unit Weight}} \equiv \frac{(P_{thrust} - P_{drag})}{W}$$

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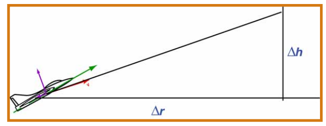
Steady Rate of Climb



Climb rate

$$\dot{h} = V \sin\gamma = V\left[\left(\frac{T}{W}\right) - \frac{\left(C_{D_o} + \varepsilon C_L^2\right)\bar{q}}{(W/S)}\right]$$

$$L = C_L \bar{q} S = W \cos\gamma$$

$$C_L = \left(\frac{W}{S}\right)\frac{\cos\gamma}{\bar{q}}$$

$$V = \sqrt{2\left(\frac{W}{S}\right)\frac{\cos\gamma}{C_L \rho}}$$

Note significance of thrust-to-weight ratio and wing loading

$$\dot{h} = V\left[\left(\frac{T}{W}\right) - \frac{C_{D_o}\bar{q}}{(W/S)} - \frac{\varepsilon (W/S)\cos^2\gamma}{\bar{q}}\right]$$

$$= V\left(\frac{T(h)}{W}\right) - \frac{C_{D_o}\rho(h)V^3}{2(W/S)} - \frac{2\varepsilon(W/S)\cos^2\gamma}{\rho(h)V}$$

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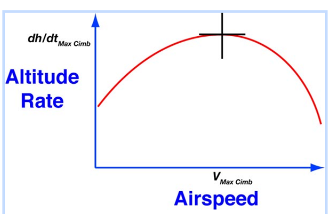

Condition for Maximum Steady Rate of Climb

$$\dot{h} = V\left(\frac{T}{W}\right) - \frac{C_{D_o}\rho V^3}{2(W/S)} - \frac{2\varepsilon (W/S)\cos^2\gamma}{\rho V}$$

Necessary condition for a maximum with respect to airspeed

$$\frac{\partial \dot{h}}{\partial V} = 0 = \left[\left(\frac{T}{W}\right) + V\left(\frac{\partial T/\partial V}{W}\right)\right] - \frac{3C_{D_o}\rho V^2}{2(W/S)} + \frac{2\varepsilon(W/S)\cos^2\gamma}{\rho V^2}$$

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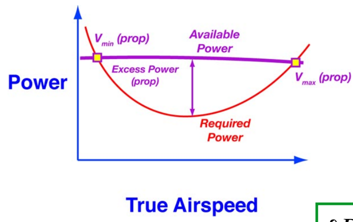

Maximum Steady Rate of Climb:

Propeller-Driven Aircraft

- At constant power

$$\frac{\partial P_{thrust}}{\partial V} = 0 = \left[\left(\frac{T}{W}\right) + V\left(\frac{\partial T / \partial V}{W}\right)\right]$$

- With $\cos^2\gamma \sim 1$, optimality condition reduces to

$$\frac{\partial \dot{h}}{\partial V} = 0 = -\frac{3C_{D_o}\rho V^2}{2(W/S)} + \frac{2\varepsilon(W/S)}{\rho V^2}$$

- Airspeed for maximum rate of climb at maximum power, P_{max}

$$V^4 = \left(\frac{4}{3}\right)\frac{\varepsilon(W/S)^2}{C_{D_o}\rho^2}; \quad V = \sqrt{2\frac{(W/S)}{\rho}\sqrt{\frac{\varepsilon}{3C_{D_o}}}} = V_{ME}$$

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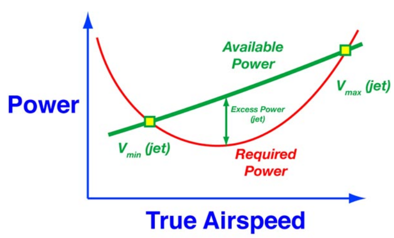

Maximum Steady Rate of Climb:

Jet-Driven Aircraft

Condition for a maximum at constant thrust and $\cos^2\gamma \sim 1$

$$\frac{\partial \dot{h}}{\partial V} = 0$$

$$-\frac{3C_{D_o}\rho}{2(W/S)}V^4 + \left(\frac{T}{W}\right)V^2 + \frac{2\varepsilon(W/S)}{\rho} = 0$$

$$-\frac{3C_{D_o}\rho}{2(W/S)}\left(V^2\right)^2 + \left(\frac{T}{W}\right)\left(V^2\right) + \frac{2\varepsilon(W/S)}{\rho} = 0$$

Quadratic in V^2

Airspeed for maximum rate of climb at maximum thrust, T_{max}

$$0 = ax^2 + bx + c \text{ and } \quad V = +\sqrt{x}$$

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Optimal Climbing Flight

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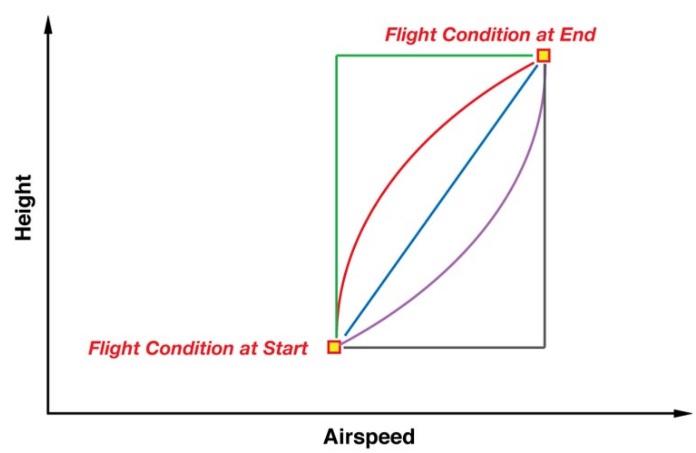
What is the Fastest Way to Climb from One Flight Condition to Another?



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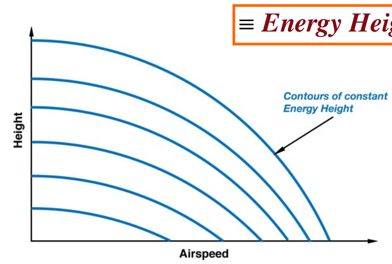
Energy Height

- **Specific Energy**
  - **= (Potential + Kinetic Energy) per Unit Weight**
  - **= Energy Height**

$$\textbf{Specific Energy} \equiv \frac{Total\ Energy}{Unit\ Weight} = \frac{mgh + mV^2/2}{mg} = h + \frac{V^2}{2g}$$

$$\equiv \textbf{Energy Height}, E_h, \quad ft\ or\ m$$



Can trade altitude for airspeed with no change in energy height if thrust and drag are zero

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Specific Excess Power

Rate of change of Specific Energy

$$\frac{dE_h}{dt} = \frac{d}{dt}\left(h + \frac{V^2}{2g}\right) = \frac{dh}{dt} + \left(\frac{V}{g}\right)\frac{dV}{dt}$$

$$= V\sin\gamma + \left(\frac{V}{g}\right)\left(\frac{T - D - mg\sin\gamma}{m}\right) = V\frac{(T-D)}{W}$$

$$= \textbf{Specific Excess Power}\ (SEP)$$

$$= \frac{\text{Excess Power}}{\text{Unit Weight}} \equiv \frac{\left(P_{thrust} - P_{drag}\right)}{W}$$

$$= V\frac{(C_T - C_D)\frac{1}{2}\rho(h)V^2 S}{W}$$

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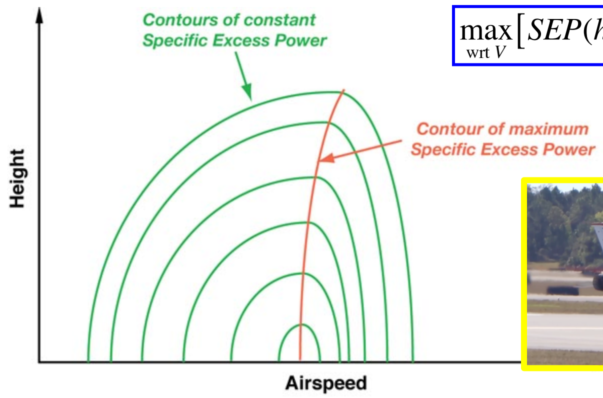
Contours of Constant Specific Excess Power

- Specific Excess Power is a function of altitude and airspeed
- *SEP* is maximized at each altitude, *h*, when

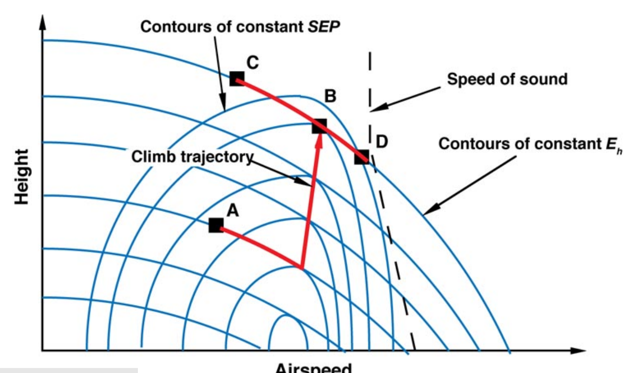
$$\max_{\text{wrt } V}\left[SEP(h)\right] \qquad \frac{d\left[SEP(h)\right]}{dV}=0$$



Subsonic Minimum-Time Energy Climb

Objective: Minimize time to climb to desired altitude and airspeed

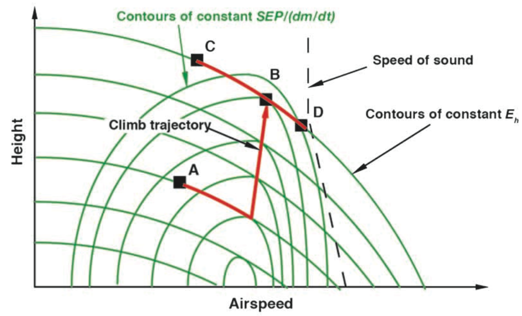
- **Minimum-Time Strategy:**
  - Zoom climb/dive to intercept $SEP_{max}(h)$ contour
  - Climb at $SEP_{max}(h)$
  - Zoom climb/dive to intercept target $SEP_{max}(h)$ contour



Bryson, Desai, Hoffman, 1969

Subsonic Minimum-Fuel Energy Climb

Objective: Minimize fuel to climb to desired altitude and airspeed



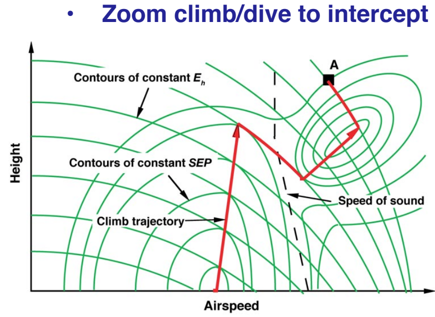
- **Minimum-Fuel Strategy:**
  - Zoom climb/dive to intercept [*SEP*(*h*)/(*dm/dt*)]$_{max}$ contour
  - Climb at [*SEP*(*h*)/(*dm/dt*)]$_{max}$
  - Zoom climb/dive to intercept target[*SEP*(*h*)/(*dm/dt*)]$_{max}$ contour

Bryson, Desai, Hoffman, 1969

Supersonic Minimum-Time Energy Climb

Objective: Minimize time to climb to desired altitude and airspeed

- **Minimum-Time Strategy:**
  - Intercept subsonic *SEP*$_{max}$(*h*) contour
  - Climb at *SEP*$_{max}$(*h*) to intercept matching zoom climb/dive contour
  - Zoom climb/dive to intercept supersonic *SEP*$_{max}$(*h*) contour
  - Climb at *SEP*$_{max}$(*h*) to intercept target *SEP*$_{max}$(*h*) contour
  - Zoom climb/dive to intercept target *SEP*$_{max}$(*h*) contour



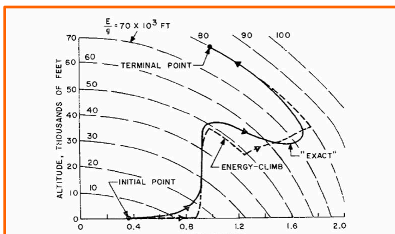


Fig. 5 Comparison of exact and energy-state minimum time-to-climb paths (airplane 1).

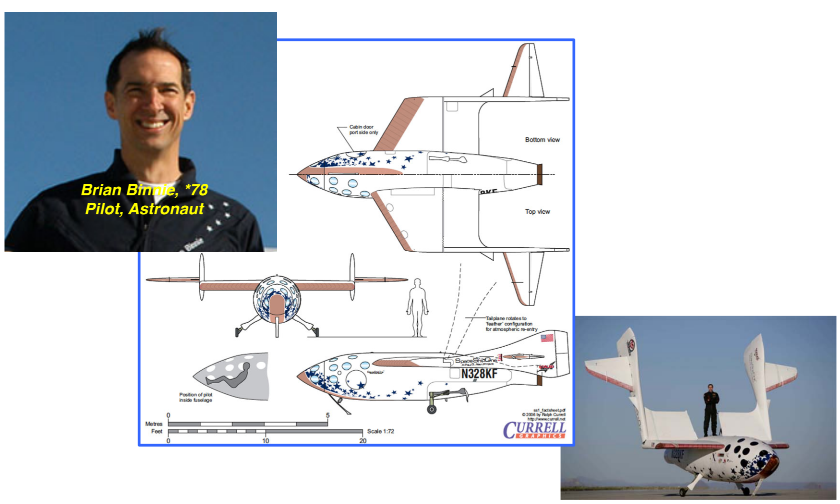
Bryson, Desai, Hoffman, 1969

Checklist

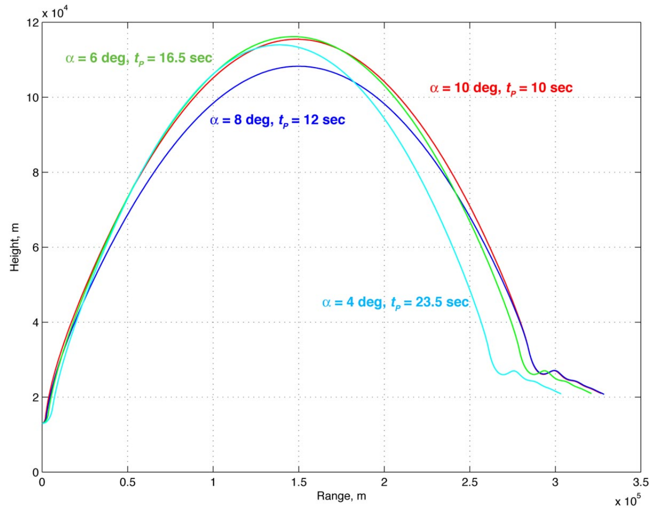
- ❑ *Energy height?*
- ❑ *Contours?*
- ❑ *Subsonic minimum-time climb?*
- ❑ *Supersonic minimum-time climb?*
- ❑ *Minimum-fuel climb?*

$$\frac{dE_h}{dm_{fuel}} = \frac{dE_h}{dt}\frac{dt}{dm_{fuel}} = \frac{1}{\dot{m}_{fuel}}\left[\frac{dh}{dt} + \left(\frac{V}{g}\right)\frac{dV}{dt}\right]$$

SpaceShipOne
Ansari X Prize, December 17, 2003



SpaceShipOne Altitude vs. Range
MAE 331 Assignment #4, 2010



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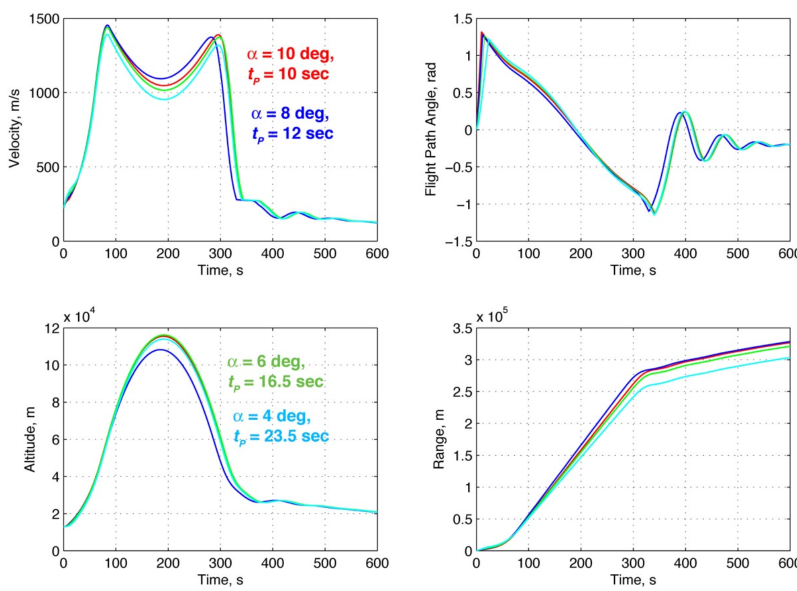
SpaceShipOne State Histories



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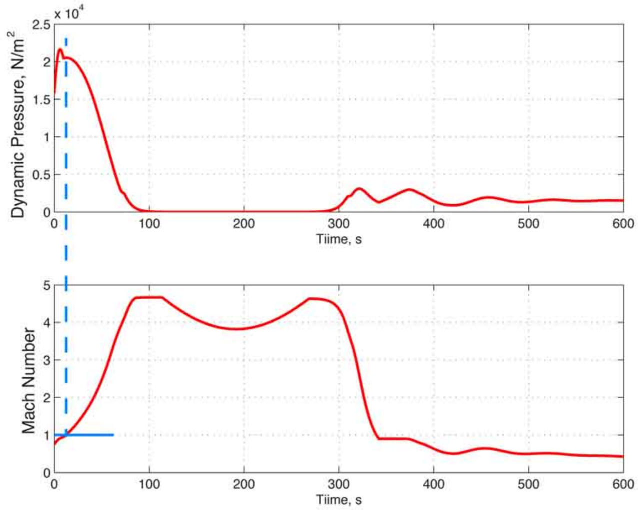
SpaceShipOne Dynamic Pressure and Mach Number Histories



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The Maneuvering Envelope

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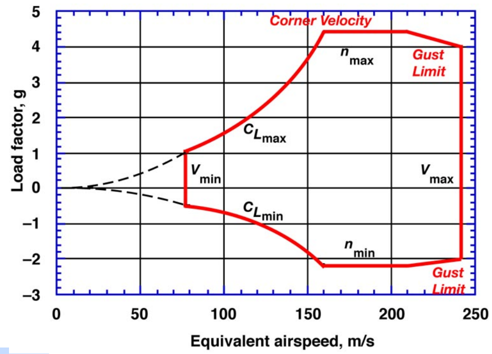
Typical Maneuvering Envelope: *V-n* Diagram

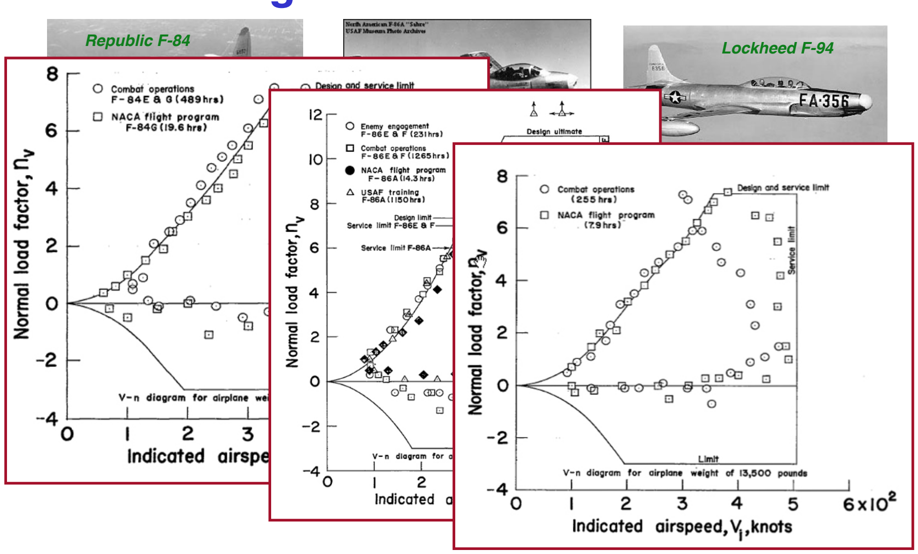
- **Maneuvering envelope**: limits on normal load factor and allowable equivalent airspeed
  - Structural factors
  - Maximum and minimum achievable lift coefficients
  - Maximum and minimum airspeeds
  - Protection against overstressing due to gusts
  - Corner Velocity: Intersection of maximum lift coefficient and maximum load factor



- Typical positive load factor limits
  - Transport: > 2.5
  - Utility: > 4.4
  - Aerobatic: > 6.3
  - Fighter: > 9

- Typical negative load factor limits
  - Transport: < –1
  - Others: < –1 to –3

Maneuvering Envelopes (*V-n Diagrams*) for Three Fighters of the Korean War Era



Turning Flight

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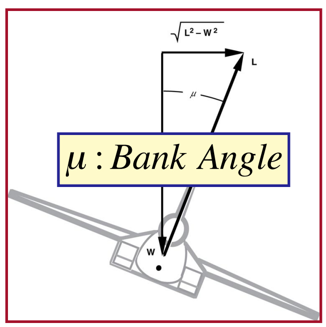
Level Turning Flight

- Level flight = constant altitude
- Sideslip angle = 0
- Vertical force equilibrium

$$L\cos\mu = W$$

- Load factor

$$n = L/W = L/mg = \sec\mu, \text{"}g\text{"}s$$



- Thrust required to maintain level flight

$$T_{req} = \left(C_{D_o} + \varepsilon C_L{}^2\right)\frac{1}{2}\rho V^2 S = D_o + \frac{2\varepsilon}{\rho V^2 S}\left(\frac{W}{\cos\mu}\right)^2$$

$$= D_o + \frac{2\varepsilon}{\rho V^2 S}(nW)^2$$

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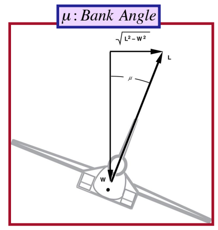
Maximum Bank Angle in Steady Level Flight



Bank angle

$$\cos\mu = \frac{W}{C_L \bar{q} S} = \frac{1}{n} = W\sqrt{\frac{2\varepsilon}{\left(T_{req} - D_o\right)\rho V^2 S}}$$

$$\mu = \cos^{-1}\left(\frac{W}{C_L \bar{q} S}\right) = \cos^{-1}\left(\frac{1}{n}\right) = \cos^{-1}\left[W\sqrt{\frac{2\varepsilon}{\left(T_{req} - D_o\right)\rho V^2 S}}\right]$$

Bank angle is limited by

$$C_{L_{max}} \; or \; T_{max} \; or \; n_{max}$$

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Turning Rate and Radius in Level Flight

Turning rate

$$\dot{\xi} = \frac{C_L \bar{q} S \sin\mu}{mV} = \frac{W \tan\mu}{mV} = \frac{g \tan\mu}{V} = \frac{\sqrt{L^2 - W^2}}{mV} = \frac{W\sqrt{n^2 - 1}}{mV} = \frac{\sqrt{\left(T_{req} - D_o\right)\rho V^2 S / 2\varepsilon - W^2}}{mV}$$

Turning rate is limited by

$$C_{L_{max}} \; or \; T_{max} \; or \; n_{max}$$

Turning radius

$$R_{turn} = \frac{V}{\dot{\xi}} = \frac{V^2}{g\sqrt{n^2 - 1}}$$

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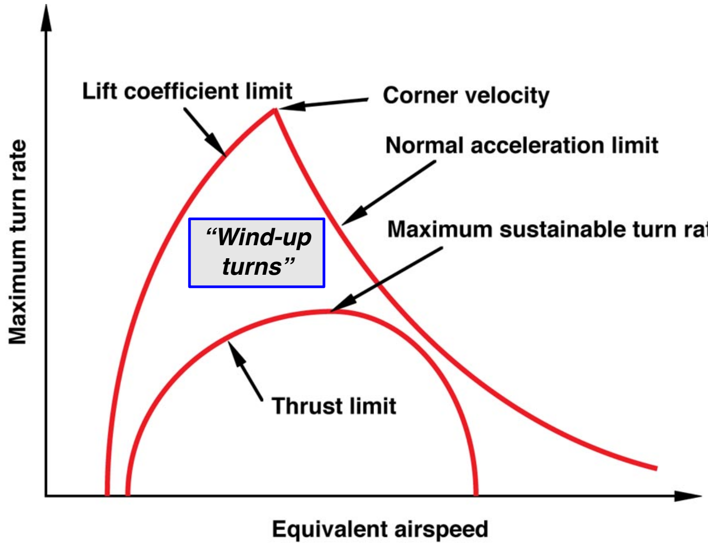
Maximum Turn Rates



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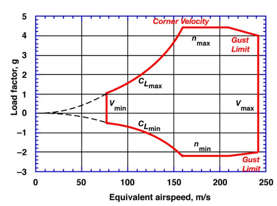
Corner Velocity Turn



- **Corner velocity**

$$V_{corner} = \sqrt{\frac{2n_{\max}W}{C_{L_{mas}}\rho S}}$$

- **For steady climbing or diving flight**

$$\sin\gamma = \frac{T_{\max} - D}{W}$$

- **Turning radius**

$$R_{turn} = \frac{V^2\cos^2\gamma}{g\sqrt{n_{\max}^2 - \cos^2\gamma}}$$

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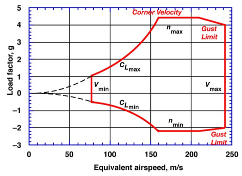

Corner Velocity Turn

- **Turning rate**

$$\dot{\xi} = \sqrt{\frac{g\left(n_{max}^2 - \cos^2\gamma\right)}{V\cos\gamma}}$$

- **Time to complete a full circle**

$$t_{2\pi} = \frac{V\cos\gamma}{g\sqrt{n_{max}^2 - \cos^2\gamma}}$$

- **Altitude gain/loss**

$$\Delta h_{2\pi} = t_{2\pi} V \sin\gamma$$

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Checklist

- ❑ ***V-n diagram?***
- ❑ ***Maneuvering envelope?***
- ❑ ***Level turning flight?***
- ❑ ***Limiting factors?***
- ❑ ***Wind-up turn?***
- ❑ ***Corner velocity?***

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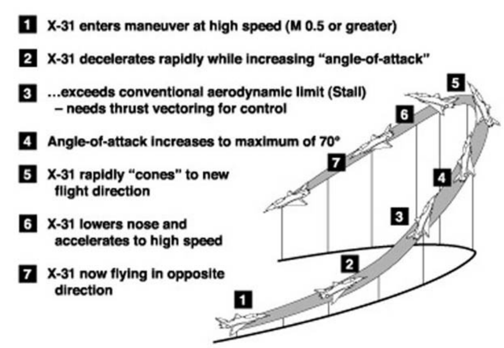
Herbst Maneuver

- Minimum-time reversal of direction
- Kinetic-/potential-energy exchange
- Yaw maneuver at low airspeed
- *X-31* performing the maneuver



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Next Time: Aircraft Equations of Motion

Reading:
Flight Dynamics,
Section 3.1, 3.2, pp. 155-161

Learning Objectives

What use are the equations of motion?
How is the angular orientation of the airplane described?
What is a cross-product-equivalent matrix?
What is angular momentum?
How are the inertial properties of the airplane described?
How is the rate of change of angular momentum calculated?

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Supplemental Material

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Gliding Flight of the P-51 Mustang

Maximum Range Glide

Loaded Weight = 9,200 lb (3,465 kg)

$$(L/D)_{max} = \frac{1}{2\sqrt{\varepsilon C_{D_o}}} = 16.31$$

$$\gamma_{MR} = -\cot^{-1}\left(\frac{L}{D}\right)_{max} = -\cot^{-1}(16.3) = -3.5°$$

$$(C_D)_{L/D_{max}} = 2C_{D_o} = 0.0326$$

$$(C_L)_{L/D_{max}} = \sqrt{\frac{C_{D_o}}{\varepsilon}} = 0.531$$

$$V_{L/D_{max}} = \frac{76.49}{\sqrt{\rho}} \text{ m/s}$$

$$\dot{h}_{L/D_{max}} = V \sin\gamma = -\frac{4.68}{\sqrt{\rho}} \text{ m/s}$$

$$R_{h_o=10\,km} = (16.31)(10) = 163.1\ km$$

Maximum Endurance Glide

Loaded Weight = 9,200 lb (3,465 kg)

$$S = 21.83 \text{ m}^2$$

$$C_{D_{ME}} = 4C_{D_o} = 4(0.0163) = 0.0652$$

$$C_{L_{ME}} = \sqrt{\frac{3C_{D_o}}{\varepsilon}} = \sqrt{\frac{3(0.0163)}{0.0576}} = 0.921$$

$$(L/D)_{ME} = 14.13$$

$$\dot{h}_{ME} = -\sqrt{\frac{2}{\rho}\left(\frac{W}{S}\right)}\left(\frac{C_{D_{ME}}}{C_{L_{ME}}^{3/2}}\right) = -\frac{4.11}{\sqrt{\rho}} \text{ m/s}$$

$$\gamma_{ME} = -4.05°$$

$$V_{ME} = \frac{58.12}{\sqrt{\rho}} \text{ m/s}$$

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