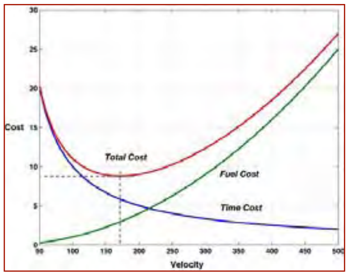
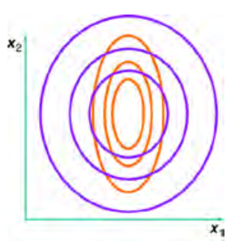


Introduction to Optimization

Robert Stengel
Robotics and Intelligent Systems,
MAE 345, Princeton University, 2017

Optimization problems and criteria
Cost functions
Static optimality conditions
Examples of static optimization



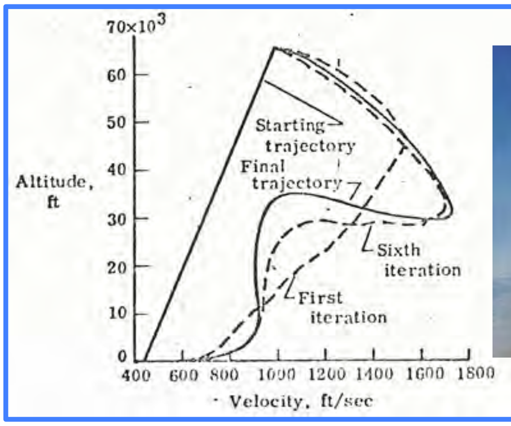
Typical Optimization Problems

- Minimize the probable error in an estimate of the dynamic state of a system
- Maximize the probability of making a correct decision
- Minimize the time or energy required to achieve an objective
- Minimize the regulation error in a controlled system

- *Estimation*
- *Control*

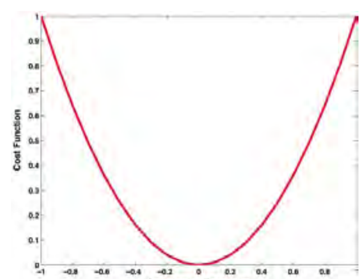
Optimization Implies Choice

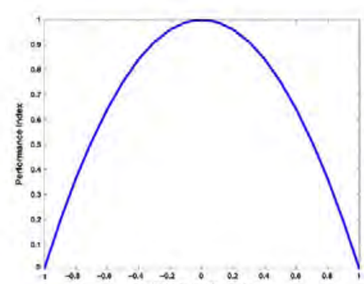
- Choice of **best strategy**
- Choice of **best design parameters**
- Choice of **best control history**
- Choice of **best estimate**
- Optimization provided by selection of the best control variable



Criteria for Optimization

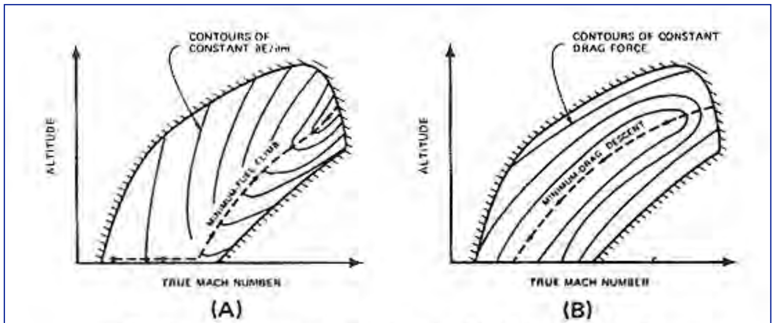
- Names for criteria
  - Figure of merit
  - Performance index
  - Utility function
  - Value function
  - Fitness function
  - Cost function, *J*
    - Optimal cost function = *J**
    - Optimal control = u*
- Different criteria lead to different optimal solutions
- Types of Optimality Criteria
  - Absolute
  - Regulatory
  - Feasible





Minimize Absolute Criteria

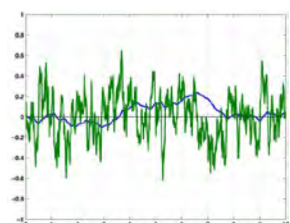
Achieve a specific objective, such as minimizing the required time, fuel, or financial cost to perform a task



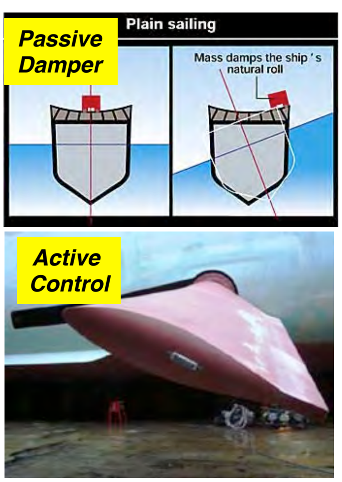
Fuel-optimal energy transitions. (A) Minimum fuel energy climb. (B) Minimum drag energy descent.

What is the control variable?

Optimal System Regulation

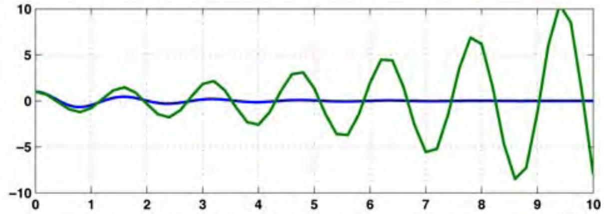


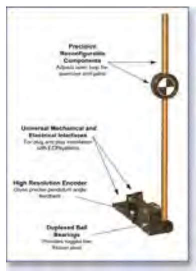
Design controller to minimize tracking error, Δx, in presence of random disturbances

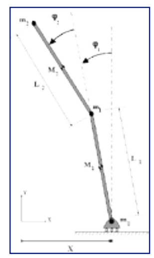


Feasible Control Logic

Find feedback control structure that guarantees stability (i.e., that prevents divergence)

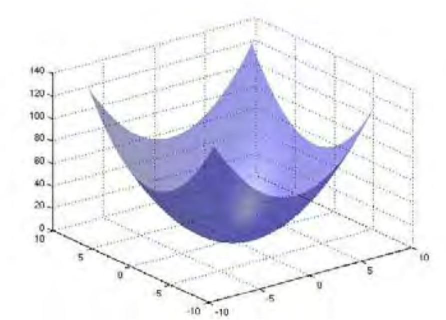




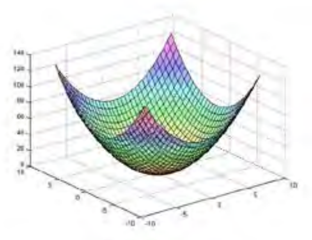


Single Inverted Pendulum	Double Inverted Pendulum
http://www.youtube.com/watch?v=mi-tek7HvZs	*http://www.youtube.com/watch?v=8HDDzKxNMEY*

Desirable Characteristics of a Cost Function

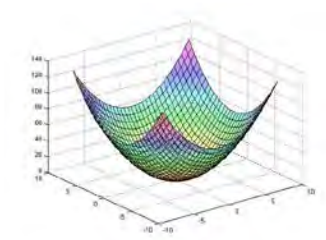


- **Scalar**
- **Clearly defined (preferably unique) maximum or minimum**
  - **Local**
  - **Global**
- **Preferably *positive-definite* (i.e., always a positive number)**

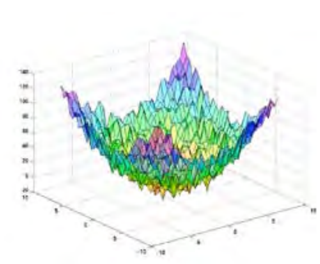


Static vs. Dynamic Optimization

- **Static**
  - Optimal state, **x***, and control, **u***, are fixed, i.e., they do not change over time: $J^* = J(\mathbf{x}^*, \mathbf{u}^*)$
    - Functional minimization (or maximization)
    - Parameter optimization
- **Dynamic**
  - Optimal state and control vary over time: $J^* = J[\mathbf{x}^*(t), \mathbf{u}^*(t)]$
    - Optimal trajectory
    - Optimal feedback strategy
- **Optimized cost function, J^*, is a scalar, real number in both cases**



Deterministic vs. Stochastic Optimization

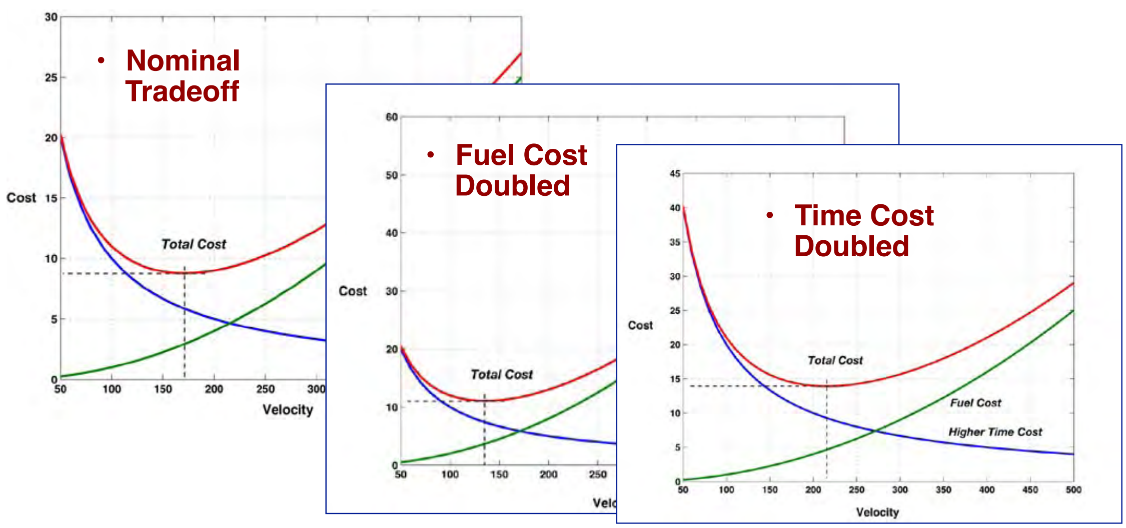


- **Deterministic**
  - System model, parameters, initial conditions, and disturbances are known without error
  - Optimal control operates on the system with certainty
    - $J^* = J(\mathbf{x}^*, \mathbf{u}^*)$
- **Stochastic**
  - Uncertainty in system model, parameters, initial conditions, disturbances, and resulting cost function
  - Optimal control minimizes the expected value of the cost:
    - *Optimal cost* = $E\{J[\mathbf{x}^*, \mathbf{u}^*]\}$
- **Cost function is a scalar, real number in both cases**

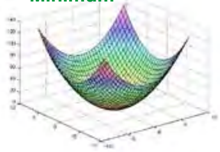
Cost Function with a Single Control Parameter

- **Tradeoff between two types of cost: Minimum-cost cruising speed**
  - **Fuel cost proportional to velocity-squared**
  - **Cost of time inversely proportional to velocity**
- **Control parameter: Velocity**

Tradeoff Between Time- and Fuel-Based Costs

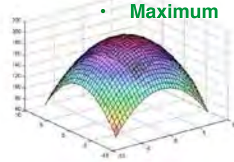


• Minimum

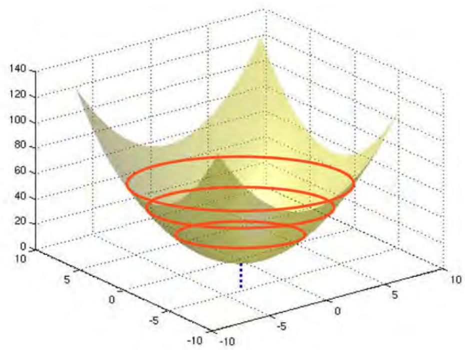


Cost Functions with Two Control Parameters

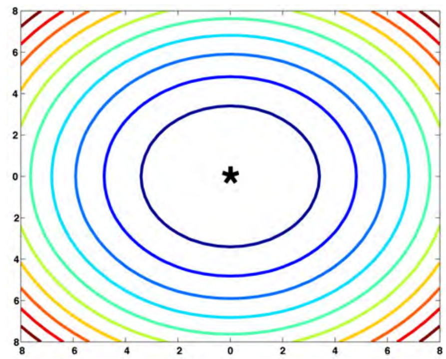
• Maximum

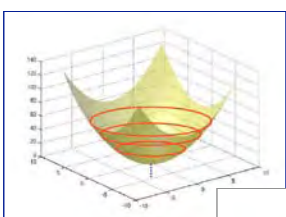


3-D plot of equal-cost contours (iso-contours)



2-D plot of equal-cost contours (iso-contours)

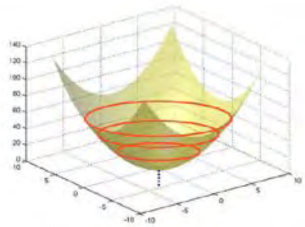




Real-World Topography

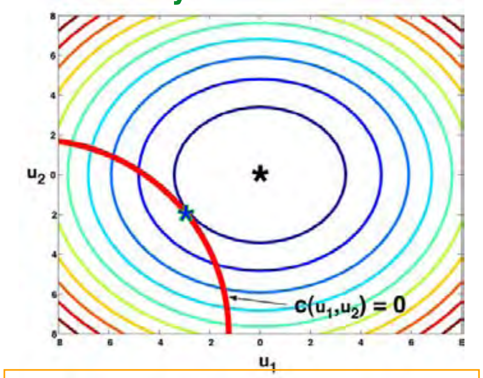
Local vs. global maxima/minima

Robustness of estimates



Cost Functions with Equality Constraints

Stay on the trail



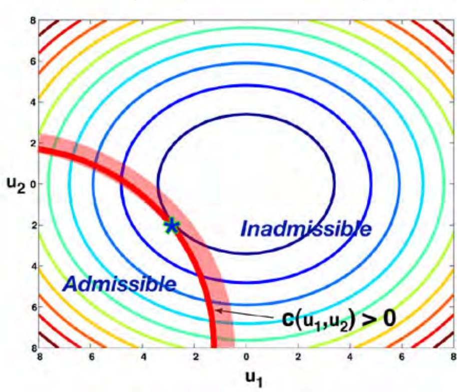
Equality constraint may allow control dimension to be reduced

$$c(u_1,u_2)=0 \rightarrow u_2 = fcn(u_1)$$

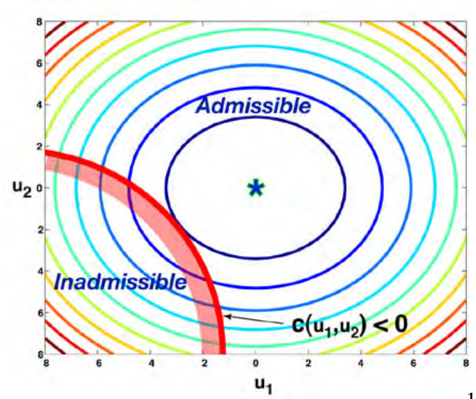
$$J(u_1,u_2)=J\left[u_1, fcn(u_1)\right]=J'(u_1)$$

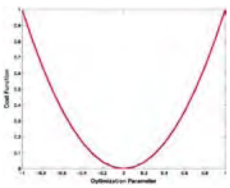
Cost Functions with Inequality Constraints

Person: Stay outside the fence



Horse: Stay inside the fence





Necessary Condition for Static Optimality

Single control

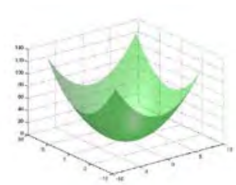
$$\left.\frac{dJ}{du}\right|_{u=u^*} = 0$$

i.e., the slope is zero at the optimum point

Example:

$$J = (u-4)^2$$
$$\frac{dJ}{du} = 2(u-4)$$
$$= 0 \quad when\ u^* = 4$$

Necessary Condition for Static Optimality



Multiple controls

$$\left.\frac{\partial J}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}^*} = \left.\begin{bmatrix} \frac{\partial J}{\partial u_1} & \frac{\partial J}{\partial u_2} & \cdots & \frac{\partial J}{\partial u_m} \end{bmatrix}\right|_{\mathbf{u}=\mathbf{u}^*} = \mathbf{0}$$

Gradient

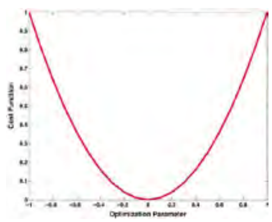
i.e., all slopes are concurrently zero at the optimum point

Example:

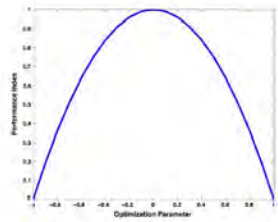
$$J = (u_1-4)^2 + (u_2-8)^2$$
$$dJ/du_1 = 2(u_1-4) = 0 \quad when\ u_1^* = 4$$
$$dJ/du_2 = 2(u_2-8) = 0 \quad when\ u_2^* = 8$$
$$\left.\frac{\partial J}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}^*} = \left.\begin{bmatrix} \frac{\partial J}{\partial u_1} & \frac{\partial J}{\partial u_2} \end{bmatrix}\right|_{\mathbf{u}=\mathbf{u}^*=\begin{bmatrix} 4 \\ 8 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

… But the Slope can be Zero for More than One Reason

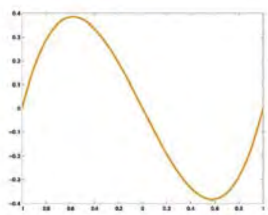
Minimum



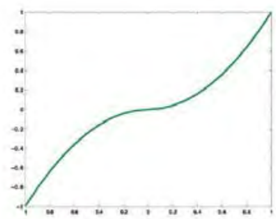
Maximum

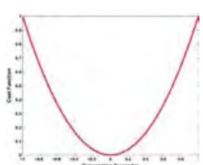


Either

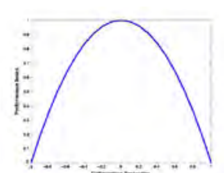


Inflection Point





Sufficient Condition for Static Optimum



- **Single control**

Minimum

Satisfy necessary condition plus

$$\left.\frac{d^2 J}{du^2}\right|_{u=u^*} > 0$$

i.e., curvature is positive at optimum

Example:

$$J = (u-4)^2$$

$$\frac{dJ}{du} = 2(u-4)$$

$$\frac{d^2 J}{du^2} = 2 > 0$$

Maximum

Satisfy necessary condition plus

$$\left.\frac{d^2 J}{du^2}\right|_{u=u^*} < 0$$

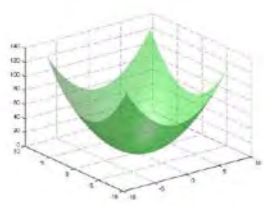
i.e., curvature is negative at optimum

Example:

$$J = -(u-4)^2$$

$$\frac{dJ}{du} = -2(u-4)$$

$$\frac{d^2 J}{du^2} = -2 < 0$$



Sufficient Condition for Static Minimum

Multiple controls

- Satisfy necessary condition
  - plus

$$\left.\frac{\partial J}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}^*} = \left[\begin{array}{cccc} \frac{\partial J}{\partial u_1} & \frac{\partial J}{\partial u_2} & \cdots & \frac{\partial J}{\partial u_m} \end{array}\right]_{\mathbf{u}=\mathbf{u}^*} = \mathbf{0}$$

$$\left.\frac{\partial^2 J}{\partial \mathbf{u}^2}\right|_{\mathbf{u}=\mathbf{u}^*} = \left[\begin{array}{cccc} \frac{\partial^2 J}{\partial u_1^2} & \frac{\partial^2 J}{\partial u_1 \partial u_2} & \cdots & \frac{\partial^2 J}{\partial u_1 \partial u_m} \\ \frac{\partial^2 J}{\partial u_2 \partial u_1} & \frac{\partial^2 J}{\partial u_2^2} & \cdots & \frac{\partial^2 J}{\partial u_2 \partial u_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 J}{\partial u_m \partial u_1} & \frac{\partial^2 J}{\partial u_2 \partial u_m} & \cdots & \frac{\partial^2 J}{\partial u_m^2} \end{array}\right]_{\mathbf{u}=\mathbf{u}^*} > \mathbf{0}$$

Hessian matrix

- … what does it mean for a matrix to be "greater than zero"?

$$\frac{\partial^2 J}{\partial \mathbf{u}^2} \triangleq \mathbf{Q} > \mathbf{0}$$ **if Its Quadratic Form, $\mathbf{x}^T\mathbf{Q}\mathbf{x}$, is Greater than Zero**

$\mathbf{x}^T\mathbf{Q}\mathbf{x} \triangleq$ Quadratic form

$\mathbf{Q}$: **Defining matrix** of the quadratic form

$$\left[(1\times n)(n\times n)(n\times 1)\right] = \left[(1\times 1)\right]$$

- dim(Q) = n x n
- Q is symmetric
- $x^T Qx$ is a scalar

Quadratic Form of Q is Positive* if Q is Positive Definite

- Q is positive-definite if
  - All leading principal minor determinants are positive
  - All eigenvalues are real and positive

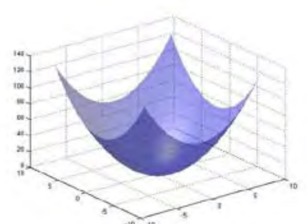
- 3 x 3 example

$$\mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix}$$

$$q_{11} > 0, \quad \begin{vmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{vmatrix} > 0$$

* *except at* **x = 0**

$$\det(s\mathbf{I} - \mathbf{Q}) = (s - \lambda_1)(s - \lambda_2)(s - \lambda_3)$$
$$\lambda_1, \lambda_2, \lambda_3 \geq 0$$



Minimized Cost Function, *J**

- Gradient is zero at the minimum
- Hessian matrix is positive-definite at the minimum
- Expand the cost in a *Taylor series*

$$J(\mathbf{u}^* + \Delta\mathbf{u}) \approx J(\mathbf{u}^*) + \Delta J(\mathbf{u}^*) + \Delta^2 J(\mathbf{u}^*) + \ldots$$

$$\Delta J(\mathbf{u}^*) = \Delta\mathbf{u}^T \left.\frac{\partial J}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}^*} = 0$$

$$\Delta^2 J(\mathbf{u}^*) = \frac{1}{2}\Delta\mathbf{u}^T \left[\left.\frac{\partial^2 J}{\partial \mathbf{u}^2}\right|_{\mathbf{u}=\mathbf{u}^*}\right]\Delta\mathbf{u} \geq 0$$

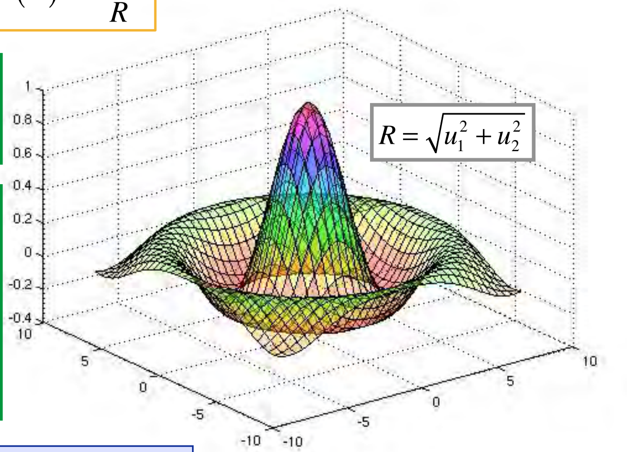
- First variation is zero at the minimum
- Second variation is positive at the minimum

How Many Maxima/Minima does the "Mexican Hat" Have?

$$z = \mathrm{sinc}(R) \triangleq \frac{\sin R}{R}$$

$$\left.\frac{\partial J}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}^*} = \left[\begin{array}{cccc} \frac{\partial J}{\partial u_1} & \frac{\partial J}{\partial u_2} & \cdots & \frac{\partial J}{\partial u_m} \end{array}\right]_{\mathbf{u}=\mathbf{u}^*} = \mathbf{0}$$

$$\left.\frac{\partial^2 J}{\partial \mathbf{u}^2}\right|_{\mathbf{u}=\mathbf{u}^*} = \left[\begin{array}{cccc} \frac{\partial^2 J}{\partial u_1^2} & \frac{\partial^2 J}{\partial u_1 \partial u_2} & \cdots & \frac{\partial^2 J}{\partial u_1 \partial u_m} \\ \frac{\partial^2 J}{\partial u_2 \partial u_1} & \frac{\partial^2 J}{\partial u_2^2} & \cdots & \frac{\partial^2 J}{\partial u_2 \partial u_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 J}{\partial u_m \partial u_1} & \frac{\partial^2 J}{\partial u_2 \partial u_m} & \cdots & \frac{\partial^2 J}{\partial u_m^2} \end{array}\right]_{\mathbf{u}=\mathbf{u}^*} \gtrless 0$$



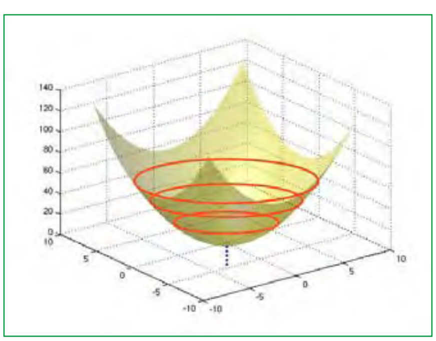
$$R = \sqrt{u_1^2 + u_2^2}$$

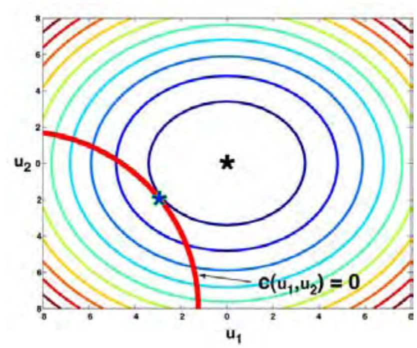
One maximum

Wolfram Alpha
maximize(sinc(sqrt(x^2+y^2)))

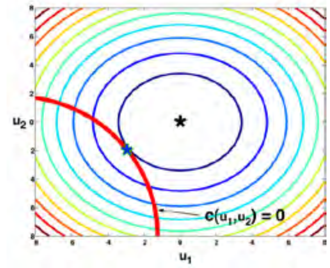
Static Cost Functions with Equality Constraints

- **Minimize $J(\mathbf{u}')$, subject to $\mathbf{c}(\mathbf{u}') = 0$**
  - **dim(c) = [n x 1]**
  - **dim(u') = [($m + n$) x 1]**





Two Approaches to Static Optimization with a Constraint



$$\mathbf{u}' = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

1. Use constraint to reduce control dimension

$$\textit{Example}: \quad \min_{u_1,u_2} J \quad \textit{subject to}$$

$$c(\mathbf{u}') = c(u_1, u_2) = 0 \rightarrow u_2 = fcn(u_1)$$

$$J(\mathbf{u}') = J(u_1, u_2) = J[u_1, fcn(u_1)] = J'(u_1)$$

2. Augment the cost function to recognize the constraint

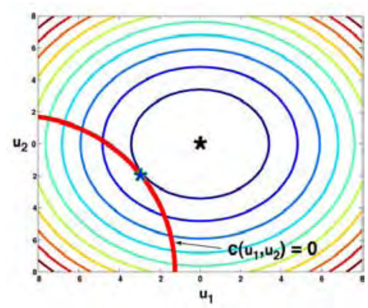
$$J_A(\mathbf{u}') = J(\mathbf{u}') + \boldsymbol{\lambda}^T \mathbf{c}(\mathbf{u}')$$

$\boldsymbol{\lambda}$, an unknown constant

$\boldsymbol{\lambda}$ has the same dimension as the constraint

$$\dim(\boldsymbol{\lambda}) = \dim(\mathbf{c}) = n \times 1$$

Solution: First Approach



Cost function

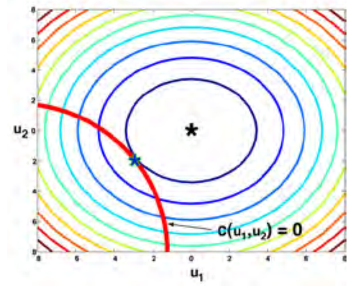
$$J = u_1^2 - 2u_1u_2 + 3u_2^2 - 40$$

Constraint

$$c = u_2 - u_1 - 2 = 0$$

$$\therefore u_2 = u_1 + 2$$

Solution Example: Reduced Control Dimension



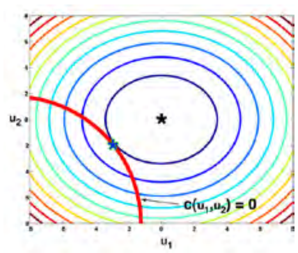
Cost function and gradient with substitution

$$
\begin{aligned}
J &= u_1^2 - 2u_1u_2 + 3u_2^2 - 40 \\
&= u_1^2 - 2u_1(u_1+2) + 3(u_1+2)^2 - 40 \\
&= 2u_1^2 + 8u_1 - 28
\end{aligned}
$$

$$\frac{\partial J}{\partial u_1} = 4u_1 + 8 = 0; \quad u_1 = -2$$

Optimal solution

$$u_1^* = -2$$
$$u_2^* = 0$$
$$J^* = -36$$



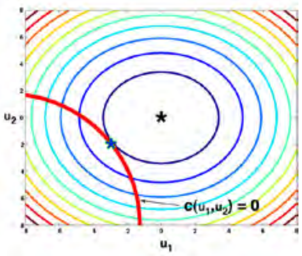
Solution: Second Approach

- **Partition u ‘into a state, x, and a control, u, such that**
  - **dim(x) = dim[c(x)] = [*n* x 1]**
  - **dim(u) = [*m* x 1]**
- **Add constraint to the cost function, weighted by Lagrange multiplier, λ**
  - **dim(λ) = [*n* x 1]**
- **c is required to be zero when J_A is a minimum**

$$\mathbf{u}' = \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix}$$

$$
\begin{aligned}
J_A(\mathbf{u}') &= J(\mathbf{u}') + \boldsymbol{\lambda}^T \mathbf{c}(\mathbf{u}') \\
J_A(\mathbf{x},\mathbf{u}) &= J(\mathbf{x},\mathbf{u}) + \boldsymbol{\lambda}^T \mathbf{c}(\mathbf{x},\mathbf{u})
\end{aligned}
$$

$$\mathbf{c}(\mathbf{u}') = \mathbf{c}\begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix} = \mathbf{0}$$



Solution: Adjoin Constraint with Lagrange Multiplier

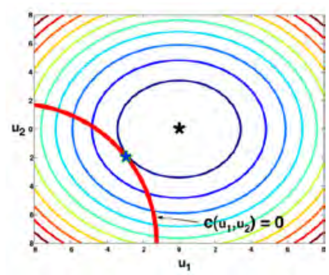
Gradient with respect to x, u, and λ is zero at the optimum point

n equations
$$\frac{\partial J_A}{\partial \mathbf{x}} = \frac{\partial J}{\partial \mathbf{x}} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{c}}{\partial \mathbf{x}} = \mathbf{0}$$

m equations
$$\frac{\partial J_A}{\partial \mathbf{u}} = \frac{\partial J}{\partial \mathbf{u}} + \boldsymbol{\lambda}^T \frac{\partial \mathbf{c}}{\partial \mathbf{u}} = \mathbf{0}$$

n equations
$$\frac{\partial J_A}{\partial \boldsymbol{\lambda}} = \mathbf{c} = \mathbf{0}$$

Simultaneous Solutions for State and Control

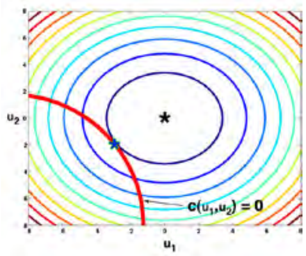


- (2*n* + *m*) values must be found (x, λ, u)
- Use first equation to find form of optimizing Lagrange multiplier (*n* scalar equations)
- Second and third equations provide (*n* + *m*) scalar equations that specify the state and control

$$\boldsymbol{\lambda}^{*T} = -\frac{\partial J}{\partial \mathbf{x}}\left(\frac{\partial \mathbf{c}}{\partial \mathbf{x}}\right)^{-1}$$
$$\boldsymbol{\lambda}^{*} = -\left[\left(\frac{\partial \mathbf{c}}{\partial \mathbf{x}}\right)^{-1}\right]^T \left(\frac{\partial J}{\partial \mathbf{x}}\right)^T$$

$$\frac{\partial J}{\partial \mathbf{u}} + \boldsymbol{\lambda}^{*T} \frac{\partial \mathbf{c}}{\partial \mathbf{u}} = \mathbf{0}$$
$$\frac{\partial J}{\partial \mathbf{u}} - \frac{\partial J}{\partial \mathbf{x}}\left(\frac{\partial \mathbf{c}}{\partial \mathbf{x}}\right)^{-1} \frac{\partial \mathbf{c}}{\partial \mathbf{u}} = \mathbf{0}$$

$$\mathbf{c}(\mathbf{x},\mathbf{u}) = \mathbf{0}$$



Solution Example: Second Approach

Cost function

$$J = u^2 - 2xu + 3x^2 - 40$$

Constraint

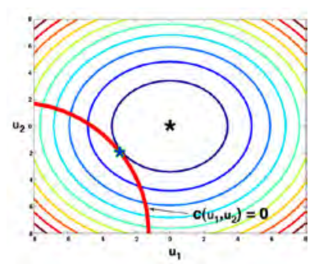
$$c = x - u - 2 = 0$$

Partial derivatives

$$\frac{\partial J}{\partial x} = -2u + 6x$$
$$\frac{\partial J}{\partial u} = 2u - 2x$$

$$\frac{\partial c}{\partial x} = 1$$
$$\frac{\partial c}{\partial u} = -1$$

Solution Example: Second Approach



- **From first equation** $\lambda * = 2u - 6x$
- **From second equation** $(2u - 2x) + (2u - 6x)(-1)$ $\therefore x = 0$
- **From constraint** $u = -2$
  - **Optimal solution**

$$x* = 0$$
$$u* = -2$$
$$J* = -36$$

Next Time:
Numerical Optimization