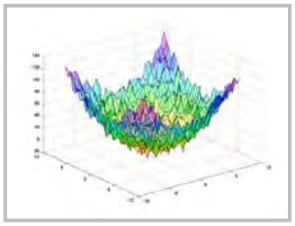
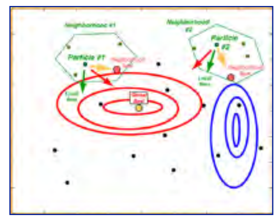


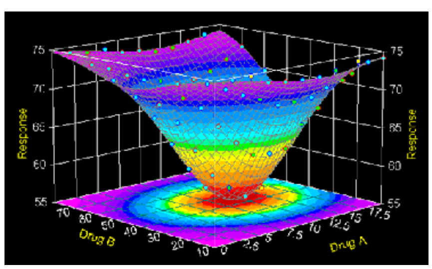
Numerical Optimization

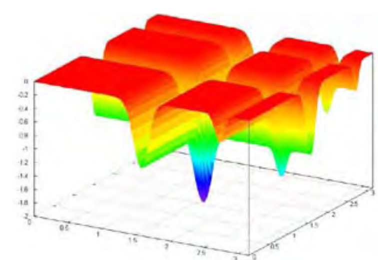
Robert Stengel
Robotics and Intelligent Systems MAE 345, Princeton University, 2017

- Gradient search
- Gradient-free search
  - Grid-based search
  - Random search
  - Downhill simplex method
- Monte Carlo evaluation
- Simulated annealing
- Genetic algorithms
- Particle swarm optimization

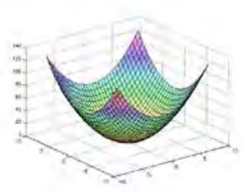


Numerical Optimization





- Previous examples with simple cost function, *J*, could be evaluated analytically
- What if *J* is too complicated to find an analytical solution for the minimum?
- … or *J* has multiple minima?
- Use numerical optimization to find *local* and/or *global* solutions



Two Approaches to Numerical Minimization

1) **Slope and curvature of surface**
   a) **Evaluate gradient , $\partial J/\partial u$, and search for zero**
   b) **Evaluate Hessian, $\partial^2 J/\partial u^2$, and search for positive value**

$$\left(\frac{\partial J}{\partial \mathbf{u}}\right)_o = \left.\frac{\partial J}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}_0} = \textit{starting guess}$$

$$\left(\frac{\partial J}{\partial \mathbf{u}}\right)_n = \left(\frac{\partial J}{\partial \mathbf{u}}\right)_{n-1} + \Delta\left(\frac{\partial J}{\partial \mathbf{u}}\right)_n = \left.\frac{\partial J}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}_n} \quad \textit{such that} \quad \left|\frac{\partial J}{\partial \mathbf{u}}\right|_n < \left|\frac{\partial J}{\partial \mathbf{u}}\right|_{n-1}$$

… until gradient is close enough to zero

2) **Evaluate cost, J , and search for smallest value**

$$J_o = J(\mathbf{u}_o) = \textit{starting guess}$$
$$J_1 = J_o + \Delta J_1(\mathbf{u}_o + \Delta\mathbf{u}_1) \; \textit{such that} \quad J_1 < J_o$$
$$J_2 = J_1 + \Delta J_2(\mathbf{u}_1 + \Delta\mathbf{u}_2) \; \textit{such that} \quad J_2 < J_1$$

Stop when difference between J_n and J_{n-1} is negligible

Gradient/Hessian Search to Minimize a Quadratic Function

Cost function, gradient, and Hessian matrix

$$J = \frac{1}{2}(\mathbf{u}-\mathbf{u}^*)^T \mathbf{R}(\mathbf{u}-\mathbf{u}^*), \quad \mathbf{R} > 0$$
$$= \frac{1}{2}\left(\mathbf{u}^T\mathbf{R}\mathbf{u} - \mathbf{u}^T\mathbf{R}\mathbf{u}^* - \mathbf{u}^{*T}\mathbf{R}\mathbf{u} + \mathbf{u}^{*T}\mathbf{R}\mathbf{u}^*\right)$$

$$\frac{\partial J}{\partial \mathbf{u}} = (\mathbf{u}-\mathbf{u}^*)^T \mathbf{R} = \mathbf{0} \; \textit{when} \; \mathbf{u} = \mathbf{u}^*$$
$$\frac{\partial^2 J}{\partial \mathbf{u}^2} = \mathbf{R} = \textit{symmetric constant} > \mathbf{0}$$

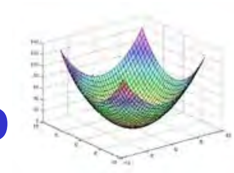
Guess a starting value of u, u_o

$$\left.\frac{\partial J}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}_o} = (\mathbf{u}_o-\mathbf{u}^*)^T \mathbf{R} = (\mathbf{u}_o-\mathbf{u}^*)^T \left.\frac{\partial^2 J}{\partial \mathbf{u}^2}\right|_{\mathbf{u}=\mathbf{u}_o}$$
$$(\mathbf{u}_o-\mathbf{u}^*)^T = \left.\frac{\partial J}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}_o} \mathbf{R}^{-1} \quad (\textit{row})$$

Solve for u*

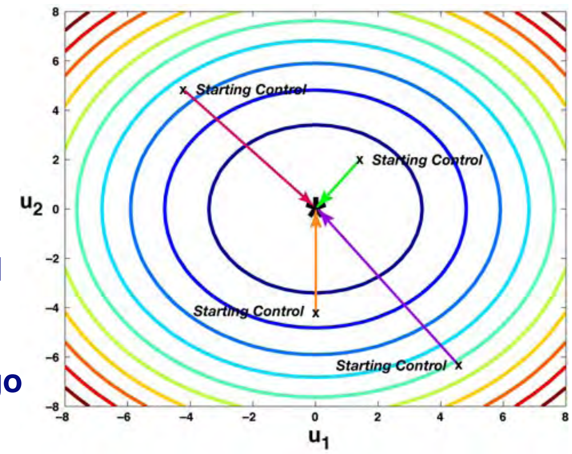
$$\boxed{\mathbf{u}^* = \mathbf{u}_o - \mathbf{R}^{-1}\left[\left.\frac{\partial J}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}_o}\right]^T \quad (\textit{column})}$$

Optimal Value of Quadratic Function Found in a One Step



$$\mathbf{u}^* = \mathbf{u}_o - \mathbf{R}^{-1}\left[\left.\frac{\partial J}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}_o}\right]^T$$
$$= \mathbf{u}_o - \left[\left.\frac{\partial^2 J}{\partial \mathbf{u}^2}\right|_{\mathbf{u}=\mathbf{u}_o}\right]^{-1}\left[\left.\frac{\partial J}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}_o}\right]^T$$

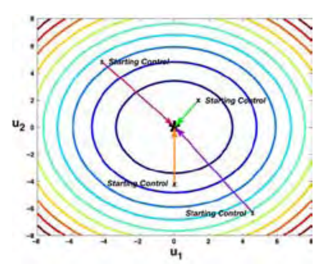
- **Gradient establishes general search direction**
- **Hessian fine-tunes direction and tells exactly how far to go**



Numerical Example

- **Cost function and derivatives**

$$J = \frac{1}{2}\left\{\left[\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right]^T \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix} \left[\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right]\right\}$$
$$\left(\frac{\partial J}{\partial \mathbf{u}}\right)^T = \left[\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right]^T \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix}; \quad \mathbf{R} = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix}$$



- **First guess at optimal control**

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_0 = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

- **Derivatives at starting point**

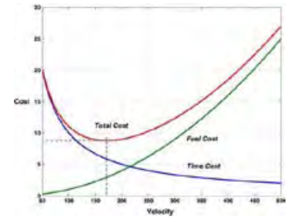
$$\left.\frac{\partial J}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}_0} = \left[\begin{pmatrix} 4 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right]^T \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix} = \begin{pmatrix} 11 \\ 42 \end{pmatrix}$$

- **Solution from starting point**

$$\mathbf{u}^* = \mathbf{u}_o - \mathbf{R}^{-1}\left[\left.\frac{\partial J}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}_o}\right]^T$$

$$\mathbf{u}^* = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^* = \begin{pmatrix} 4 \\ 7 \end{pmatrix} - \begin{bmatrix} 9/5 & -2/5 \\ -2/5 & 1/5 \end{bmatrix}\begin{pmatrix} 11 \\ 42 \end{pmatrix}$$
$$= \begin{pmatrix} 4 \\ 7 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Newton-Raphson Iteration



- **Many cost functions are not quadratic**
- **However, the surface is well-approximated by a quadratic in the vicinity of the optimum, u***

$$J(\mathbf{u}^* + \Delta\mathbf{u}) \approx J(\mathbf{u}^*) + \Delta J(\mathbf{u}^*) + \Delta^2 J(\mathbf{u}^*) + \ldots$$

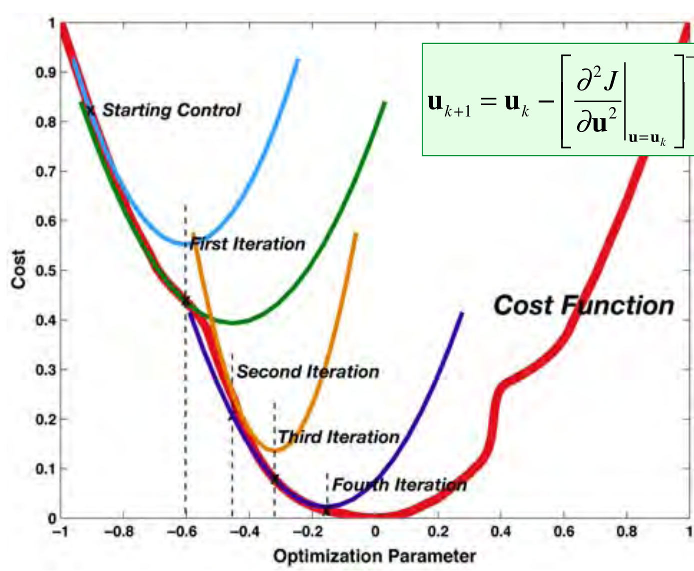
$$\Delta J(\mathbf{u}^*) = \Delta\mathbf{u}^T \left.\frac{\partial J}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}^*} = 0$$

$$\Delta^2 J(\mathbf{u}^*) = \Delta\mathbf{u}^T \left[\left.\frac{\partial^2 J}{\partial \mathbf{u}^2}\right|_{\mathbf{u}=\mathbf{u}^*}\right] \Delta\mathbf{u} \geq 0$$

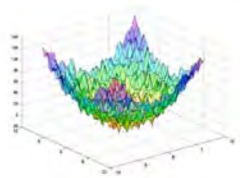
Optimal solution requires multiple steps

Newton-Raphson Iteration

Newton-Raphson algorithm is an iterative search using both the gradient and the Hessian matrix



$$\mathbf{u}_{k+1} = \mathbf{u}_k - \left[\left.\frac{\partial^2 J}{\partial \mathbf{u}^2}\right|_{\mathbf{u}=\mathbf{u}_k}\right]^{-1} \left[\left.\frac{\partial J}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}_k}\right]^T$$



Difficulties with Newton-Raphson Iteration

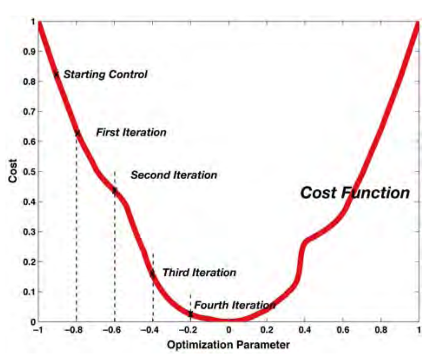
$$\mathbf{u}_{k+1} = \mathbf{u}_k - \left[\boxed{\left. \frac{\partial^2 J}{\partial \mathbf{u}^2} \right|_{\mathbf{u}=\mathbf{u}_k}} \right]^{-1} \left[\left. \frac{\partial J}{\partial \mathbf{u}} \right|_{\mathbf{u}=\mathbf{u}_k} \right]^T$$

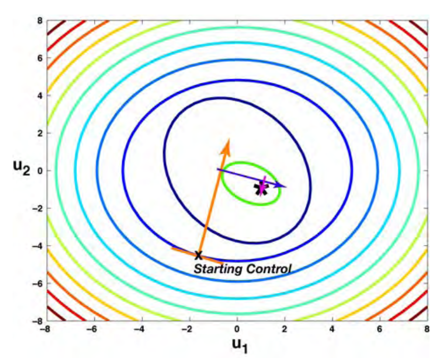
- **Good when close to the optimum, but …**
- **Hessian matrix (i.e., the curvature) may be**
  - **Hard to estimate, e.g., large effects of small errors**
  - **Locally misleading, e.g., wrong curvature**
- **Gradient searches focus on local minima**

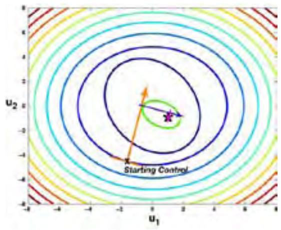
Steepest-Descent Algorithm Multiplies Gradient by a Scalar Constant

$$\mathbf{u}_{k+1} = \mathbf{u}_k - \varepsilon \left[\left. \frac{\partial J}{\partial \mathbf{u}} \right|_{\mathbf{u}=\mathbf{u}_k} \right]^T$$

- **Replace Hessian matrix by a scalar constant**
- **Gradient is orthogonal to equal-cost contours**



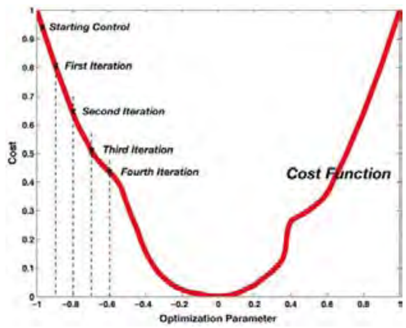




Choice of Steepest-Descent Constant

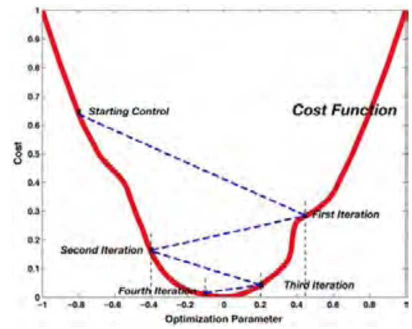
If gain is too small

Convergence is slow



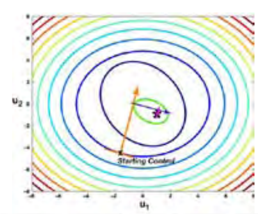
If gain is too large

Convergence oscillates or may fail



Solution: Make gain adaptive

Optimal Steepest-Descent Gain



Find optimal gain by evaluating cost, J, for intermediate solutions (with same $\partial J/\partial \mathbf{u}$)

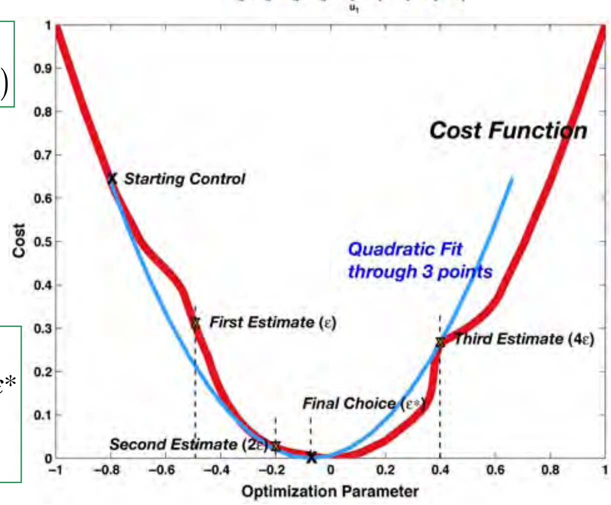
Adjustment rule for ε

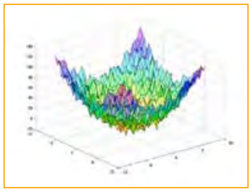
- Starting estimate, J_0
- First estimate, J_1, using ε
- Second estimate, J_2, using 2ε

If $J_2 > J_1$

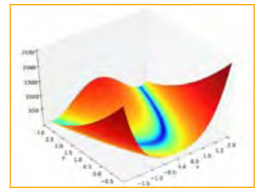
- Quadratic fit through three points to find ε^*
- Else, third estimate, J_3, using 4ε
- …

Use optimal gain, ε^*, on each major iteration





Gradient Search Issues



Steepest Descent

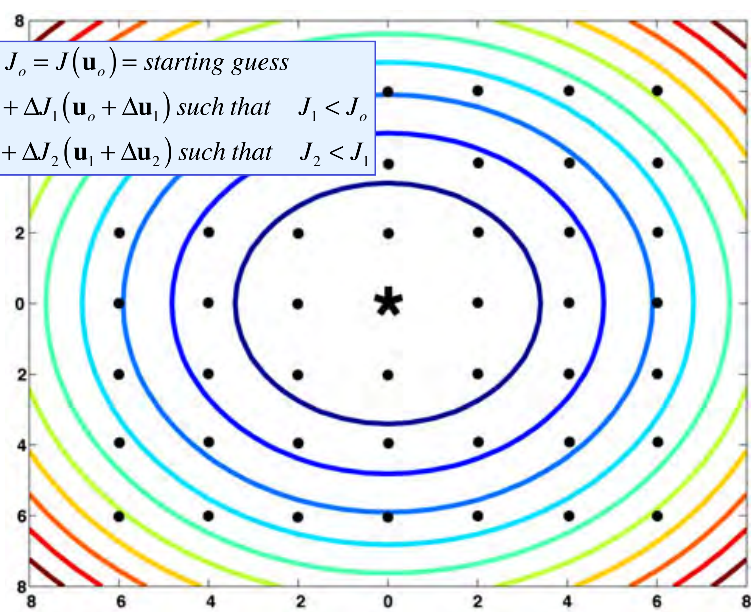
$$\mathbf{u}_{k+1} = \mathbf{u}_k - \varepsilon \left[\left. \frac{\partial J}{\partial \mathbf{u}} \right|_{\mathbf{u}=\mathbf{u}_k} \right]^T$$

Newton Raphson

$$\mathbf{u}_{k+1} = \mathbf{u}_k - \left[\left. \frac{\partial^2 J}{\partial \mathbf{u}^2} \right|_{\mathbf{u}=\mathbf{u}_k} \right]^{-1} \left[\left. \frac{\partial J}{\partial \mathbf{u}} \right|_{\mathbf{u}=\mathbf{u}_k} \right]^T$$

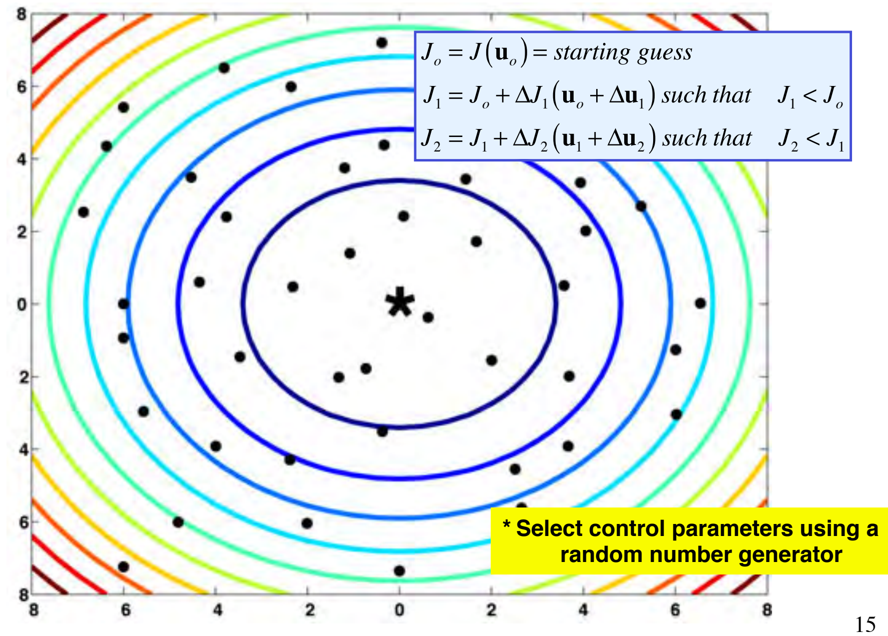
- **Need to evaluate gradient (and possibly Hessian matrix)**
- **Not global: gradient searches focus on local minima**
- **Convergence may be difficult with “noisy” or complex cost functions**

Gradient-Free Search: Grid-Based Search

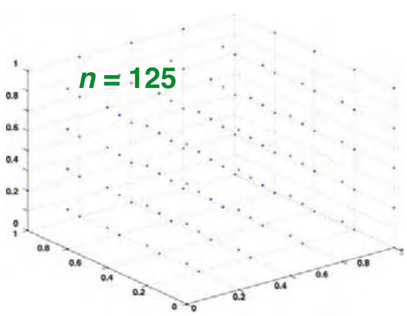


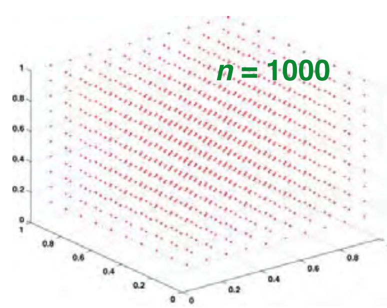
$$J_o = J(\mathbf{u}_o) = \textit{starting guess}$$
$$J_1 = J_o + \Delta J_1(\mathbf{u}_o + \Delta\mathbf{u}_1) \textit{ such that } \quad J_1 < J_o$$
$$J_2 = J_1 + \Delta J_2(\mathbf{u}_1 + \Delta\mathbf{u}_2) \textit{ such that } \quad J_2 < J_1$$

Gradient-Free Search: Random Search



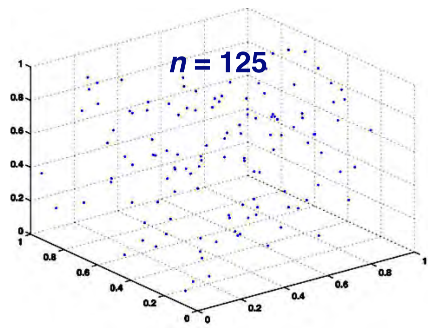
Three-Parameter Grid Search

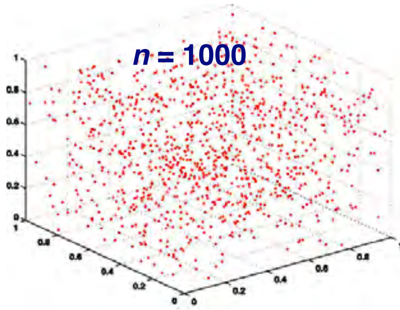




- Regular spacing
- Fixed resolution
- Trials grow as m^n, where
  - n = Number of parameters
  - m = Resolution

Three-Parameter Random Field Search

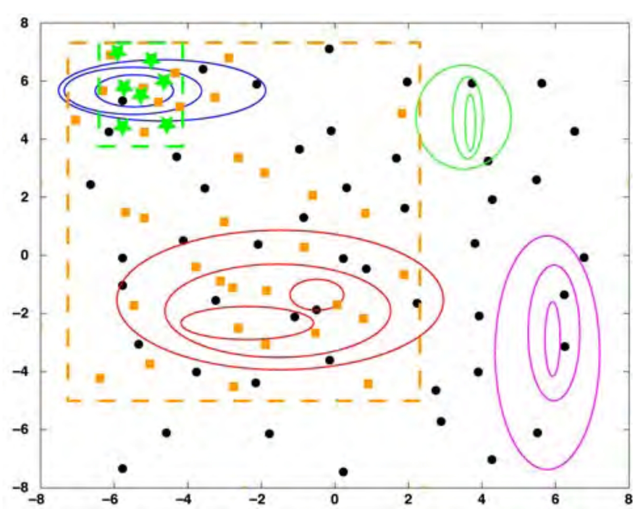




Variable spacing and resolution
Arbitrary number of trials
Random space-filling

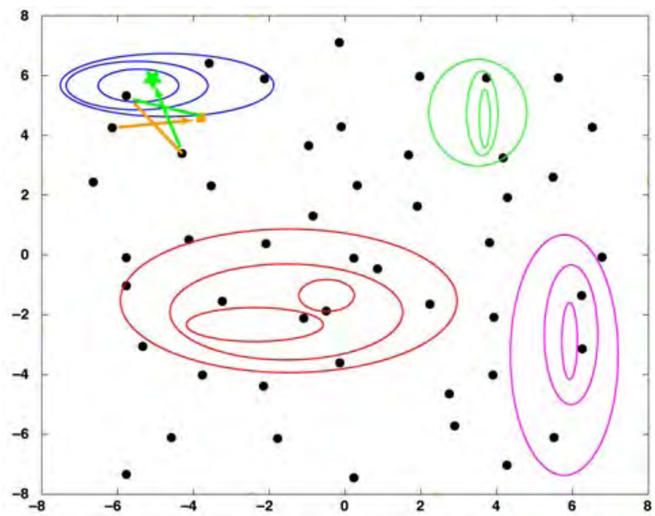
Directed (Structured) Search for Minimum Cost

Continuation of grid-based or random search
Localize areas of low cost
Increase sampling density in those areas



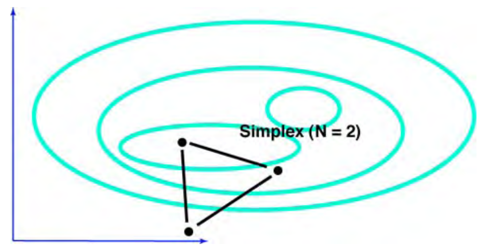
Directed (Structured) Search for Minimum Cost

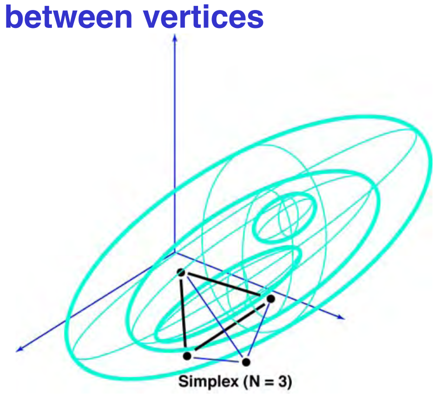
- Interpolate or extrapolate from one or more starting points



Downhill Simplex Search (Nelder-Mead Algorithm)

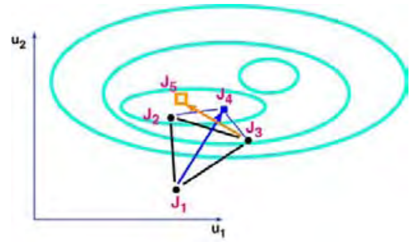
- Simplex: *N*-dimensional figure in control space defined by
  - *N* + 1 vertices
  - (*N* + 1) *N* / 2 straight edges between vertices





https://en.wikipedia.org/wiki/Nelder-Mead_method

Search Procedure for Downhill Simplex Method

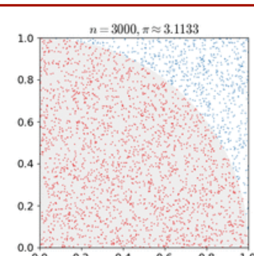


- **Select starting set of vertices**
- **Evaluate cost at each vertex**
- **Determine vertex with largest cost (e.g., J_1 at right)**
- **Project search from this vertex through middle of opposite face (or edge for $N = 2$)**
  - **Reflection [equal distance along direction]**
  - **Expansion [longer distance along direction]**
  - **Contraction [shorter distance along direction]**
  - **Shrink [replace all but best point with points contracted toward best point]**
- **Evaluate cost at new vertex (e.g., J_4 at right)**
- **Drop J_1 vertex, and form simplex with new vertex**
- **Repeat until cost is "small enough" (termination)**
- **MATLAB implementation: *fminsearch***

Monte Carlo Evaluation of Systems and Cost Functions

- **Multiple evaluations of a function with uncertain parameters using**
  - **Random number generators, and**
  - **Assumed or measured statistics of parameters**
- **Not an exhaustive evaluation of all parameters**

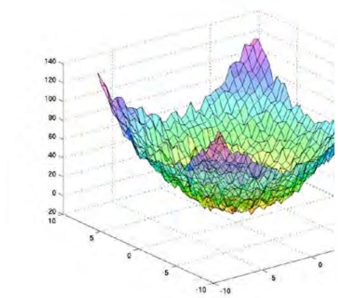
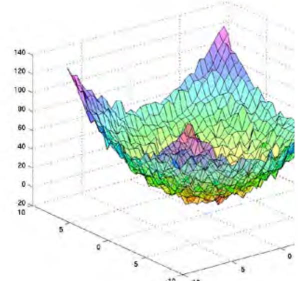
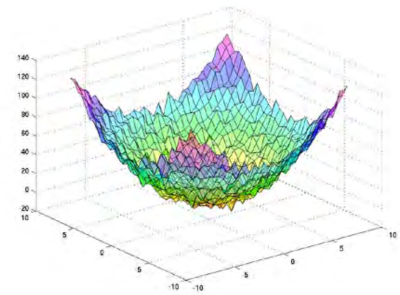
Example (from Wikipedia): Evaluation of π from percentage of points that fall within the unit circle (30,000 trials)



Monte Carlo Evaluation of Systems and Cost Functions

- **Example:** 2-D quadratic function with added Gaussian noise
- Each trial generates a different result $(\sigma_z = 4)$

```
[X,Y] = meshgrid(-8:.5:8);
   Z  =  X.^2 + Y.^2;
   Z1  =  Z + 4*randn(33);
   surf(X,Y,Z1)
   colormap hsv
   alpha(.4)
```


```
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   Z  =  X.^2 + Y.^2;
   Z1  =  Z + 4*randn(33);
   surf(X,Y,Z1)
   colormap hsv
   alpha(.4)
```
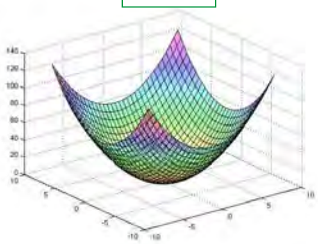
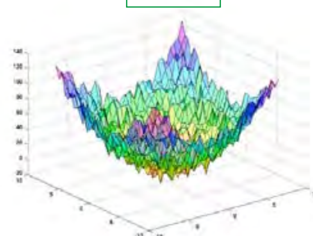

Effect of Increasing Noise on Cost Function

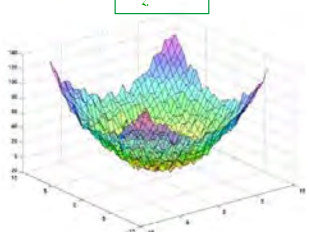
$\sigma_z = 0$



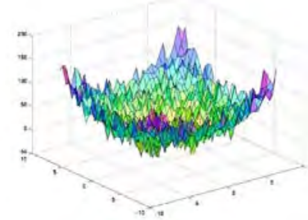
$\sigma_z = 8$



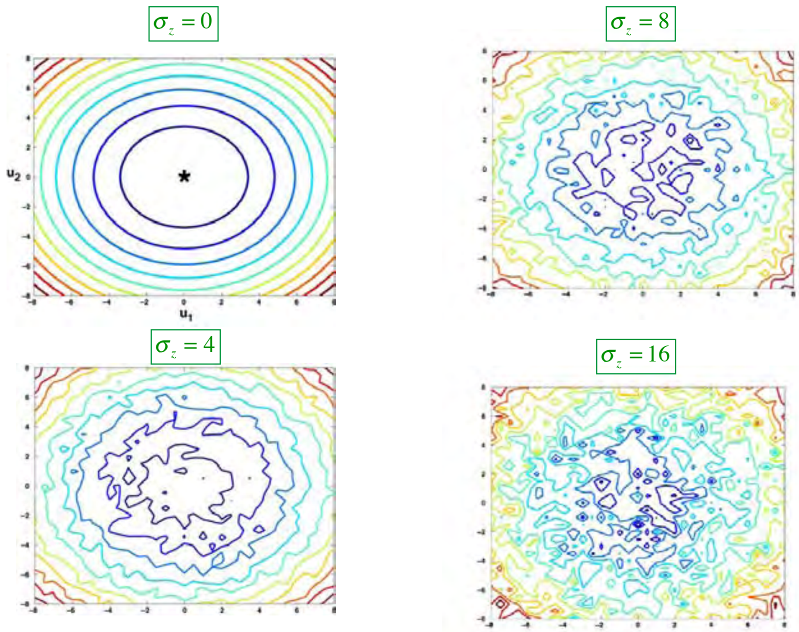
$\sigma_z = 4$



$\sigma_z = 16$

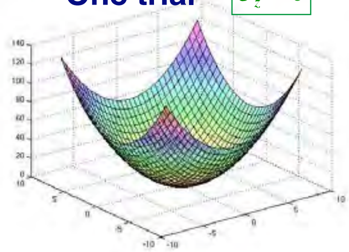


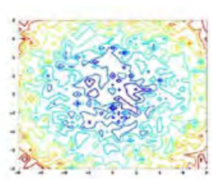
Iso-Cost Contours Lose Structure with Increasing Noise

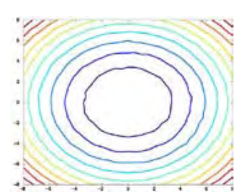


Effect of Averaging on Noisy Cost Function

- **One trial** $\sigma_z = 0$

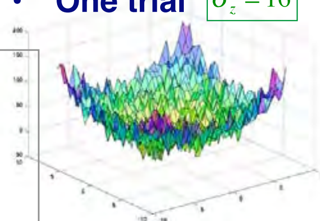




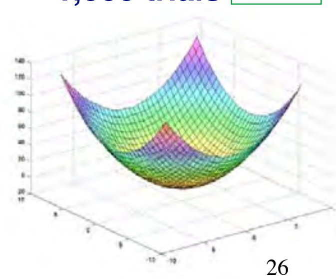


```
[X,Y] = meshgrid(-8:.5:8);
  Z   =  X.^2 + Y.^2;
  Z1  =  Z + 16*randn(33);
  Z2  =  Z1;
%  Averaged Z1
  for k = 1:1000
     Z1  =  Z + 16*randn(33);
     Z2  =  Z1 * (1/(k + 1)) + Z2 * (k/(k+1));
  end
  figure
  surf(X,Y,Z2)
  colormap hsv
  alpha(.4)
```

- **One trial** $\sigma_z = 16$



- **1,000 trials** $\sigma_z = 16$



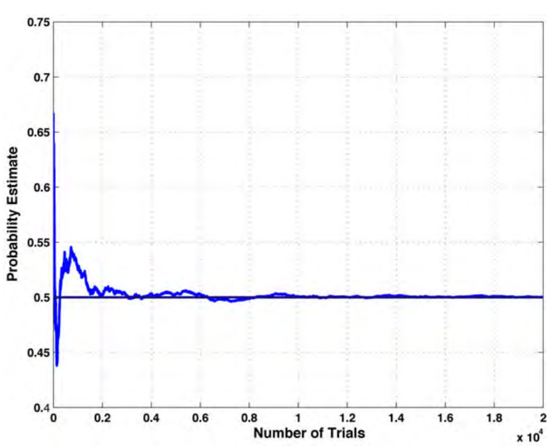
Estimating the Probability of Coin Flips

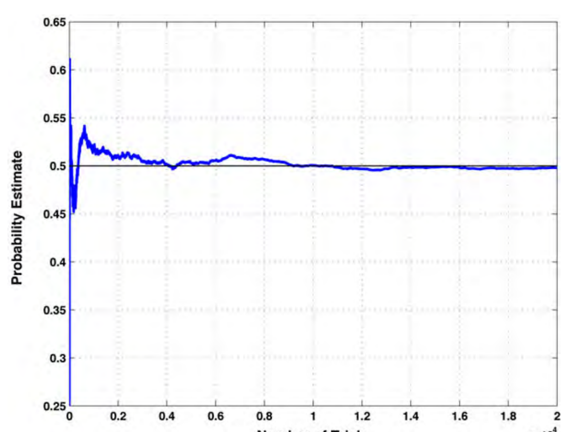
- **Single coin**
  - **Exhaustive search: Correct answer in 2 trials**
  - **Random search (20,000 trials)**
- **21 coins**
  - **Exhaustive search: Correct answer in $n^m = 2^{21}$ = 2,097,152 trials**
  - **Random search (20,000 trials)**

```
% Single coin
x    = [];
prob = round(rand);
for k = 1:20000
    prob   = round(rand) * (1/(k + 1)) + prob * (k/(k+1));
    x      = [x prob];
end
figure
plot(x), grid

% 21 coins
y    = [];
prob = round(rand);
for k = 1:20000
    for j = 1:21
        coin(j) = round(rand);
    end
    score = sum(coin);
    if score > 10
        result = 1;
    else result = 0;
    end
    prob = result * (1/(k + 1)) + prob * (k/(k+1));
    y    = [y prob];
end
figure
plot(y), grid
```


Random Search Excels When There are Many Uncertain Parameters

- **Single coin**
  - **Exhaustive search: Correct answer in 2 trials**
  - **Random search (20,000 trials)**
- **21 coins**
  - **Exhaustive search: Correct answer in $n^m = 2^{21}$ = 2,097,152 trials**
  - **Random search (20,000 trials)**





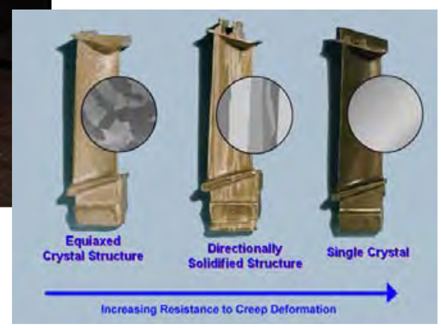
Physical Annealing

- **Produce a strong, hard object made of crystalline material**
  - **High temperature allows molecules to redistribute to relieve stress, remove dislocations**
  - **Gradual cooling allows large, strong crystals to form**
  - **Low temperature "working"(e.g., squeezing, bending, drawing, shearing, and hammering) produces desired crystal structure and shape**

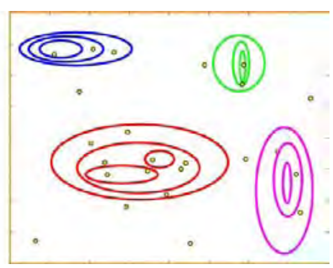
Turbojet Engine Turbines

Turbine Blade Casting

Single-Crystal Turbine Blade



Simulated Annealing Algorithm

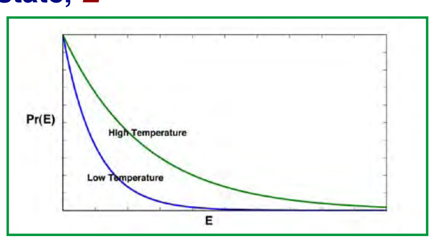


- **Goal:** **Find global minimum among local minima**
- **Approach:** **Randomized search, with convergence that emulates *physical annealing***
  - **Evaluate cost, J_k**
  - **Accept if $J_k < J_{k-1}$**
  - **Accept with probability $Pr(E)$ if $J_k > J_{k-1}$**
- **Probability distribution of energy state, E *(Boltzmann Distribution)***

$$\Pr(E) \propto e^{-E/kT}$$

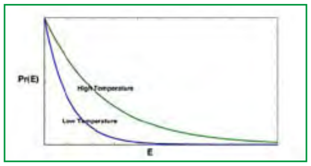
k : Boltzmann's constant

T : Temperature



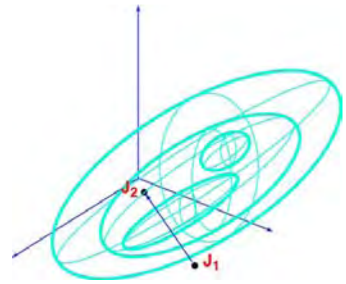
- **Algorithm's "cooling schedule" accepts many bad guesses at first, fewer as iteration number, k, increases**
- **MATLAB implementation: *simulannealbnd (Global Optimization Toolbox)***

https://en.wikipedia.org/wiki/Simulated_annealing



Application of Annealing Principle to Search

- If cost decreases ($J_2 < J_1$), always accept new point
- If cost increases ($J_2 > J_1$), accept new point with probability proportional to *Boltzmann factor*

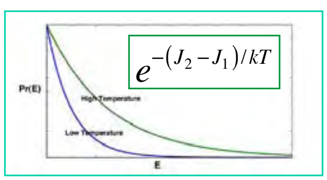


$$e^{-(J_2 - J_1)/kT}$$

- Occasional diversion from convergent path intended to prevent entrapment by a local minimum
- As search progresses, decrease kT, making probability of accepting a cost increase smaller

Realistic Bird Flight Animation by SA
http://www.youtube.com/watch?v=SoM1nS3uSrY

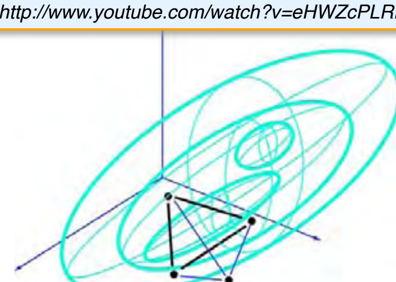
SA Face Morphing
http://www.youtube.com/watch?v=SP3nQKnzexs



Combination of Simulated Annealing with Downhill Simplex Method

SA Mona Lisa
http://www.youtube.com/watch?v=eHWZcPLRMjQ

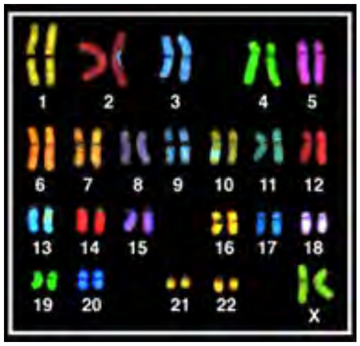
- Introduce random "wobble" to simplex search
  - Add random components to costs evaluated at vertices
  - Project new vertex as before based on modified costs
  - With large T, this becomes a random search
  - Decrease random components on a "cooling" schedule

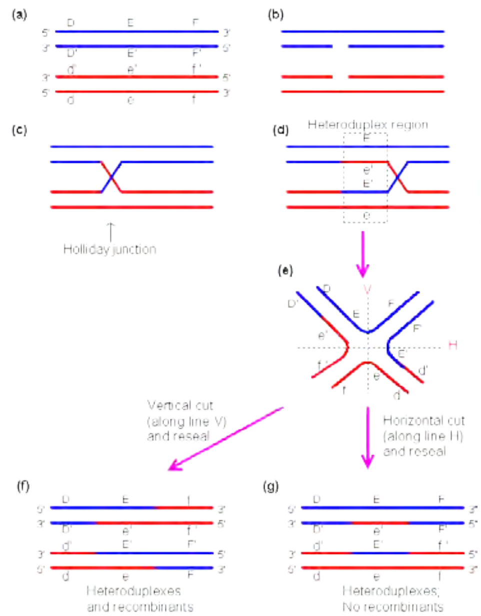


- Same annealing strategy as before
  - If cost decreases ($J_2 < J_1$), always accept new point
  - If cost increases ($J_2 > J_1$), accept new point probabilistically
  - As search progresses, decrease T

$$J_{1_{SA}} = J_1 + \Delta J_1(rng)$$
$$J_{2_{SA}} = J_2 + \Delta J_2(rng)$$
$$J_{3_{SA}} = J_3 + \Delta J_3(rng)$$
$$\ldots = \ldots$$

Genetic Coding:
Replication, Recombination, and Mutation of Chromosomes





Broad Characteristics of Genetic Algorithms

- Search based on the coding of a parameter set, <u>not the parameters</u> themselves
- Search evolves from a population of points
- "Blind" search, i.e., without gradient
- Probabilistic transitions from one control state to another (using random number generator)
- Control parameters assembled as genes of a single chromosome strand (Example: four 4-bit parameters = four "genes")

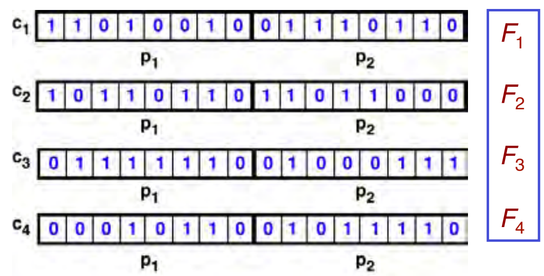
p_1				p_2				p_3				p_4			
1	1	0	1	0	0	1	0	0	1	1	1	0	1	1	0

John Holland, 1975

Progression of a Genetic Algorithm

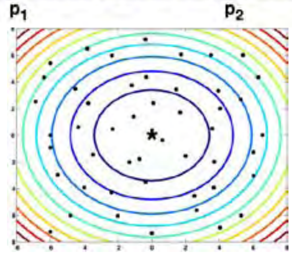
Most fit chromosome evolves from a sequence of reproduction, crossover, and mutation

- **Initialize algorithm with *N* (even) random chromosomes, c_n (two 8-bit genes or parameters in example)**
- **Evaluate fitness, F_n, of each chromosome**
- **Compute total fitness, F_{total}, of chromosome population**



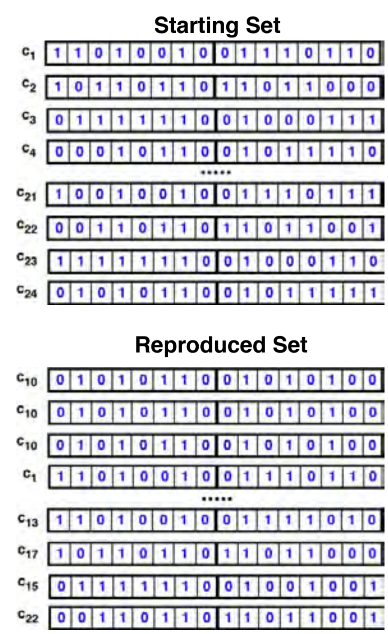
$$F_{total} = \sum_{n=1}^{N} F_n$$

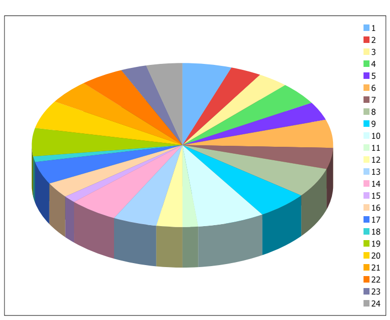
Bigger *F* is better

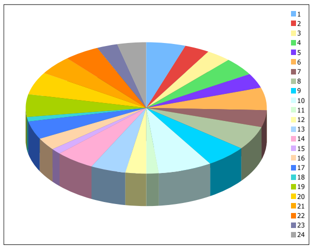


Genetic Algorithm: Reproduction

- **Reproduce *N* additional copies of the *N* originals with probabilistic weighting based on relative fitness, F_n/F_{total}, of originals (Survival of the fittest)**
- **Roulette wheel selection:**
  - **$\Pr(c_n) = F_n/F_{total}$**
  - **Multiple copies of most-fit chromosomes**
  - **No copies of least-fit chromosomes**

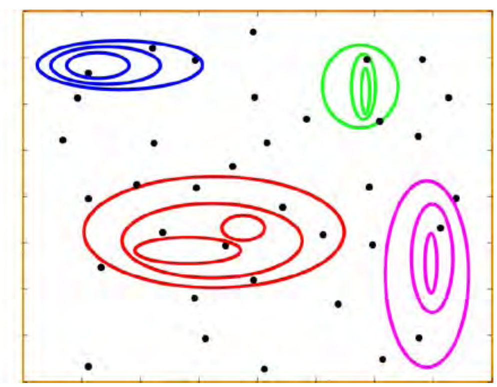




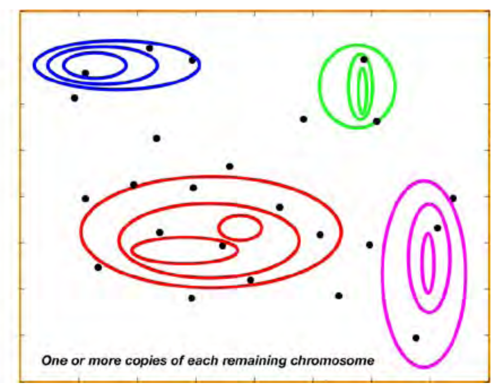


Reproduction Eliminates Least Fit Chromosomes Probabilistically

Starting Set

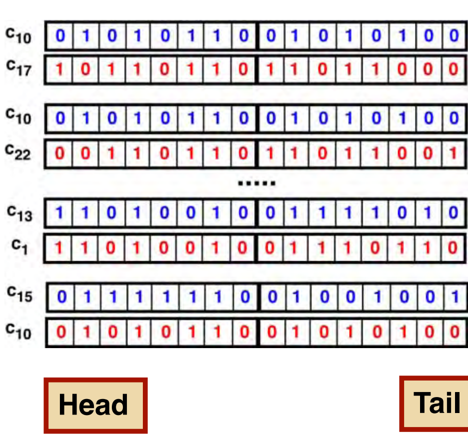


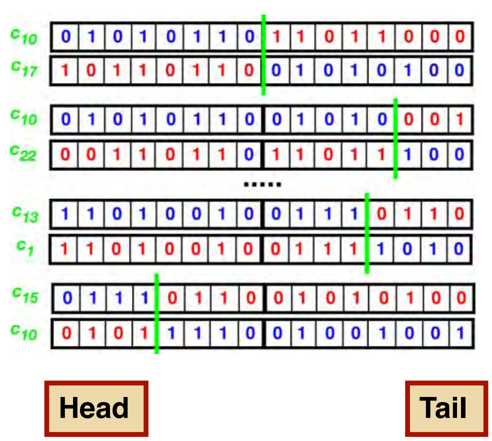
Reproduced Set



Genetic Algorithm: Crossover

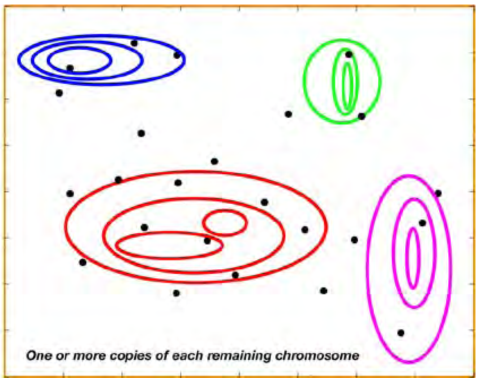
- Arrange *N* new chromosomes in *N*/2 pairs chosen at random
- Interchange tails that are cut at random locations



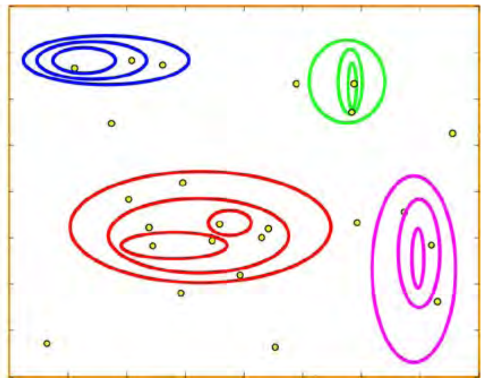


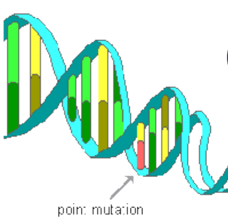
Crossover Creates New Chromosome Population Containing Old Gene Sequences

Reproduced Set



Crossover Set

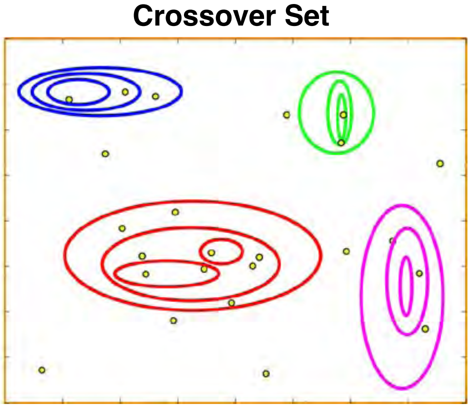




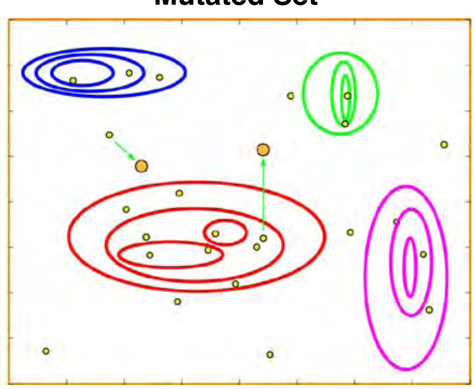
Genetic Algorithm: Mutation

Flip a bit, 0 -> 1 or 1 -> 0, at random every 1,000 to 5,000 bits

Crossover Set



Mutated Set



1	1	0	1	0	0	1	0	0	1	1	1	0	1	1	0

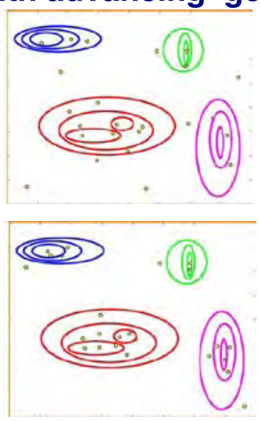
p_1 p_2

1	1	0	1	0	0	1	0	1	1	1	1	0	1	1	0

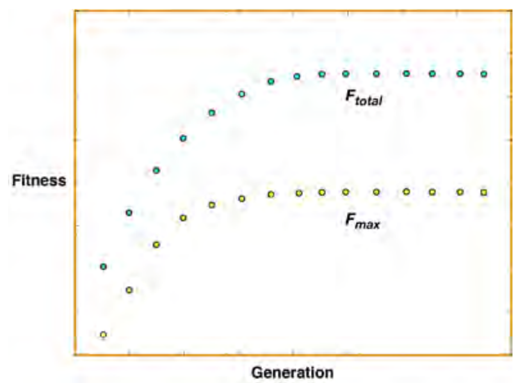
p_1 p_2

Create New Generations By Reproduction, Crossover, and Mutation Until Solution Converges

Chromosomes narrow in on best values with advancing generations



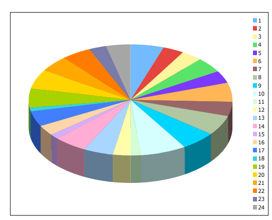
***F_{max}* and *F_{total}* increase with advancing generations**



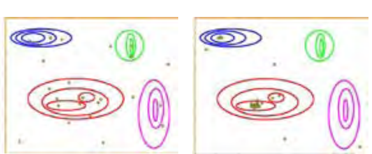
MATLAB implementation: *ga*
(Global Optimization Toolbox)

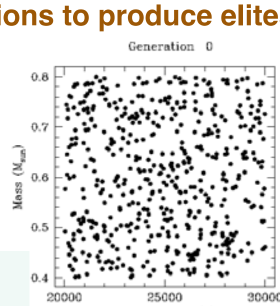
Comments on GA

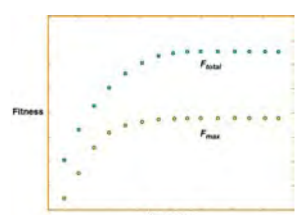
GA Mona Lisa
http://www.youtube.com/watch?v=rGt3iMAJVT8



- Short, fit genes tend to survive crossover
- Random location of crossover
  - produces large and small variations in genes
  - interchanges genes in chromosomes
- Multiple copies of best genes evolve
- Alternative implementations
  - Real numbers rather than binary numbers
  - Retention of "elite" chromosomes
  - Clustering in "fit" regions to produce elites



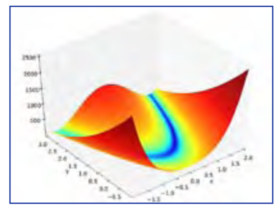




https://en.wikipedia.org/wiki/Genetic_algorithm

Particle Swarm Optimization

- **Converse of the GA: Uses multiple cost evaluations to guide parameter search directly**
- **Stochastic, population-based algorithm**
- **Search for optimizing parameters modeled on social behavior of groups that possess cognitive consistency**
- **Particles = Parameter vectors**
- **Particles have position and velocity**
- **Projection of own best (Local best)**
- **Knowledge of swarm's best**
  - **Neighborhood best**
  - **Global best**



Peregrine Falcon Hunting Murmuration of Starlings in Rome
https://www.youtube.com/watch?v=V-mCuFYfJdI

Particle Swarm Optimization

$$\boxed{\text{Find } \min_{\mathbf{u}} J(\mathbf{u}) = J^*(\mathbf{u}^*)}$$

$$\textbf{Jargon:} \quad \text{argmin} J(\mathbf{u}) = \mathbf{u}^*$$

$$i.e., \ \textbf{argument} \text{ of } J \text{ that minimizes } J$$

Recursive algorithm to find best particle or configuration of particles

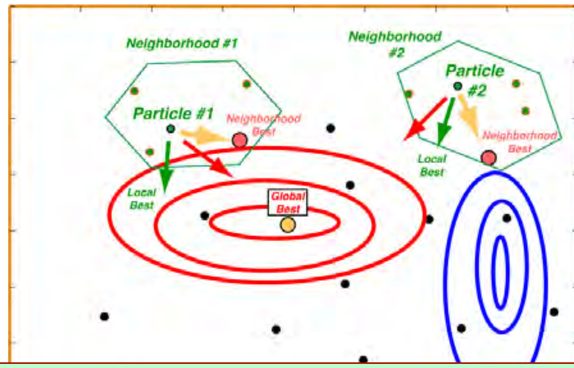
$$\mathbf{u}: \ \text{Parameter vector} \sim \ \textbf{"Position"} \text{ of the particles}$$

$$\mathbf{v}: \ \textbf{"Velocity"} \text{ of } \mathbf{u}$$

$$\dim(\mathbf{u}) = \dim(\mathbf{v}) = \ \text{Number of particles}$$

https://en.wikipedia.org/wiki/Particle_swarm_optimization

Particle Swarm Optimization



- **Local best**: RNG, downhill simplex, or SA step for **each** particle
- **Neighborhood best**: argmin of closest ***n*** neighboring points
- **Global best**: argmin of **all** particles

$$\mathbf{u}_k = \mathbf{u}_{k-1} + a\mathbf{v}_{k-1}$$

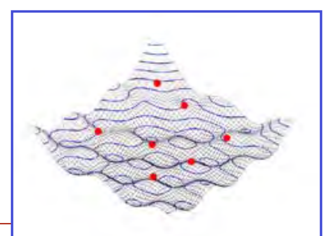
$$\mathbf{v}_k = b\mathbf{v}_{k-1} + c\left(\mathbf{u}_{best_{local_{k-1}}} - \mathbf{u}_{k-1}\right) + d\left(\mathbf{u}_{best_{neighborhood_{k-1}}} - \mathbf{u}_{k-1}\right) + e\left(\mathbf{u}_{best_{global_{k-1}}} - \mathbf{u}_{k-1}\right)$$

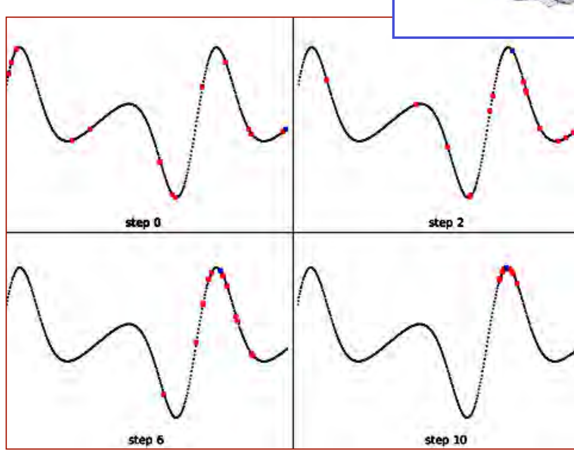
$\mathbf{u}_0$: Starting value from random number generator

$\mathbf{v}_0$: Zero

a, b, c, d : Search tuning parameters

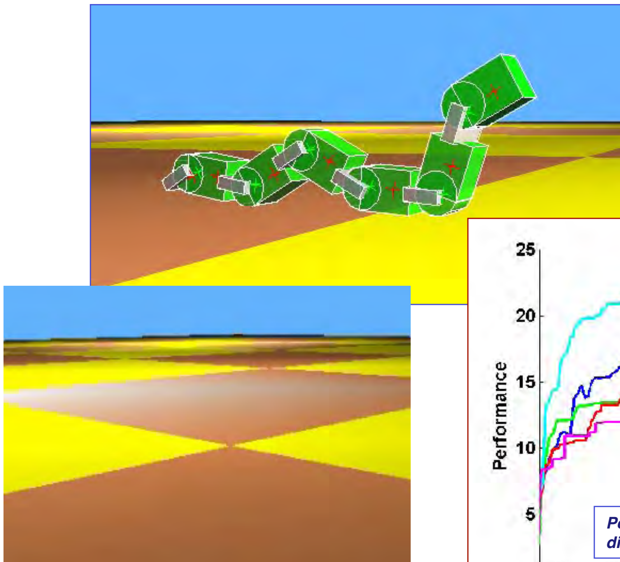
Particle Swarm Optimization



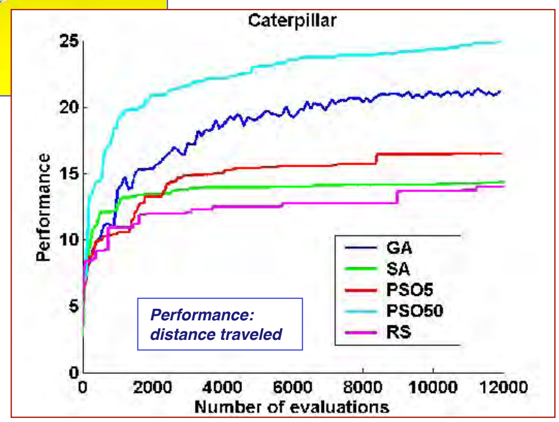


MATLAB implementation: *particleswarm* *(Global Optimization Toolbox)*

Comparison of Algorithms in Caterpillar Gait-Training Example



Y. Bourquin, U. Sussex, BIRG

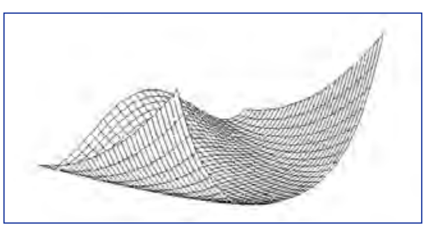


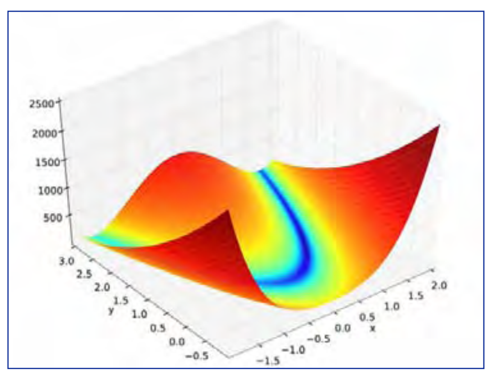
Next Time: Dynamic Optimal Control

Supplemental Material

Rosenbrock Function

Typical test function for numerical optimization algorithms





$$J(u_1, u_2) = \left(1 - u_1\right)^2 + 100\left(u_2 - u_1^2\right)^2$$

Wolfram Alpha

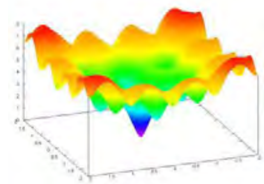
Minimize[(1-u1)^2+100 (u2 - u1^2)^2, u1, u2]

Cost Function and Gradient Searches

- Evaluate *J* and search for smallest value

$$J_o = J(\mathbf{u}_o) = \textit{starting guess}$$
$$J_1 = J_o + \Delta J_1(\mathbf{u}_o + \Delta\mathbf{u}_1) \textit{ such that } \quad J_1 < J_o$$
$$J_2 = J_1 + \Delta J_2(\mathbf{u}_1 + \Delta\mathbf{u}_2) \textit{ such that } \quad J_2 < J_1$$

Stop when difference between J_n and J_{n-1} is negligible

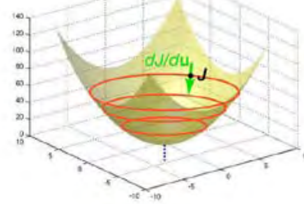


- ***J*** **is a scalar**
- ***J*** **provides** **no search direction**

- Evaluate $\partial J/\partial$u and search for zero

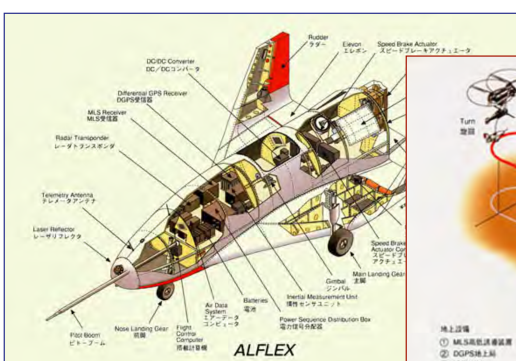
$$\left(\frac{\partial J}{\partial \mathbf{u}}\right)_o = \left.\frac{\partial J}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}_0} = \textit{starting guess}$$
$$\left(\frac{\partial J}{\partial \mathbf{u}}\right)_n = \left(\frac{\partial J}{\partial \mathbf{u}}\right)_{n-1} + \Delta\left(\frac{\partial J}{\partial \mathbf{u}}\right)_n = \left.\frac{\partial J}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}_n} \textit{ such that } \quad \left|\frac{\partial J}{\partial \mathbf{u}}\right|_n < \left|\frac{\partial J}{\partial \mathbf{u}}\right|_{n-1}$$

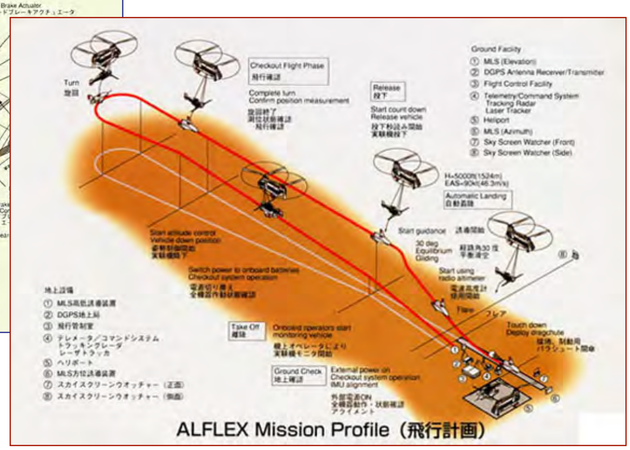
… until gradient is close enough to zero



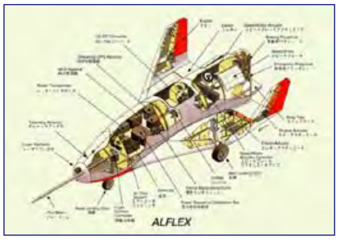
- **$\partial J/\partial$u is a vector**
- **$\partial J/\partial$u indicates feasible search direction**

Comparison of SA, DS, and GA in Designing a PID Controller: ALFLEX Reentry Test Vehicle





Motoda, Stengel, and Miyazawa, 2002



Parameter Uncertainties and Touchdown Requirements for ALFLEX Reentry Test Vehicle

Table 4 Uncertain parameters for ALFLEX model

Category	Number of parameters
Mass parameters	5
Aerodynamics	27
Actuator dynamics	9
Sensor dynamics and error	38
Atmospheric condition	6
Initial condition and error at release	18

Table 5 Requirements of ALFLEX touchdown performance

Touchdown states	Requirement
Position,[a] m	$X > 0$, $\lvert Y \rvert < 18$
Velocity, m/s	$V_G < 62$, $\dot{Z} < 3$
Attitude, deg	$\Theta < 23$, $\lvert\Phi\rvert < 10$, $\lvert\Psi\rvert < 8$
Side slip, deg	$\lvert\beta_G\rvert < 8$

[a]Runway coordination; the origin is at the runway threshold, the X axis is directed along the runway centerline, and the Z axis is directed downward.

ALFLEX Pitch Attitude Control Logic

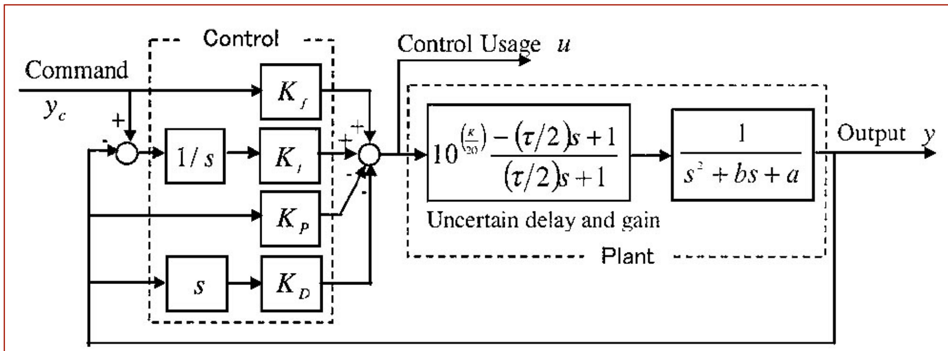


Fig. 5 Simplified model of pitch attitude control.

Comparison of SA, DS, and GA in Designing a PID Controller

Table 2 Comparison of three optimization methods

Parameter	Simulated annealing	Downhill–simplex	Genetic algorithm
Best design vector ***d******			
K_f	0.866	2.95	0.423
K_P	3.88	4.33	4.11
K_I	1.04	2.24	1.08
K_D	3.05	3.31	3.18
Total simulation number	31,998	13,604	121,552
Number of evaluated design vectors	66	51	745

Table 3 Results of 10,000 Monte Carlo evaluations using optimized design parameters

Method	Cost function J	[Confidence interval]
Simulated annealing	0.0135	[0.012, 0.016]
Downhill–simplex	0.0278	[0.025, 0.031]
Genetic algorithm	0.0133	[0.012, 0.015]

Genetic Algorithm Applications

GA Mona Lisa, 2
http://www.youtube.com/watch?v=A8x4Lyj33Ro&NR=1

Learning Network Weights for a Flapping Wing Neural-Controller
http://www.youtube.com/watch?v=BfY4jRtcE4c&feature=related

Virtual Creature Evolution
http://www.youtube.com/watch?v=oquKOVfzGfk&NR=1

Evolution of Locomotion
http://www.youtube.com/watch?v=STkfUZtR-Vs&feature=related

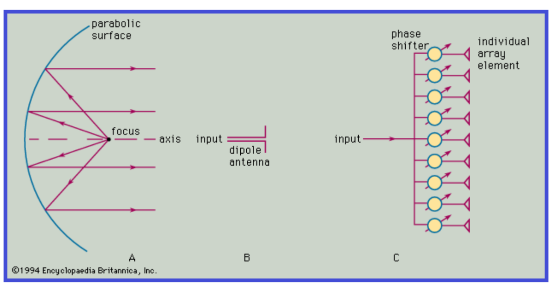
Examples of Particle Swarm Optimization

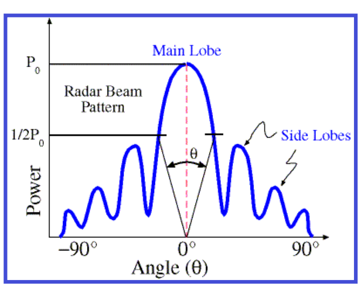
Robot Swarm Animation
http://www.youtube.com/watch?v=RLIA1EKfSys

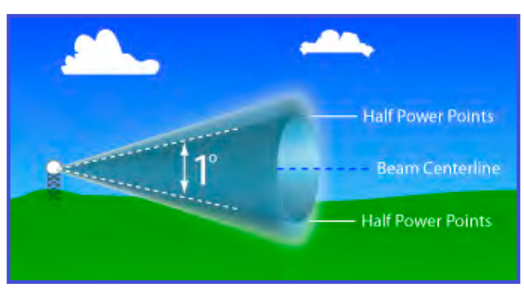
Swarm-Bots Finding a Path and Retrieving Object
http://www.youtube.com/watch?v=Xs_Y22N1r_A

Learning Robot Control System Gains
http://www.youtube.com/watch?v=itf8NHF1bS0&feature=related

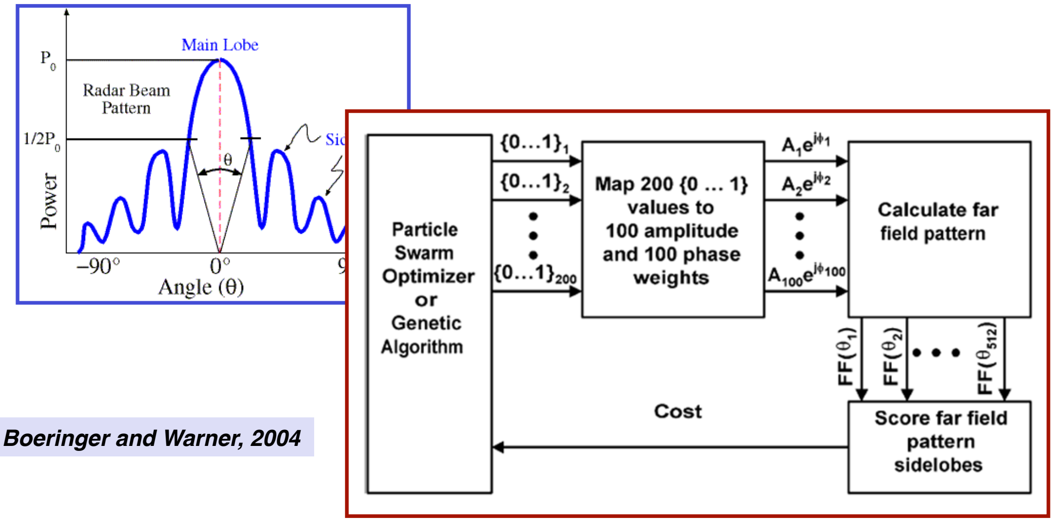
Parabolic and Phased-Array Radar Antenna Patterns





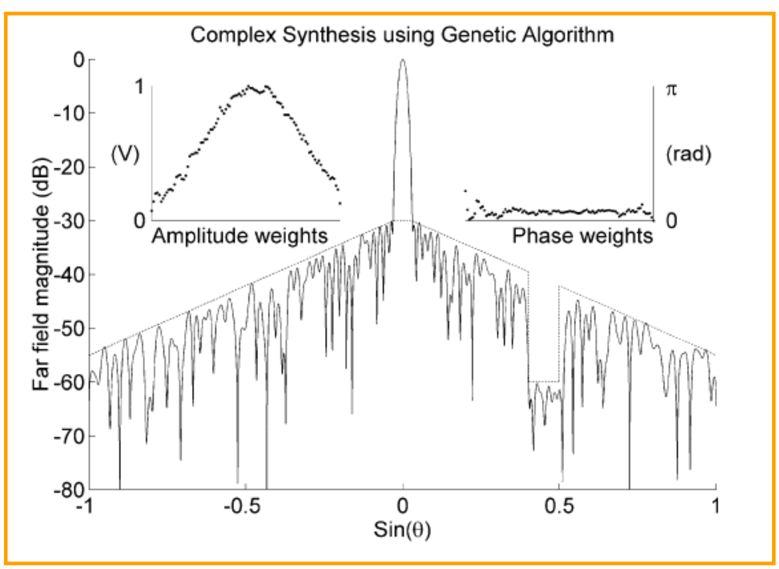


Phased-Array Antenna Design Using Genetic Algorithm or Particle Swarm Optimization

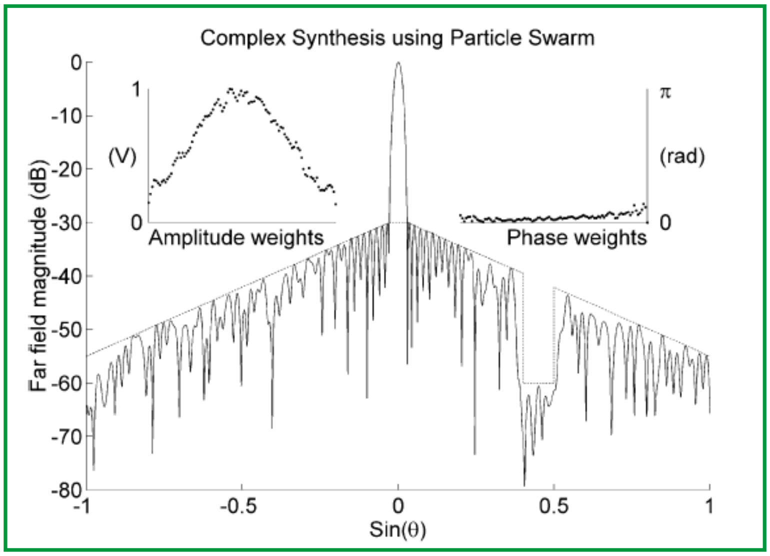


Boeringer and Warner, 2004

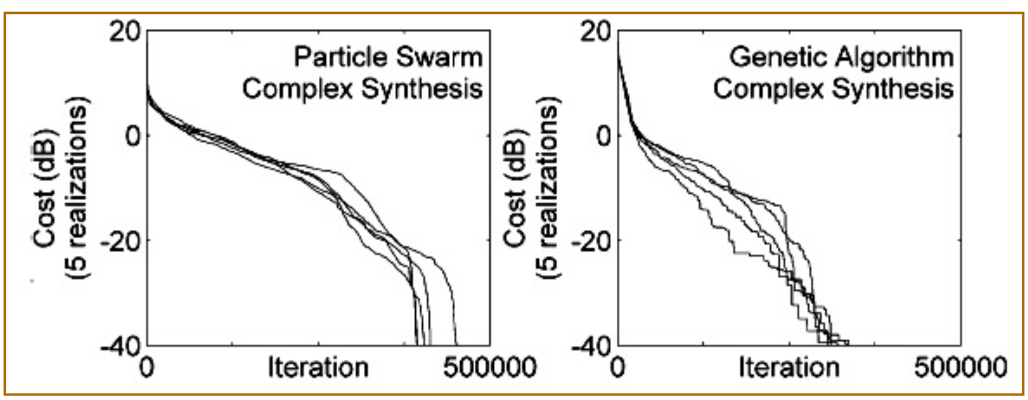
Phased-Array Antenna Design Using Genetic Algorithm



Phased-Array Antenna Design Using Particle Swarm Optimization



Comparison of Phased-Array Antenna Designs



Summary of Gradient-Free Optimization Algorithms

- Grid search
  - Uniform coverage of search space
- Random Search
  - Arbitrary placement of test parameters
- Downhill Simplex Method
  - Robust search of difficult cost function topology
- Simulated Annealing
  - Structured random search with convergence feature
- Genetic Algorithm
  - Coding of the parameter set
- Particle Swarm Optimization
  - Intuitively appealing, efficient heuristic