Translational and Rotational Dynamics

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Robotics and Intelligent Systems MAE 345,
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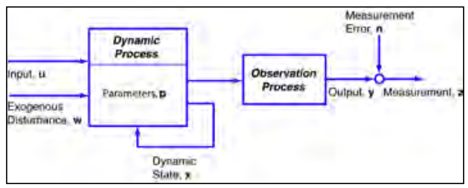
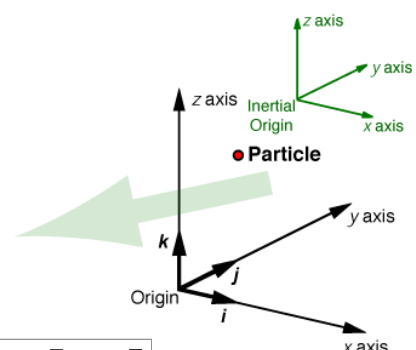


Reference Frame

- Newtonian (Inertial) Frame of Reference
  - Unaccelerated Cartesian frame
    - Origin referenced to inertial (non-moving) frame
  - Right-hand rule
  - Origin can translate at constant linear velocity
  - Frame cannot rotate with respect to inertial origin



- Position: 3 dimensions
  - What is a non-moving frame?

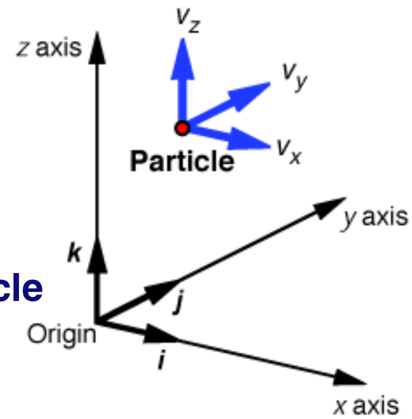
$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- Translation = Linear motion

Velocity and Momentum of a Particle

- **Velocity of a particle**

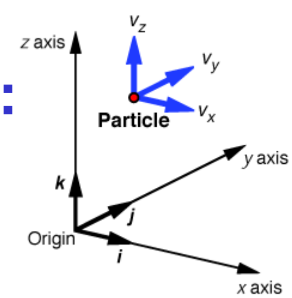
$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$



- **Linear momentum of a particle**

$$\mathbf{p} = m\mathbf{v} = m\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

Newton's Laws of Motion: Dynamics of a Particle

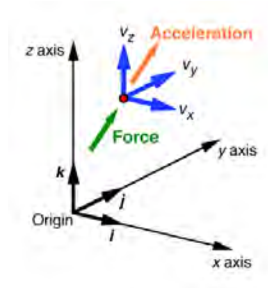


First Law

. If no force acts on a particle,
it remains at rest or continues to move in straight line at constant velocity,
. Inertial reference frame
. Momentum is conserved

$$\frac{d}{dt}(m\mathbf{v}) = 0 \quad ; \quad m\mathbf{v}\big|_{t_1} = m\mathbf{v}\big|_{t_2}$$

Newton's Laws of Motion:
Dynamics of a Particle



Second Law

- Particle acted upon by force
- Acceleration proportional to and in direction of force
- Inertial reference frame
- Ratio of force to acceleration is particle mass

$$\frac{d}{dt}(m\mathbf{v}) = m\frac{d\mathbf{v}}{dt} = m\mathbf{a} = \mathbf{Force}$$

$$\mathbf{Force} = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \textbf{force vector}$$

$$\therefore \quad \frac{d\mathbf{v}}{dt} = \frac{1}{m}\mathbf{Force} = \frac{1}{m}\mathbf{I}_3\mathbf{Force}$$

$$= \begin{bmatrix} 1/m & 0 & 0 \\ 0 & 1/m & 0 \\ 0 & 0 & 1/m \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$$

Newton's Laws of Motion:
Dynamics of a Particle

Third Law

For every action, there is an equal and opposite reaction

Force on rocket motor = –Force on exhaust gas

$$\mathrm{F}_{\mathbf{R}} = -\mathbf{F}_E$$

One-Degree-of-Freedom Example of Newton's Second Law

2nd-order, linear, time-invariant ordinary differential equation

$$\frac{d^2x(t)}{dt^2} \triangleq \ddot{x}(t) = \dot{v}_x(t) = \frac{f_x(t)}{m}$$

$\triangleq$ "Defined as"

Corresponding set of 1st-order equations
(State-Space Model)

$$\frac{dx_1(t)}{dt} \triangleq \dot{x}_1(t) \triangleq x_2(t) \triangleq v_x(t)$$
$$\frac{dx_2(t)}{dt} \triangleq \ddot{x}_1(t) = \dot{x}_2(t) = \dot{v}_x(t) = \frac{f_x(t)}{m}$$

$x_1(t) \triangleq x(t)$, Displacement

$x_2(t) \triangleq \frac{dx(t)}{dt}$, Rate

State-Space Model is a Set of 1st-Order Ordinary Differential Equations

State, control, and output vectors for the example

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}; \quad \mathbf{u}(t) = u(t) = f_x(t); \quad \mathbf{y}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Stability and control-effect matrices

$$\mathbf{F} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad \mathbf{G} = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$$

Dynamic equation

$$\dot{\mathbf{x}}(t) = \mathbf{F}\,\mathbf{x}(t) + \mathbf{G}\,\mathbf{u}(t)$$

State-Space Model of the 1-DOF Example

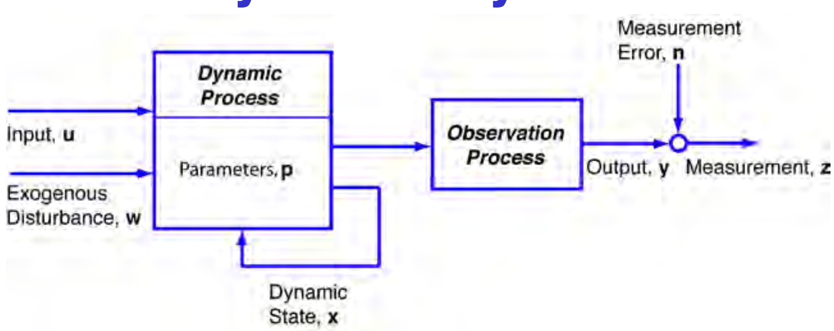
Output equation

$$\mathbf{y}(t) = \mathbf{H_x}\mathbf{x}(t) + \mathbf{H_u}\mathbf{u}(t)$$

Output coefficient matrices

$$\mathbf{H_x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \mathbf{H_u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Dynamic System



Dynamic Process: Current state may depend on prior state

x : state — *dim* = (*n* x 1)
u : input — *dim* = (*m* x 1)
w : disturbance — *dim* = (*s* x 1)
p : parameter — *dim* = (ℓ x 1)

t : time (independent variable, 1 x 1)

Observation Process: Measurement may contain error or be incomplete

y : output (error-free) — *dim* = (*r* x 1)
n : measurement error — *dim* = (*r* x 1)
z : measurement — *dim* = (*r* x 1)

State-Space Model of Three-Degree-of-Freedom Dynamics

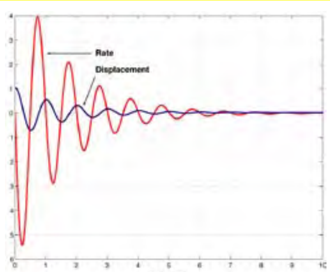
$$\dot{\mathbf{x}}(t) \triangleq \begin{bmatrix} \dot{\mathbf{r}}(t) \\ \dot{\mathbf{v}}(t) \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \hdashline \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ \hdashline f_x / m \\ f_y / m \\ f_z / m \end{bmatrix} = \mathbf{Fx}(t) + \mathbf{Gu}(t)$$

$$\mathbf{x}(t) \triangleq \begin{bmatrix} \mathbf{r}(t) \\ \mathbf{v}(t) \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ \hdashline v_x \\ v_y \\ v_z \end{bmatrix}$$

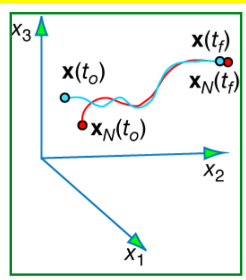
$$= \left[\begin{array}{ccc:ccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x \\ y \\ z \\ \hdashline v_x \\ v_y \\ v_z \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hdashline 1/m & 0 & 0 \\ 0 & 1/m & 0 \\ 0 & 0 & 1/m \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$$

What Use are the Dynamic Equations?

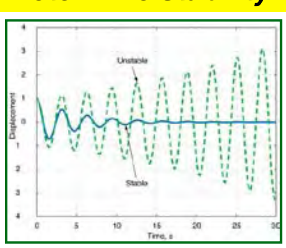
Compute time response



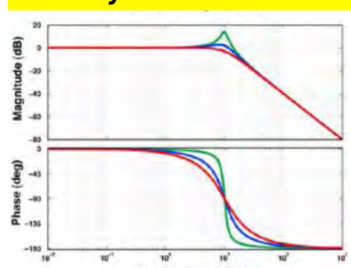
Compute trajectories



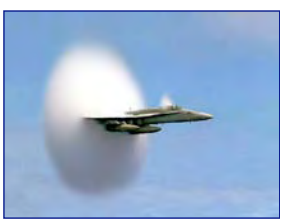
Determine stability



Identify modes of motion



Forces



External Forces:
Aerodynamic/Hydrodynamic

$$\mathbf{f}_{aero} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} C_X \\ C_Y \\ C_Z \end{bmatrix} \frac{1}{2} \rho V^2 A$$

ρ = air density, function of height

$= \rho_{sealevel} e^{-\beta h}$

V = airspeed

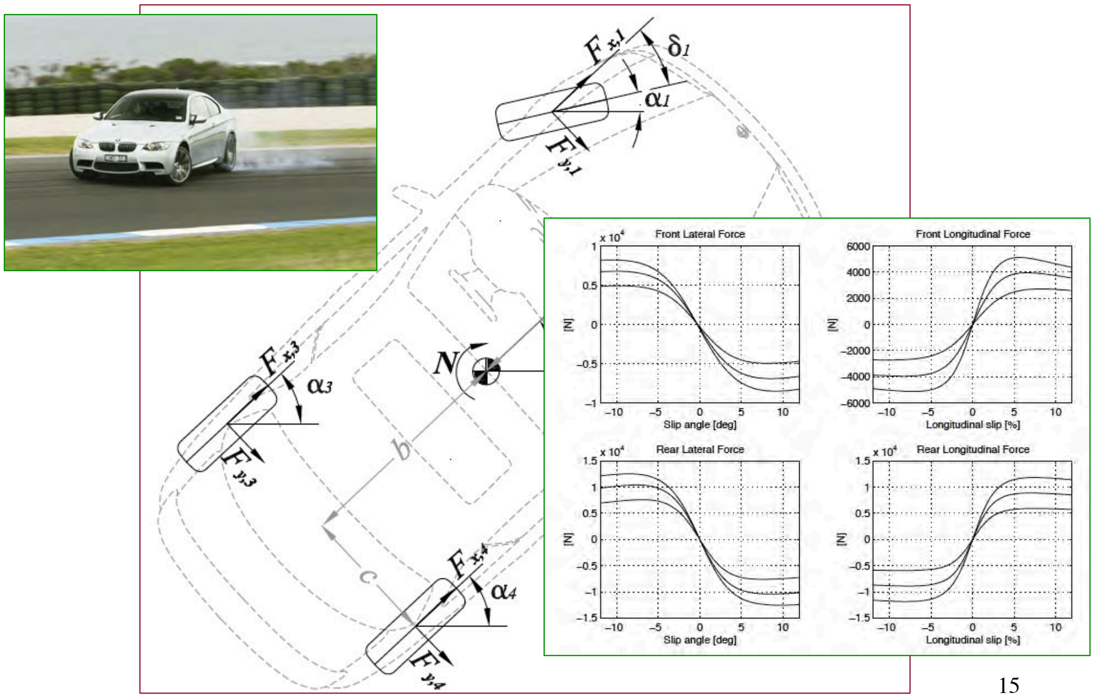
$= \left[v_x^2 + v_y^2 + v_z^2 \right]^{1/2} = \left[\mathbf{v}^T \mathbf{v} \right]^{1/2}$

A = reference area

$\begin{bmatrix} C_X \\ C_Y \\ C_Z \end{bmatrix} =$ dimensionless aerodynamic coefficients

Inertial frame or body frame?

External Forces: Friction



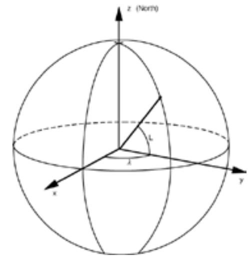
External Forces: Gravity

- Flat-earth approximation
  - g is gravitational acceleration
  - *m*g is gravitational force
  - Independent of position
  - *z* measured up

$$m\mathbf{g}_{flat} = m\begin{bmatrix} 0 \\ 0 \\ -g_o \end{bmatrix}; \quad g_o = 9.807\ m/s^2$$

External Forces: Gravity

$$r = \left(x^2 + y^2 + z^2\right)^{1/2}$$
$$L = \text{Latitude}$$
$$\lambda = \text{Longitude}$$



- **Spherical earth, inertial frame**
  - "Inverse-square" gravitation
  - Non-linear function of position
  - μ = 3.986 x 10^{14} m/s

$$\mathbf{g}_{round} = \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} = \mathbf{g}_{gravity}$$
$$= -\frac{\mu}{r^3}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{\mu}{r^2}\begin{bmatrix} x/r \\ y/r \\ z/r \end{bmatrix} = -\frac{\mu}{r^2}\begin{bmatrix} \cos L \cos\lambda \\ \cos L \sin\lambda \\ \sin L \end{bmatrix}$$

- **Spherical earth, rotating frame**
  - "Inverse-square" gravitation
  - "Centripetal acceleration"
  - Ω = 7.29 x 10^{-5} rad/s

$$\mathbf{g}_r = \mathbf{g}_{gravity} + \mathbf{g}_{rotation}$$
$$= -\frac{\mu}{r^3}\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \Omega^2 \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

State-Space Model with Round-Earth Gravity Model (Non-Rotating Frame)

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mu/r^3 & 0 & 0 \\ 0 & \mu/r^3 & 0 \\ 0 & 0 & \mu/r^3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\mu/r^3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\mu/r^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu/r^3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix}$$

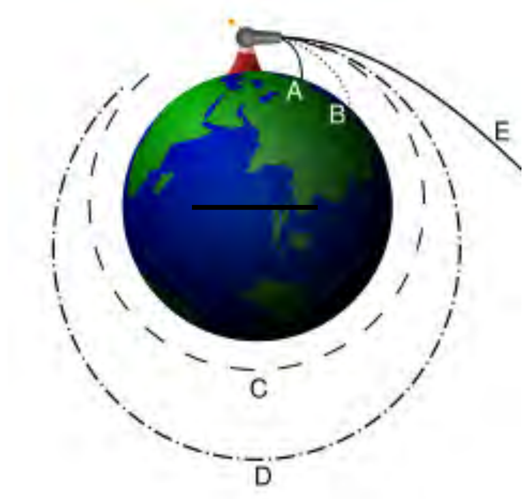
Inverse-square gravity model introduces nonlinearity

Vector-Matrix Form of Round-Earth Dynamic Model

$$\begin{bmatrix} \dot{\mathbf{r}} \\ \hline \dot{\mathbf{v}} \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{I}_3 \\ \hline \dfrac{-\mu}{r^3}\mathbf{I}_3 & \mathbf{0} \end{array}\right] \begin{bmatrix} \mathbf{r} \\ \hline \mathbf{v} \end{bmatrix}$$

What other forces might be considered, and where would they appear in the model?

Point-Mass Motions of Spacecraft

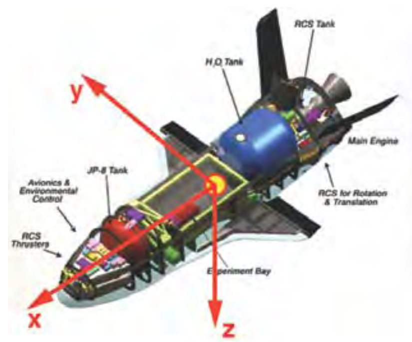


- **For short distance and low speed, flat-Earth frame of reference and gravity are sufficient**
- **For long distance and high speed, round-Earth frame and inverse-square gravity are needed**

Mass of an Object

$$m = \int_{Body} dm = \int_{x_{\min}}^{x_{\max}} \int_{y_{\min}}^{y_{\max}} \int_{z_{\min}}^{z_{\max}} \rho(x,y,z)\, dx\, dy\, dz$$

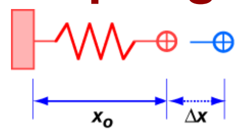
$$\rho(x,y,z) = density\ of\ the\ body$$



Density of object may vary with *(x,y,z)*

More Forces

External Forces:
Linear Springs



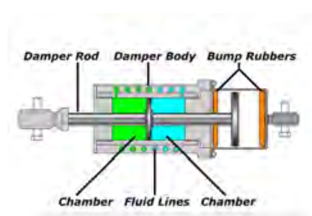
Scalar, linear spring

$$f = -k\left(x - x_o\right) = -k\Delta x; \quad k = spring\ constant$$

Uncoupled, linear vector spring

$$\mathbf{f}_S = -\begin{bmatrix} k_x\left(x - x_o\right) \\ k_y\left(y - y_o\right) \\ k_z\left(z - z_o\right) \end{bmatrix} = -\begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}$$

$\left(x_o, y_o, z_o\right)$: Equilibrium (reference) position



External Forces:
Viscous Dampers

Scalar, linear damper

$$f = -d\left(v - v_o\right) = -d\Delta v; \quad d = damping\ constant$$

Uncoupled, linear vector damper

$$\mathbf{f}_D = -\begin{bmatrix} d_x\left(v_x - v_{x_o}\right) \\ d_y\left(v_y - v_{y_o}\right) \\ d_z\left(v_z - v_{z_o}\right) \end{bmatrix} \cong -\begin{bmatrix} d_x & 0 & 0 \\ 0 & d_y & 0 \\ 0 & 0 & d_z \end{bmatrix}\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$\left(v_{x_o}, v_{y_o}, v_{z_o}\right)$: Equilibrium (reference) velocity [usually = 0]

State-Space Model with Linear Spring and Damping

$$
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-k_x}{m} & 0 & 0 & \frac{-d_x}{m} & 0 & 0 \\ 0 & \frac{-k_y}{m} & 0 & 0 & \frac{-d_y}{m} & 0 \\ 0 & 0 & \frac{-k_z}{m} & 0 & 0 & \frac{-d_z}{m} \end{bmatrix} \begin{bmatrix} (x-x_o) \\ (y-y_o) \\ (z-z_o) \\ v_x \\ v_y \\ v_z \end{bmatrix}
$$

Spring Effects

Damping Effects

State-Space Model with Linear Spring and Damping

Stability Effects

Reference Effects

$$
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{-k_x}{m} & 0 & 0 & \frac{-d_x}{m} & 0 & 0 \\ 0 & \frac{-k_y}{m} & 0 & 0 & \frac{-d_y}{m} & 0 \\ 0 & 0 & \frac{-k_z}{m} & 0 & 0 & \frac{-d_z}{m} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{k_x}{m} & 0 & 0 & \frac{d_x}{m} & 0 & 0 \\ 0 & \frac{k_y}{m} & 0 & 0 & \frac{d_y}{m} & 0 \\ 0 & 0 & \frac{k_z}{m} & 0 & 0 & \frac{d_z}{m} \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ z_o \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Vector-Matrix Form

$$\begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_3 \\ -\mathbf{K}/m & -\mathbf{D}/m \end{bmatrix} \begin{bmatrix} (\mathbf{r} - \mathbf{r}_o) \\ \mathbf{v} \end{bmatrix}$$

$$\begin{bmatrix} \dot{\mathbf{r}} \\ \dot{\mathbf{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_3 \\ -\mathbf{K}/m & -\mathbf{D}/m \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \mathbf{v} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & -\mathbf{I}_3 \\ \mathbf{K}/m & \mathbf{D}/m \end{bmatrix} \begin{bmatrix} \mathbf{r}_o \\ \mathbf{0} \end{bmatrix}$$

Rotational Motion

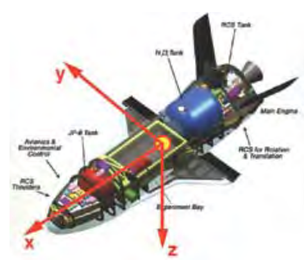
Assignment # 2

Document the physical characteristics and flight behavior of a Syma X11 quadcopter

https://www.youtube.com/watch?v=EwO6U7DbqSo

https://www.youtube.com/watch?v=kyiuy2CHzj0

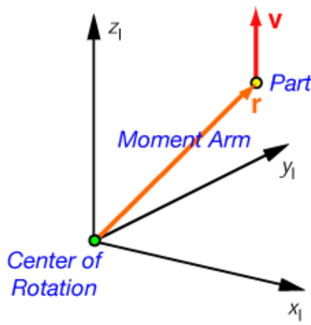
Center of Mass



$$\mathbf{r}_{cm} \triangleq \begin{bmatrix} x_{cm} \\ y_{cm} \\ z_{cm} \end{bmatrix} = \frac{1}{m} \int_{Body} \mathbf{r}\, dm$$

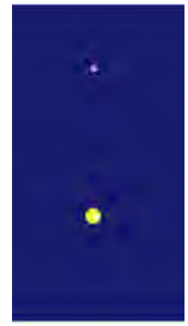
$$= \int_{x_{\min}}^{x_{\max}} \int_{y_{\min}}^{y_{\max}} \int_{z_{\min}}^{z_{\max}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \rho(x,y,z)\, dx\, dy\, dz$$

Reference point for rotational motion



Angular Momentum of a Particle

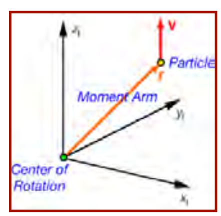
Particle in Inverse-Square Field



- **Moment of linear momentum of differential particles that make up the body**
  - **Differential mass of a particle times**
  - **Component of velocity perpendicular to moment arm from center of rotation to particle**

$$d\mathbf{h} = (\mathbf{r} \times dm\mathbf{v}) = (\mathbf{r} \times \mathbf{v})dm$$

Cross Product of Two Vectors



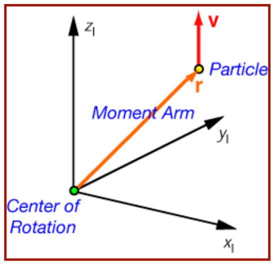
$$\mathbf{r} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix} = (yv_z - zv_y)i + (zv_x - xv_z)j + (xv_y - yv_x)k$$

(i, j, k): **Unit vectors** along (x, y, z)

This is equivalent to

$$= \begin{bmatrix} (yv_z - zv_y) \\ (zv_x - xv_z) \\ (xv_y - yv_x) \end{bmatrix} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \tilde{\mathbf{r}}\mathbf{v}$$

Cross-Product-Equivalent Matrix



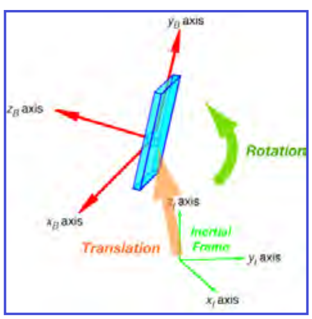
$$\mathbf{r} \times \mathbf{v} = \tilde{\mathbf{r}}\mathbf{v}$$

Cross-product equivalent of radius vector

$$\mathbf{r} \times \triangleq \tilde{\mathbf{r}} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

Velocity vector

$$\mathbf{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

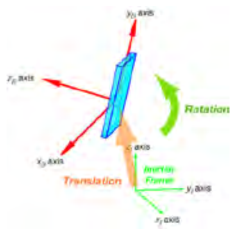


Angular Momentum of an Object

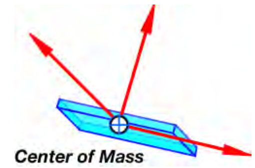
Integrate <u>moment</u> of linear momentum of differential particles over the body

$$\mathbf{h} \triangleq \int_{Body} (\mathbf{r} \times \mathbf{v})\, dm = \int_{x_{\min}}^{x_{\max}} \int_{y_{\min}}^{y_{\max}} \int_{z_{\min}}^{z_{\max}} (\mathbf{r} \times \mathbf{v})\, \rho(x,y,z)\, dx\, dy\, dz$$

$$= \int_{x_{\min}}^{x_{\max}} \int_{y_{\min}}^{y_{\max}} \int_{z_{\min}}^{z_{\max}} (\tilde{\mathbf{r}}\mathbf{v})\, \rho(x,y,z)\, dx\, dy\, dz \triangleq \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix}$$



Angular Velocity and Corresponding Velocity Increment



Angular velocity of object with respect to inertial frame of reference

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_I$$

Linear velocity increment at a point, *(x,y,z)*, due to angular rotation

$$\Delta\mathbf{v}(x,y,z) = \begin{bmatrix} \Delta v_x \\ \Delta v_y \\ \Delta v_z \end{bmatrix} = \boldsymbol{\omega}\times\mathbf{r} = -(\mathbf{r}\times\boldsymbol{\omega})$$

Angular Momentum of an Object with Respect to Its Center of Mass

- **Choose center of mass as origin about which angular momentum is calculated (= center of rotation)**
  - **i.e., r is measured from the center of mass**

$$\mathbf{h} = \int_{Body} \left[\mathbf{r}\times(\mathbf{v}_{cm}+\Delta\mathbf{v})\right]dm = \int_{Body} \left[\mathbf{r}\times\mathbf{v}_{cm}\right]dm + \int_{Body} \left[\mathbf{r}\times\Delta\mathbf{v}\right]dm$$

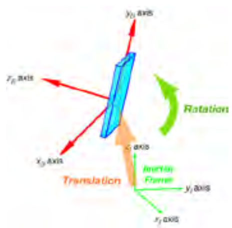
$$= \int_{Body} \left[\mathbf{r}\,dm\right]\times\mathbf{v}_{cm} - \int_{Body} \left[\mathbf{r}\times(\mathbf{r}\times\boldsymbol{\omega})\right]dm$$

$$= -\int_{Body} \tilde{\mathbf{r}}\tilde{\mathbf{r}}\,\boldsymbol{\omega}\,dm = \left[-\int_{Body} \tilde{\mathbf{r}}\tilde{\mathbf{r}}\,dm\right]\boldsymbol{\omega} \triangleq \mathbb{I}\boldsymbol{\omega}$$

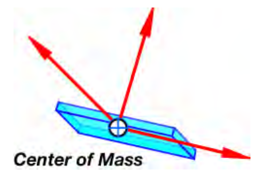
By symmetry,

$$\int_{Body} \left[\mathbf{r}\,dm\right]\times\mathbf{v}_{cm} = \mathbf{r}_{cm}\times\mathbf{v}_{cm} = 0$$

$$\mathbb{I} \triangleq \text{Inertia Matrix}$$



Angular Momentum



$$\mathbf{h} = \mathbb{I}\boldsymbol{\omega}$$

Three components of angular momentum

$$\begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} = \left[-\int_{Body} \tilde{\mathbf{r}}\,\tilde{\mathbf{r}}\,dm \right] \boldsymbol{\omega} \triangleq \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Inertia Matrix, $\mathbb{I}$

Angular rate has equal effect on all particles

$$\mathbb{I} = -\int_{Body} \tilde{\mathbf{r}}\,\tilde{\mathbf{r}}\,dm = -\int_{Body} \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} dm$$

$$= \int_{Body} \begin{bmatrix} (y^2+z^2) & -xy & -xz \\ -xy & (x^2+z^2) & -yz \\ -xz & -yz & (x^2+y^2) \end{bmatrix} dm$$

Inertia Matrix

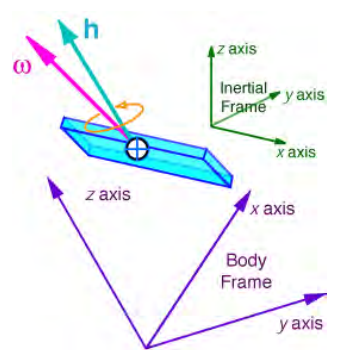
$$\mathbb{I} = \int_{Body} \begin{bmatrix} (y^2+z^2) & -xy & -xz \\ -xy & (x^2+z^2) & -yz \\ -xz & -yz & (x^2+y^2) \end{bmatrix} dm = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

- **Moments of inertia on the diagonal**
- **Products of inertia off the diagonal**
- **If <u>products</u> of inertia are zero, (*x, y, z*) are principal axes, and**

$$\mathbb{I} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

Angular Momentum and Rate are Vectors

- **Can be expressed in either an inertial or a body frame**
- **Vectors are transformed by the rotation matrix and its inverse**



$$\mathbf{h}_B = \mathbb{I}_B \boldsymbol{\omega}_B = \mathbf{H}_I^B \mathbf{h}_I = \mathbf{H}_I^B \mathbb{I}_I \boldsymbol{\omega}_I$$
$$\boldsymbol{\omega}_B = \mathbf{H}_I^B \boldsymbol{\omega}_I$$

$$\mathbf{h}_I = \mathbb{I}_I \boldsymbol{\omega}_I = \mathbf{H}_B^I \mathbf{h}_B = \mathbf{H}_B^I \mathbb{I}_B \boldsymbol{\omega}_B$$
$$\boldsymbol{\omega}_I = \mathbf{H}_B^I \boldsymbol{\omega}_B$$

Similarity Transformations for Matrices

Alternative expressions for angular momentum

$$\mathbf{h}_B = \mathbb{I}_B \boldsymbol{\omega}_B = \mathbf{H}_I^B \mathbf{h}_I = \mathbf{H}_I^B \mathbb{I}_I \boldsymbol{\omega}_I$$
$$= \mathbf{H}_I^B \mathbb{I}_I \mathbf{H}_B^I \boldsymbol{\omega}_B$$

Inertia matrices are transformed from one frame to the other by similarity transformations

$$\mathbb{I}_B = \mathbf{H}_I^B \mathbb{I}_I \mathbf{H}_B^I \qquad \mathbb{I}_I = \mathbf{H}_B^I \mathbb{I}_B \mathbf{H}_I^B$$

Newton's 2[nd] Law Applied to Rotational Motion in Inertial Frame

Rate of change of angular momentum = applied moment (or torque), m

Inertia Matrix Expressed in Inertial Frame is Not Constant if Body is Rotating

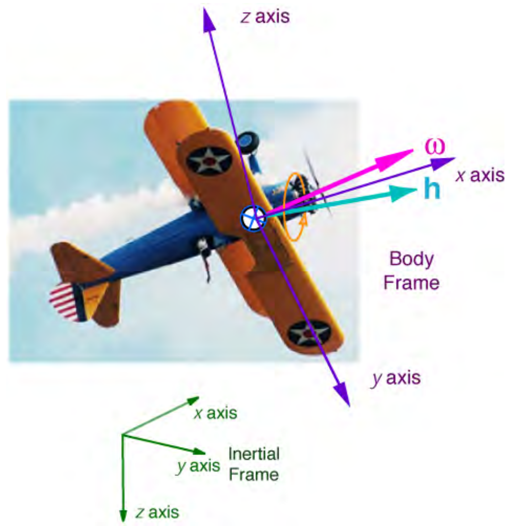
$$\frac{d\mathbf{h}_I}{dt} = \frac{d(\mathbb{I}_I \boldsymbol{\omega}_I)}{dt} = \frac{d\mathbb{I}_I}{dt}\boldsymbol{\omega}_I + \mathbb{I}_I \frac{d\boldsymbol{\omega}_I}{dt} = \mathbf{m}_I \quad [\text{moment vector}]_I$$

In a body reference frame, inertia matrix is constant

$$\frac{d\mathbf{h}_B}{dt} = \frac{d(\mathbb{I}_B \boldsymbol{\omega}_B)}{dt} = \frac{d\mathbb{I}_B}{dt}\boldsymbol{\omega}_B + \mathbb{I}_B \frac{d\boldsymbol{\omega}_B}{dt} = (\mathbf{0})\boldsymbol{\omega}_B + \mathbb{I}_B \frac{d\boldsymbol{\omega}_B}{dt}$$
$$= \mathbb{I}_B \frac{d\boldsymbol{\omega}_B}{dt} = \mathbf{m}_B \quad [\text{moment vector}]_B$$

How do We get Rid of $d(\mathbb{I}_I)/dt$ in the Angular Momentum Rate Equation?

- Write the dynamic equation in body-referenced frame
  - With constant mass, inertial properties are unchanging in body reference frame
  - ... but the frame is "non-Newtonian" or "non-inertial"



Vector Derivative Expressed in a Rotating Frame

Chain Rule

$$\frac{d\mathbf{h}_I}{dt} = \mathbf{H}_B^I \frac{d\mathbf{h}_B}{dt} + \frac{d\mathbf{H}_B^I}{dt}\mathbf{h}_B$$

$$= \mathbf{H}_B^I \frac{d\mathbf{h}_B}{dt} + \left(\boldsymbol{\omega}_I \times \mathbf{H}_B^I\right)\mathbf{h}_B = \mathbf{H}_B^I \frac{d\mathbf{h}_B}{dt} + \boldsymbol{\omega}_I \times \left(\mathbf{H}_B^I \mathbf{h}_B\right)$$

$$\frac{d\mathbf{h}_I}{dt} = \mathbf{H}_B^I \frac{d\mathbf{h}_B}{dt} + \boldsymbol{\omega}_I \times \mathbf{h}_I = \mathbf{H}_B^I \frac{d\mathbf{h}_B}{dt} + \tilde{\boldsymbol{\omega}}_I \mathbf{h}_I$$

Cross-product-equivalent matrix of angular rate:

$$\tilde{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

Similarity transformation

$$\tilde{\boldsymbol{\omega}}_I = \mathbf{H}_B^I \tilde{\boldsymbol{\omega}}_B \mathbf{H}_I^B$$

Rate of Change of Body-Axis Angular Momentum

Substitute

$$\frac{d\mathbf{h}_I}{dt} = \mathbf{H}_B^I \frac{d\mathbf{h}_B}{dt} + \mathbf{H}_B^I \tilde{\boldsymbol{\omega}}_B \mathbf{H}_I^B \mathbf{h}_I = \mathbf{m}_I = \mathbf{H}_B^I \mathbf{m}_B$$

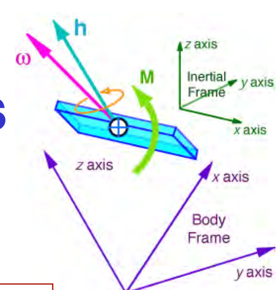
Eliminate

$$\mathbf{H}_B^I \frac{d\mathbf{h}_B}{dt} + \mathbf{H}_B^I \tilde{\boldsymbol{\omega}}_B \mathbf{H}_I^B \mathbf{h}_I = \mathbf{H}_B^I \mathbf{m}_B$$

Rearrange and Substitute

$$\frac{d\mathbf{h}_B}{dt} = \mathbf{m}_B - \tilde{\boldsymbol{\omega}}_B \mathbf{h}_B$$

Rate of Change of Body-Axis <u>Angular</u> Velocity



$$\mathbb{I}_B \frac{d\boldsymbol{\omega}_B}{dt} = \mathbf{m}_B - \tilde{\boldsymbol{\omega}}_B \mathbb{I}_B \boldsymbol{\omega}_B$$

$$\frac{d\boldsymbol{\omega}_B}{dt} = \mathbb{I}_B^{-1}\left(\mathbf{m}_B - \tilde{\boldsymbol{\omega}}_B \mathbb{I}_B \boldsymbol{\omega}_B\right)$$

Rate of Change of Body-Axis Translational Velocity

Inertial Velocity

$$\mathbf{v}_I = \mathbf{H}_B^I \mathbf{v}_B$$

Rate of change

$$\frac{d\mathbf{v}_I}{dt} = \mathbf{H}_B^I \frac{d\mathbf{v}_B}{dt} + \frac{d\left(\mathbf{H}_B^I\right)}{dt}\mathbf{v}_B = \mathbf{H}_B^I \frac{d\mathbf{v}_B}{dt} + \tilde{\boldsymbol{\omega}}_I \mathbf{v}_I$$

$$= \mathbf{H}_B^I \frac{d\mathbf{v}_B}{dt} + \mathbf{H}_B^I \tilde{\boldsymbol{\omega}}_B \mathbf{H}_I^B \mathbf{v}_I = \mathbf{H}_B^I \frac{d\mathbf{v}_B}{dt} + \mathbf{H}_B^I \tilde{\boldsymbol{\omega}}_B \mathbf{v}_B$$

$$= \frac{1}{m}\mathbf{f}_I = \frac{1}{m}\mathbf{H}_B^I \mathbf{f}_B$$

Substitution

$$\mathbf{H}_B^I \frac{d\mathbf{v}_B}{dt} + \mathbf{H}_B^I \tilde{\boldsymbol{\omega}}_B \mathbf{v}_B = \frac{1}{m}\mathbf{H}_B^I \mathbf{f}_B$$

Result

$$\frac{d\mathbf{v}_B}{dt} = \frac{1}{m}\mathbf{f}_B - \tilde{\boldsymbol{\omega}}_B \mathbf{v}_B$$

Rate of Change of Translational Position

Express derivative in inertial frame

$$\dot{\mathbf{r}}_I = \mathbf{v}_I = \mathbf{H}_B^I \mathbf{v}_B$$

Rate of Change of Angular Orientation (Euler Angles)

- **Body-axis angular rate vector components are orthogonal**

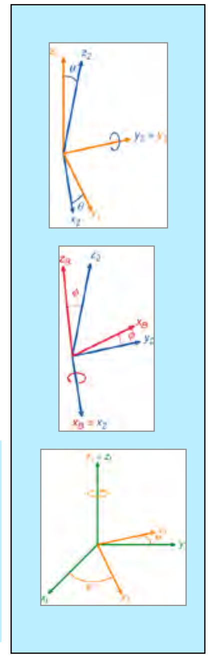
$$\boldsymbol{\omega}_B = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_B \triangleq \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

- **Euler angles form a non-orthogonal vector**

$$\boldsymbol{\Theta} \triangleq \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$

- **Therefore, Euler-angle rate vector is not orthogonal**

$$\frac{d\boldsymbol{\Theta}}{dt} = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \neq \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_I \neq \boldsymbol{\omega}_I$$

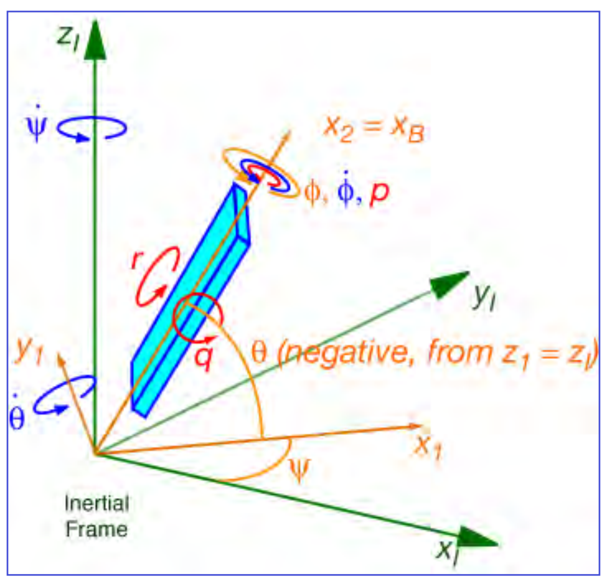


Transformation From Euler-Angle Rates to Body-Axis Rates

$\dot{\psi}$ is measured in the Inertial Frame

$\dot{\theta}$ is measured in Intermediate Frame #1

$\dot{\phi}$ is measured in Intermediate Frame #2



Sequential Transformations from Euler-Angle Rates to Body-Axis Rates

$\dot{\psi}$ is measured in the Inertial Frame
$\dot{\theta}$ is measured in Intermediate Frame #1
$\dot{\phi}$ is measured in Intermediate Frame #2

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \mathbf{I}_3 \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + \mathbf{H}_2^B \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \mathbf{H}_2^B \mathbf{H}_1^2 \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

... which is

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \triangleq \mathbf{L}_I^B \dot{\mathbf{\Theta}}$$

Inversion to Transform Body-Axis Rates to Euler-Angle Rates

Transformation is not orthonormal

$$\mathbf{L}_I^B = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix}$$

Inverse transformation is not the transpose

$$\left(\mathbf{L}_I^B\right)^{-1} \triangleq \mathbf{L}_B^I \neq \left(\mathbf{L}_I^B\right)^T$$

$$\left(\mathbf{L}_I^B\right)^{-1} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix}^{-1} = \frac{Adj\left(\mathbf{L}_I^B\right)}{\det\left(\mathbf{L}_I^B\right)}$$

Euler-Angle Rates

$$\left(\mathbf{L}_I^B\right)^{-1} = \frac{Adj\left(\mathbf{L}_I^B\right)}{\det\left(\mathbf{L}_I^B\right)} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix}$$

Euler-angle rates from body-axis rates

$$\begin{bmatrix} \dot\phi \\ \dot\theta \\ \dot\psi \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \mathbf{L}_B^I \boldsymbol{\omega}_B$$

Can the inversion become singular?
What does this mean?

Summary of Six-Degree-of-Freedom (Rigid Body) Equations of Motion

Rigid-body dynamic equations are nonlinear

$$\dot{\mathbf{r}}_I = \mathbf{H}_B^I \mathbf{v}_B$$

$$\dot{\mathbf{v}}_B = \frac{1}{m}\mathbf{f}_B - \tilde{\boldsymbol{\omega}}_B \mathbf{v}_B$$

$$\dot{\boldsymbol{\Theta}} = \mathbf{L}_B^I \boldsymbol{\omega}_B$$

$$\dot{\boldsymbol{\omega}}_B = \mathbb{I}_B^{-1}\left(\mathbf{m}_B - \tilde{\boldsymbol{\omega}}_B \mathbb{I}_B \boldsymbol{\omega}_B\right)$$

Translational position and velocity

$$\mathbf{r}_I = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \quad \mathbf{v}_B = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

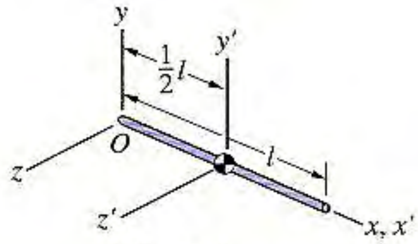
Rotational position and velocity

$$\boldsymbol{\Theta} = \begin{bmatrix} \varphi \\ \theta \\ \psi \end{bmatrix}; \quad \boldsymbol{\omega}_B = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Next Time:
Flying and Swimming Robots

Supplemental Material

Moments and Products of Inertia for Common Constant-Density Objects are Tabulated



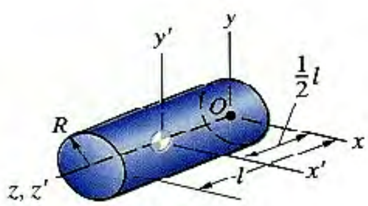
Slender bar

$$I_{x \text{ axis}} = 0, \qquad I_{y \text{ axis}} = I_{z \text{ axis}} = \frac{1}{3} ml^2,$$

$$I_{xy} = I_{yz} = I_{zx} = 0.$$

$$I_{x' \text{ axis}} = 0, \qquad I_{y' \text{ axis}} = I_{z' \text{ axis}} = \frac{1}{12} ml^2,$$

$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0.$$



Circular cylinder

$$\text{Volume} = \pi R^2 l$$

$$I_{x \text{ axis}} = I_{y \text{ axis}} = m\left(\frac{1}{3} l^2 + \frac{1}{4} R^2\right), \qquad I_{z \text{ axis}} - \frac{1}{2} mR^2,$$

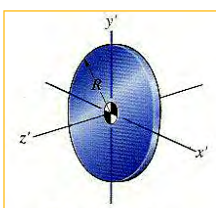
$$I_{xy} = I_{yz} = I_{zx} = 0.$$

$$I_{x' \text{ axis}} = I_{y' \text{ axis}} = m\left(\frac{1}{12} l^2 + \frac{1}{4} R^2\right), \qquad I_{z' \text{ axis}} = \frac{1}{2} mR^2,$$

$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0.$$

Moments and Products of Inertia

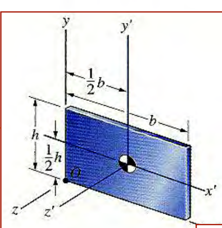
(Bedford & Fowler)



Thin circular plate

$$I_{x' \text{ axis}} = I_{y' \text{ axis}} = \frac{1}{4} mR^2, \qquad I_{z' \text{ axis}} = \frac{1}{2} mR^2,$$

$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0.$$



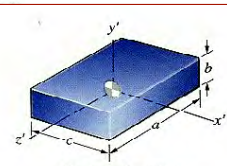
Thin rectangular plate

$$I_{x \text{ axis}} = \frac{1}{3} mh^2, \qquad I_{y \text{ axis}} = \frac{1}{3} mb^2, \qquad I_{z \text{ axis}} = \frac{1}{3} m(b^2 + h^2),$$

$$I_{xy} = \frac{1}{4} mbh, \qquad I_{yz} = I_{zx} = 0.$$

$$I_{x' \text{ axis}} = \frac{1}{12} mh^2, \qquad I_{y' \text{ axis}} = \frac{1}{12} mb^2, \qquad I_{z' \text{ axis}} = \frac{1}{12} m(b^2 + h^2),$$

$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0.$$



Rectangular prism

$$\text{Volume} = abc$$

$$I_{x' \text{ axis}} = \frac{1}{12} m(a^2 + b^2), \qquad I_{y' \text{ axis}} = \frac{1}{12} m(a^2 + c^2),$$

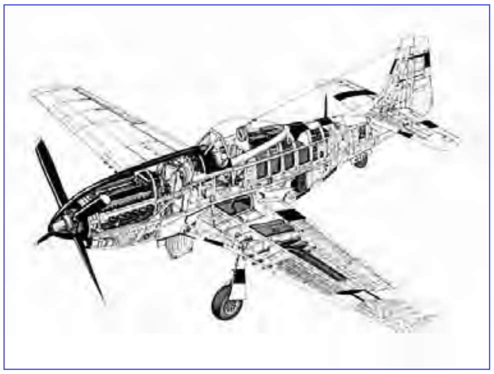
$$I_{z' \text{ axis}} = \frac{1}{12} m(b^2 + c^2), \qquad I_{x'y'} = I_{y'z'} = I_{z'x'} = 0.$$

$$\mathbb{I} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$

Construction of Inertia Matrix

Build up moments and products of inertia from components using *parallel-axis theorem*, e.g.,

$$I_{xx_{airplane}} = I_{xx_{wings}} + I_{xx_{fuselage}} + I_{xx_{horizontal\ tail}} + I_{xx_{vertical\ tail}} + \quad ...$$



... or use software, e.g., *Autodesk*, *Creo Parametric*, *SimMechanics*, ...