Dynamic Effects of Feedback Control

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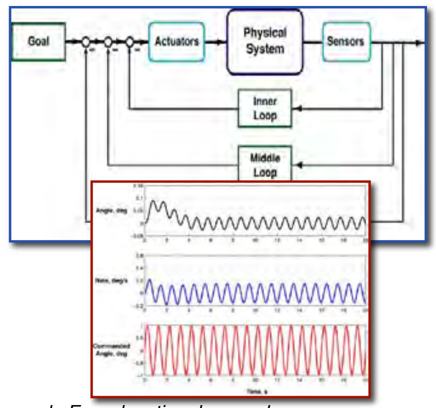
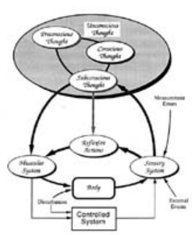
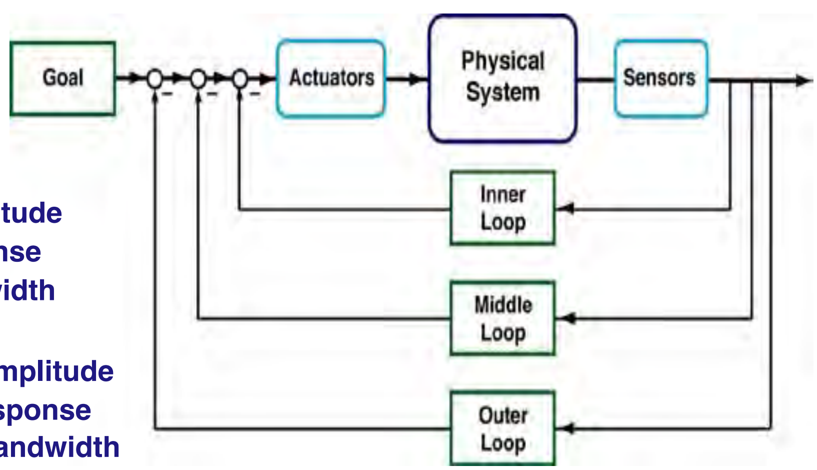
Robotics and Intelligent Systems MAE 345,
Princeton University, 2017

- Inner, Middle, and Outer Feedback Control Loops
- Step Response of Linear, Time-Invariant (LTI) Systems
- Position and Rate Control
- Transient and Steady-State Response to Sinusoidal Inputs



Outer-to-Inner-Loop Control Hierarchy





- Inner Loop
  - Small Amplitude
  - Fast Response
  - High Bandwidth
- Middle Loop
  - Moderate Amplitude
  - Medium Response
  - Moderate Bandwidth
- Outer Loop
  - Large Amplitude
  - Slow Response
  - Low Bandwidth

- Feedback
  - Error between command and feedback signal drives next inner-most loop

Natural Feedback Control

Inner Loop

Chicken Head Control - 1

http://www.youtube.com/watch?v=_dPlkFPowCc

Middle Loop

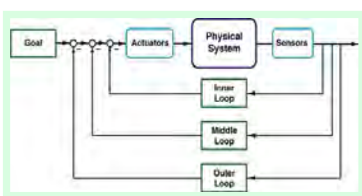
Hovering Red-Tail Hawks

http://www.youtube.com/watch?v=-VPVZMSEvwU

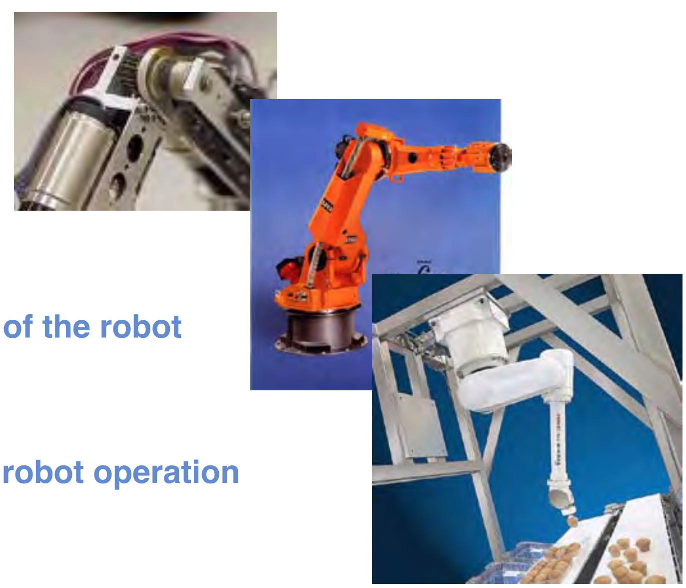
Outer Loop

Osprey Diving for Fish

http://www.youtube.com/watch?v=qrgpl9-N6jY



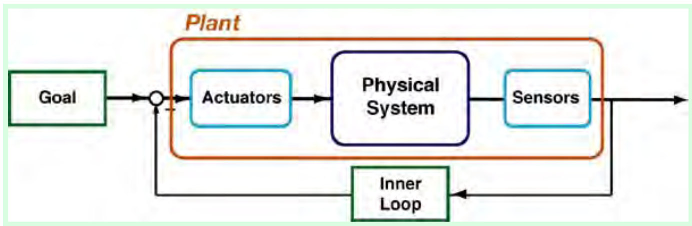
Outer-to-Inner-Loop Control Hierarchy of an Industrial Robot



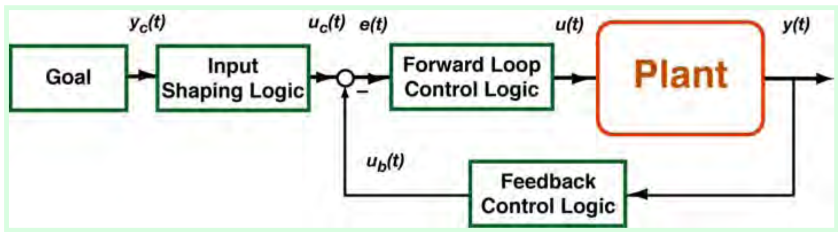
- **Inner Loop**
  - **Focus on control of individual joints**
- **Middle Loop**
  - **Focus on operation of the robot**
- **Outer Loop**
  - **Focus on goals for robot operation**

Inner-Loop Feedback Control

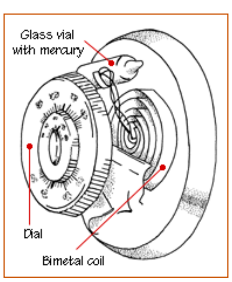
Feedback control design must account for actuator-system-sensor dynamics

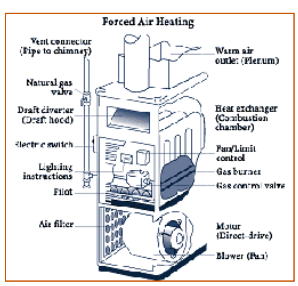


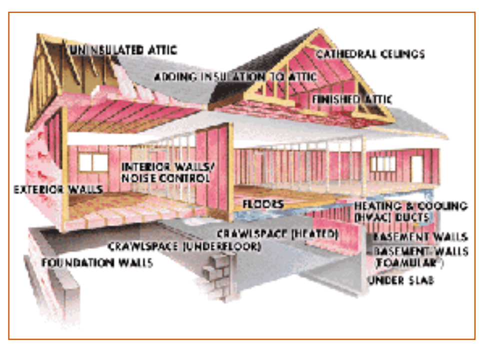
Single-Input/Single-Output Example, with forward and feedback control logic ("compensation")



Thermostatic Temperature Control

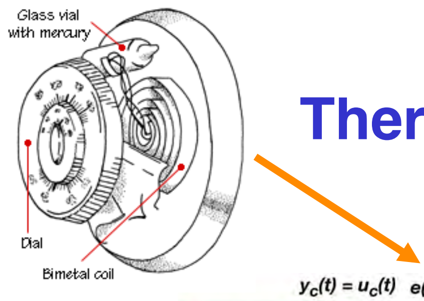




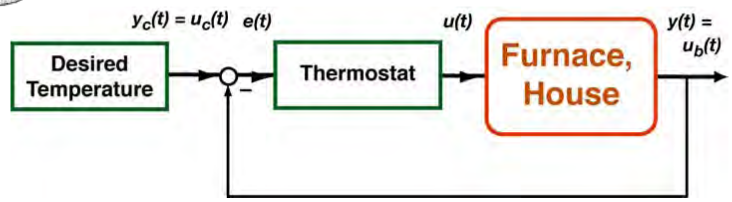


- **Dynamics**
  - Delays
  - Dead Zones
  - Saturation
  - Coupling
- **Structure**
  - Layout
  - Insulation
  - Circulation
  - Multiple Spaces
- **External Effects**
  - Solar Radiation
  - Air Temperature
  - Wind
  - Rain, Humidity

... all controlled by a simple (but nonlinear) on/off switch



Thermostat Control Logic



- **Control Law** ***[i.e., logic that drives the control variable, u(t)]***

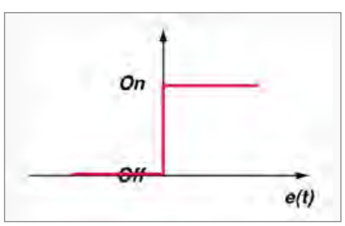
$$e(t) = y_c(t) - y(t) = u_c(t) - u_b(t)$$

$$< \text{Thermostat} >$$

$$u(t) = \begin{cases} 1\,(on), & e(t) > 0 \\ 0\,(off), & e(t) \le 0 \end{cases}$$

- **y_c: Desired output variable (command)**
- **y: Actual output**
- **u: Control variable (forcing function)**
- **e: Control error**

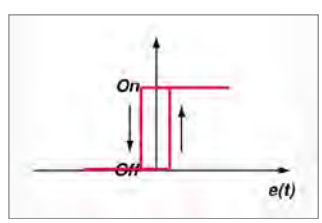
Thermostat Control Logic



$$u(t) = \begin{cases} 1\,(on), & e(t) > 0 \\ 0\,(off), & e(t) \le 0 \end{cases}$$

- **...but control signal would "chatter" with slightest change of temperature**
- **Solution: Introduce *lag* to slow the switching cycle, e.g., *hysteresis***

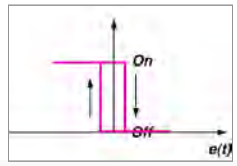
$$u(t) = \begin{cases} 1\,(on), & e(t) - T > 0 \\ 0\,(off), & e(t) + T \le 0 \end{cases}$$

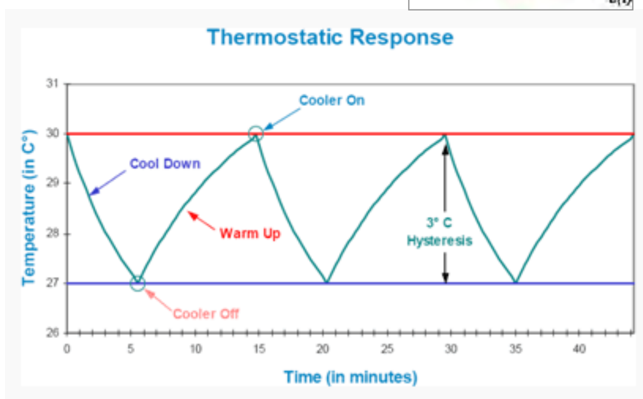


Thermostat Control Logic with Hysteresis

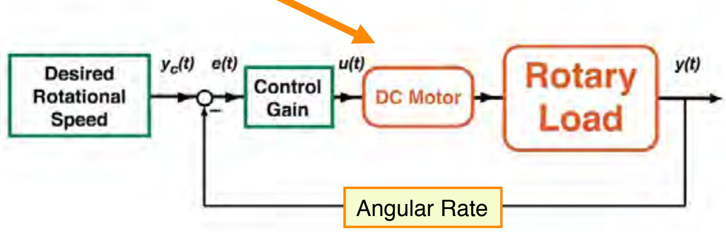
- **Hysteresis delays the response**
- **System responds with a *limit cycle***

- **Cooling control is similar with sign reversal**





Speed Control of Direct-Current Motor



Linear Feedback Control Law *(c = Control Gain)*

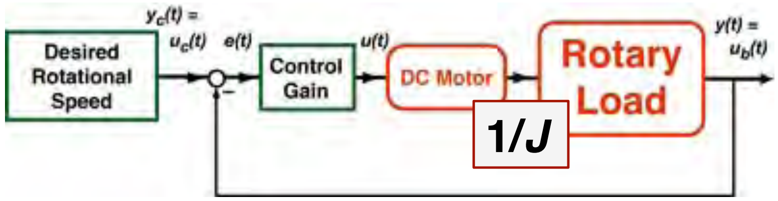
$$u(t) = c\,e(t)$$

where

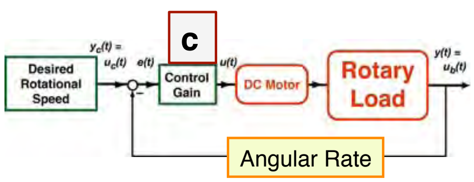
$$e(t) = y_c(t) - y(t)$$

How would *y(t)* be measured?

Characteristics of the Model



- **Simplified Dynamic Model**
  - **Rotary inertia, *J*, is the sum of motor and load inertias**
  - **Internal damping neglected**
  - **Output speed, *y(t)*, rad/s, is an integral of the control input, *u(t)***
  - **Motor control torque is proportional to *u(t)***
  - **Desired speed, $y_c(t)$, rad/s, is constant**



Model of Dynamics and Speed Control

First-order LTI ordinary differential equation

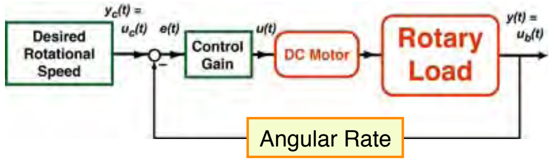
$$\boxed{\frac{dy(t)}{dt} = \frac{1}{J}u(t) = \frac{c}{J}e(t) = \frac{c}{J}\left[y_c(t) - y(t)\right], \quad y(0) \text{ given}}$$

Integral of the equation, with $y(0) = 0$

$$y(t) = \frac{1}{J}\int_0^t u(t)\,dt = \frac{c}{J}\int_0^t e(t)\,dt = \frac{c}{J}\int_0^t \left[y_c(t) - y(t)\right]dt$$

$$= -\frac{c}{J}\int_0^t \left[y(t)\right]dt + \frac{c}{J}\int_0^t \left[y_c(t)\right]dt$$

- **Positive integration of $y_c(t)$**
- **Negative feedback of $y(t)$**



Step Response of Speed Controller

Step input :

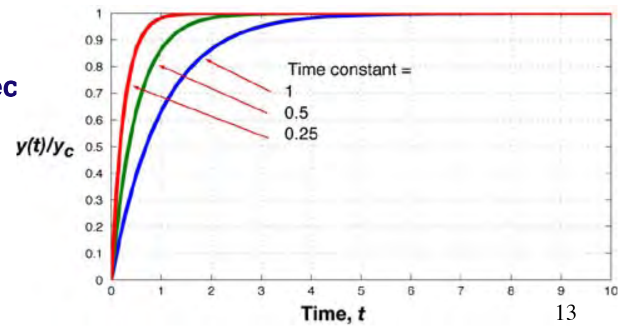
$$y_C(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

- **Solution of the integral**

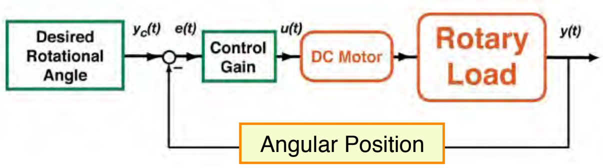
$$y(t) = y_c\left[1 - \exp^{-\left(\frac{c}{J}\right)t}\right] = y_c\left[1 - \exp^{\lambda t}\right] = y_c\left[1 - \exp^{-t/\tau}\right]$$

- **where**
  - λ **= $-c/J$ = eigenvalue or root of the system (rad/sec**
  - τ **= J/c = time constant of the response (sec)**

What does the shaft angle response look like?



Angle Control of Direct-Current Motor



- **Simplified Dynamic Model**
  - **Rotary inertia, *J*, is the sum of motor and load inertias**
  - **Output angle, *y(t)*, is a double integral of the control, *u(t)***
  - **Desired angle, $y_c(t)$, is constant**

Feedback Control Law

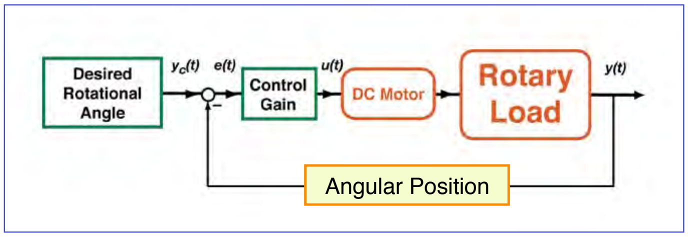
$$u(t) = c\,e(t)$$

where

$$e(t) = y_c(t) - y(t)$$

How would *y(t)* be measured?

Model of Dynamics and Angle Control



2nd-order, linear, time-invariant ordinary differential equation

$$\frac{d^2y(t)}{dt^2} = \ddot{y}(t) = \frac{1}{J}u(t) = \frac{c}{J}e(t) = \frac{c}{J}\left[y_c - y(t)\right]$$

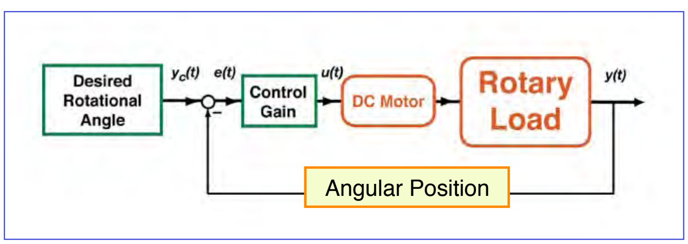
Output angle, *y(t)*, as a function of time

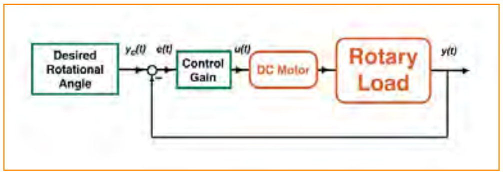
$$y(t) = \frac{c}{J}\int_0^t\int_0^t \left[y_c - y(t)\right]dt\,dt$$
$$= -\frac{c}{J}\int_0^t\int_0^t \left[y(t)\right]dt^2 + \frac{c}{J}\int_0^t\int_0^t \left[y_c\right]dt^2$$

Model of Dynamics and Angle Control

- **Corresponding set of 1st-order equations, with**
  - **Angle:** $x_1(t) = y(t)$
  - **Angular rate:** $x_2(t) = dy(t)/dt$

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = \frac{u(t)}{J} = \frac{c}{J}\left[y_c - y(t)\right] = \frac{c}{J}\left[y_c - x_1(t)\right]$$





State-Space Model of the DC Motor

Open-loop dynamic equation

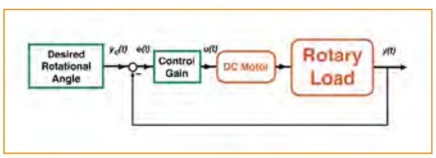
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} u(t)$$

Feedback control law

$$u(t) = c\left[y_c(t) - y_1(t)\right] = c\left[y_c(t) - x_1(t)\right]$$

Closed-loop dynamic equation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c/J & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ c/J \end{bmatrix} y_c$$

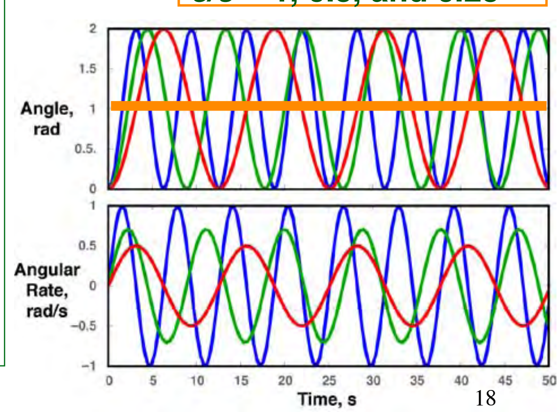


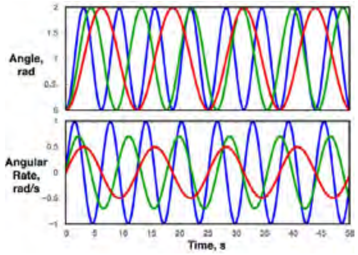
Step Response with Angle Feedback

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c/J & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ c/J \end{bmatrix} y_c$$

```
%   Step Response of Undamped Angle Control

    F1  =  [0 1;-1 0];
    G1  =  [0;1];
    F2  =  [0 1;-0.5 0];
    G2  =  [0;0.5];
    F3  =  [0 1;-0.25 0];
    G3  =  [0;0.25];
    Hx  =  [1 0;0 1];

    Sys1   =  ss(F1,G1,Hx,0);
    Sys2   =  ss(F2,G2,Hx,0);
    Sys3   =  ss(F3,G3,Hx,0);

    step(Sys1,Sys2,Sys3)
```

***c/J* = 1, 0.5, and 0.25**



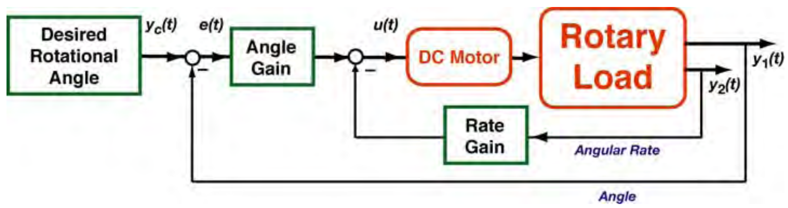


What Went Wrong?

- **No damping!**
- **Solution: Add rate feedback in the control law**

- **Control law with rate feedback**

$$u(t) = c_1\left[y_c(t) - y_1(t)\right] - c_2 y_2(t)$$



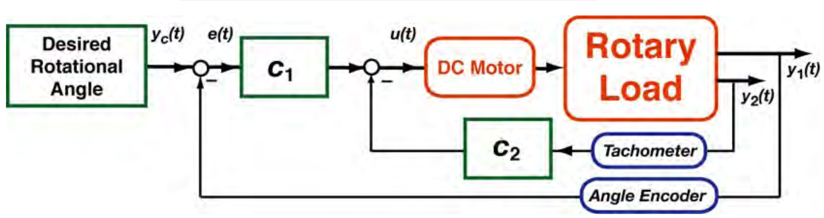
Closed-loop dynamic equation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c_1/J & -c_2/J \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ c_1/J \end{bmatrix} y_c$$

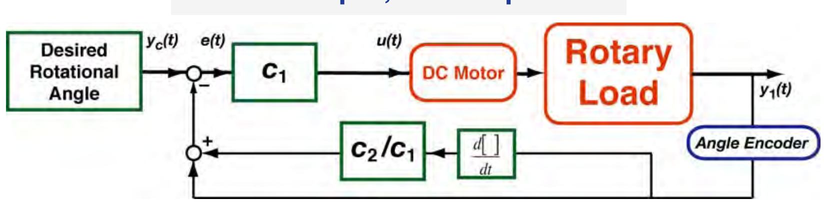
Alternative Implementations of Rate Feedback

$$u(t) = c_1[y_c(t) - y_1(t)] - c_2 y_2(t) = c_1[y_c(t) - y_1(t)] - c_2 \frac{dy_1(t)}{dt}$$

One input, two outputs



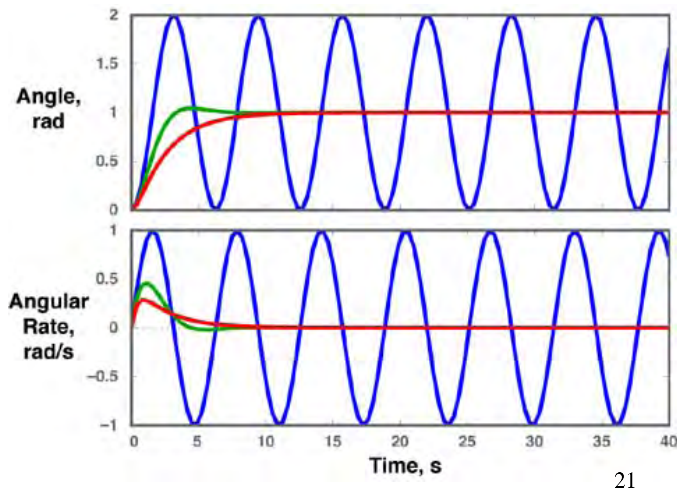
One input, one output



Step Response with Angle and Rate Feedback

$c_1/J = 1$
$c_2/J = 0,\ 1.414,\ 2.828$

```
%   Step Response of Damped
Angle Control

   F1  =  [0 1;-1 0];
   G1  =  [0;1];

   F1a =  [0 1;-1 -1.414];
   F1b =  [0 1;-1 -2.828];

   Hx  =  [1 0;0 1];

   Sys1   =  ss(F1,G1,Hx,0);
   Sys2   =  ss(F1a,G1,Hx,0);
   Sys3   =  ss(F1b,G1,Hx,0);

   step(Sys1,Sys2,Sys3)
```

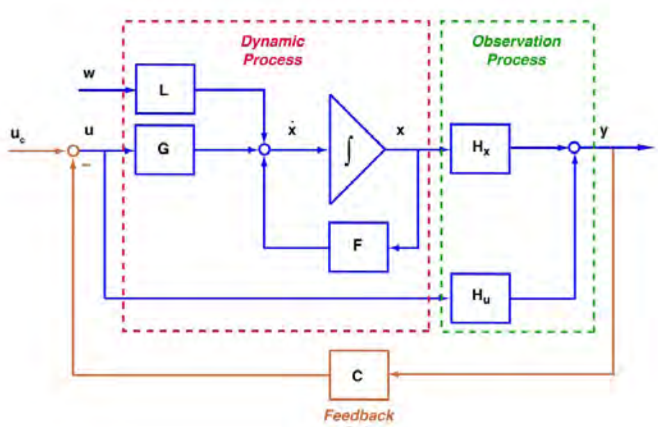


LTI Model with Feedback Control

- Command input, $\mathbf{u}_c$, has dimensions of $\mathbf{u}$

$$\mathbf{u}(t) = \mathbf{u}_c(t) - \mathbf{C}\mathbf{y}(t)$$

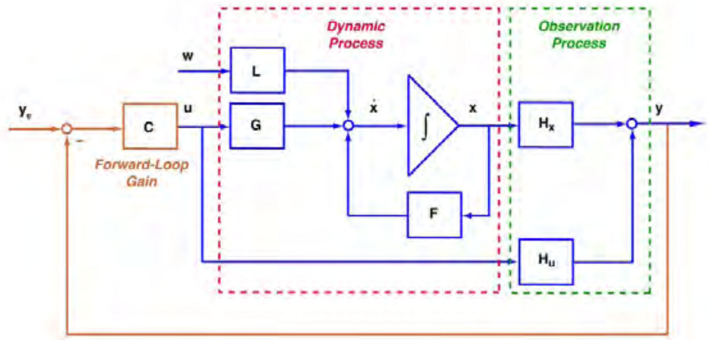
$$\dot{\mathbf{x}}(t) = \mathbf{F}\,\mathbf{x}(t) + \mathbf{G}\,\mathbf{u}(t) + \mathbf{L}\mathbf{w}(t)$$
$$\mathbf{y}(t) = \mathbf{H}_x\mathbf{x}(t) + \mathbf{H}_u\mathbf{u}(t)$$



LTI Control with Forward-Loop Gain

$$\mathbf{u}(t) = \mathbf{C}\left[\mathbf{y}_c(t) - \mathbf{y}(t)\right]$$

$$\dot{\mathbf{x}}(t) = \mathbf{F}\,\mathbf{x}(t) + \mathbf{G}\,\mathbf{u}(t) + \mathbf{L}\mathbf{w}(t)$$
$$\mathbf{y}(t) = \mathbf{H_x}\mathbf{x}(t) + \mathbf{H_u}\mathbf{u}(t)$$



With $\mathbf{C}_c = \mathbf{C}$, command input, $\mathbf{y}_c$, has dimensions of $\mathbf{y}$

Effect of Feedback Control on the LTI Model

$$\dot{\mathbf{x}}(t) = \mathbf{F}\,\mathbf{x}(t) + \mathbf{G}\,\mathbf{u}(t) = \mathbf{F}\,\mathbf{x}(t) + \mathbf{G}\left[\mathbf{u}_c(t) - \mathbf{C}\mathbf{y}(t)\right]$$
$$= \mathbf{F}_{open\ loop}\,\mathbf{x}(t) + \mathbf{G}\left\{\mathbf{u}_c(t) - \mathbf{C}\left[\mathbf{H_x}\mathbf{x}(t)\right]\right\}$$

$$= \left[\mathbf{F} - \mathbf{G}\mathbf{C}\mathbf{H_x}\right]\mathbf{x}(t) + \mathbf{G}\,\mathbf{u}_c(t)$$
$$\triangleq \mathbf{F}_{closed\ loop}\mathbf{x}(t) + \mathbf{G}\,\mathbf{u}_c(t)$$

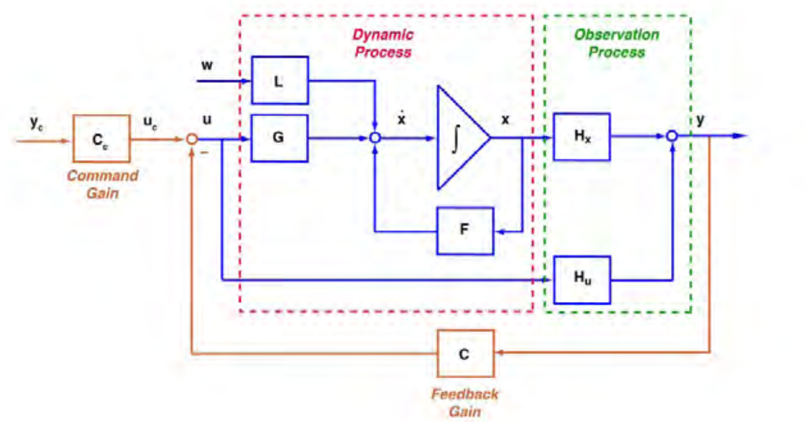
Feedback modifies the <u>stability matrix</u> of the closed-loop system

Convergence or divergence

Envelope of transient response

LTI Model with Feedback Control and Command Gain

Command input, $\mathbf{y}_c$, is "shaped" by $\mathbf{C}_c$



$$\begin{aligned}\mathbf{u}(t) &= \mathbf{u}_c(t) - \mathbf{C}\mathbf{y}(t) \\ &= \mathbf{C}_c\mathbf{y}_c(t) - \mathbf{C}\mathbf{y}(t)\end{aligned}$$

Effect of Command Gain on LTI Model

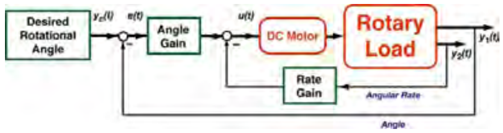
$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{F}\mathbf{x}(t) + \mathbf{G}\mathbf{u}(t) = \mathbf{F}\mathbf{x}(t) + \mathbf{G}\left\{\mathbf{C}_c\mathbf{y}_c(t) - \mathbf{C}\mathbf{y}(t)\right\} \\ &= \mathbf{F}\mathbf{x}(t) + \mathbf{G}\left\{\mathbf{C}_c\mathbf{y}_c(t) - \mathbf{C}\left[\mathbf{H}_x\mathbf{x}(t)\right]\right\} \\ &= \left[\mathbf{F} - \mathbf{G}\mathbf{C}\mathbf{H}_x\right]\mathbf{x}(t) + \mathbf{G}\mathbf{C}_c\mathbf{y}_c(t)\end{aligned}$$

Steady-state response of the system

$$\dot{\mathbf{x}}(t) = \mathbf{0}$$

$$\mathbf{x}^*(t) = -\left[\mathbf{F} - \mathbf{G}\mathbf{C}\mathbf{H}_x\right]^{-1}\mathbf{G}\mathbf{C}_c\mathbf{y}_c^*(t)$$

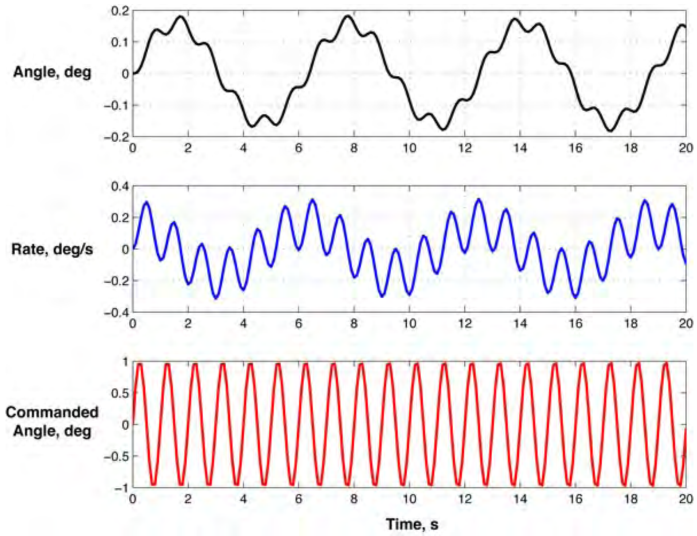
- **Command gain adjusts the steady-state response**
- **Has no effect on the stability of the system**



Response to Sine Wave Input with Angle Feedback: No Damping

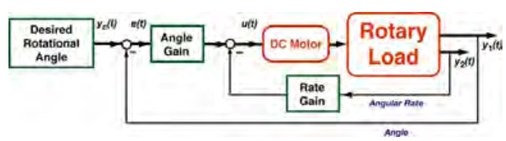
$$y_c(t) = \sin(\omega t) = \sin(6.28\, t), \text{deg}$$

$c_1 / J = 1;\ c_2 / J = 0$



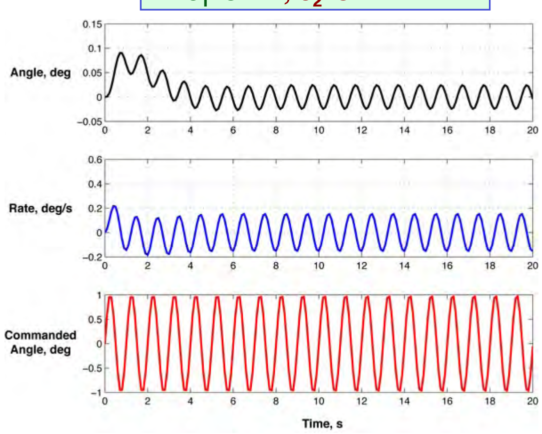
- Why are there 2 oscillations in the output?
  - *Undamped transient response to the input*
  - *Long-term dynamic response to the input*
- System has a <u>natural frequency</u> of oscillation, ω_n
- Long-term response to a sine wave is a sine wave

Response to Sine Wave Input with Rate Damping

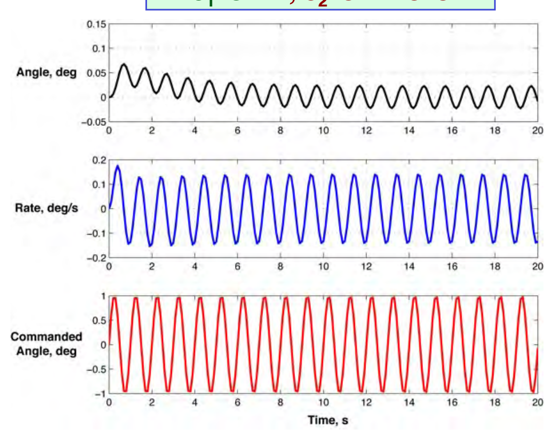


$$y_c(t) = \sin(\omega t) = \sin(6.28\, t), \text{deg}$$

$c_1 / J = 1;\ c_2 / J = 1.414$



$c_1 / J = 1;\ c_2 / J = 2.828$



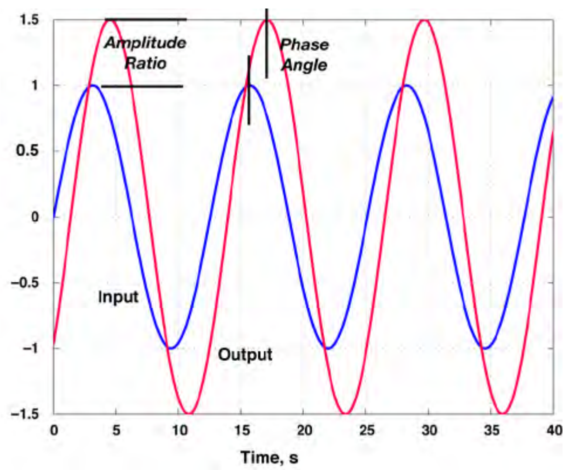
With damping, transient response decays
In this case, damping has negligible effect on long-term response

System Dynamics Produces Differences in Amplitude and Phase Angle of Input and Output

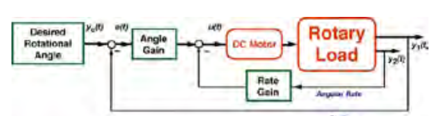
$$\textbf{Amplitude Ratio}\ (AR) = \frac{\left| y_{Output\ Peak} \right|}{\left| y_{Input\ Peak} \right|}$$

$$\textbf{Phase Angle} = 360 \frac{\left(t_{Input\ Peak} - t_{Output\ Peak} \right)}{Period\ of\ Input}, \text{deg}$$

Amplitude ratio and phase angle characterize the system model



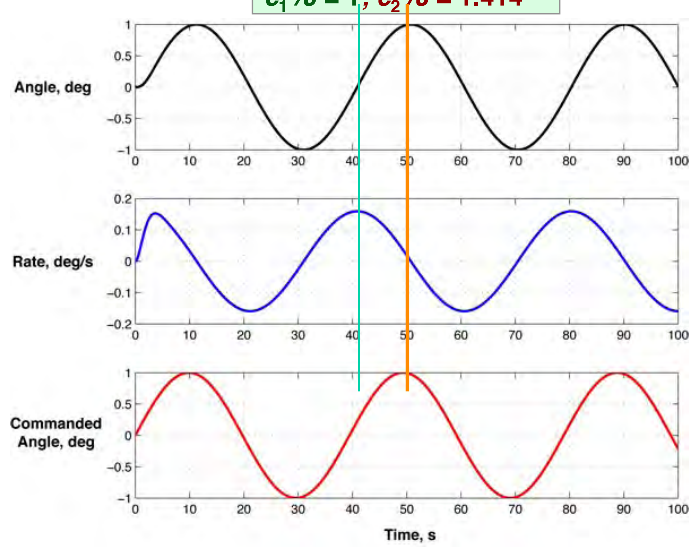
Effect of Input Frequency on Output Amplitude and Phase Angle



$$y_c(t) = \sin\left(\boldsymbol{t / 6.28}\right), \text{deg}$$

$c_1/J = 1$; $c_2/J = 1.414$

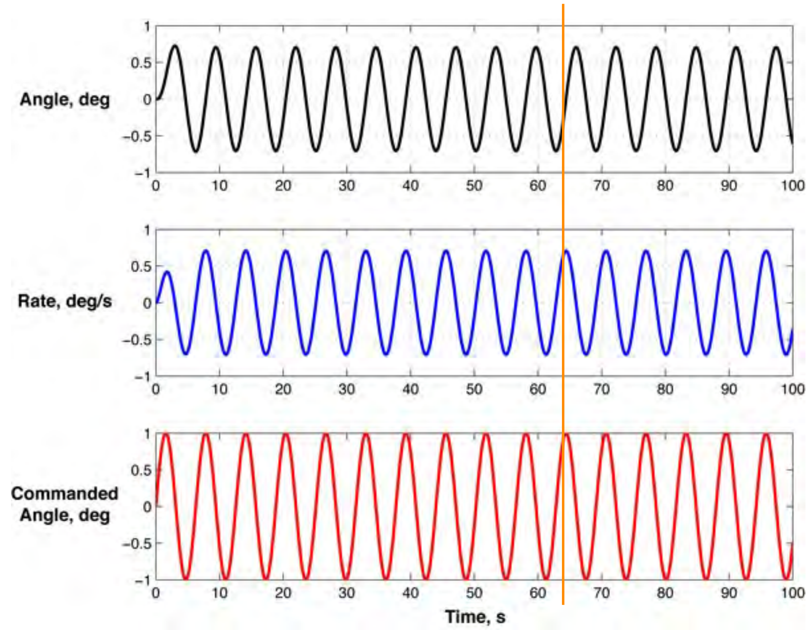
- **With low input frequency, input and output amplitudes are about the same**
- **Output angle oscillation "lags" input by a few degrees**
- **Rate oscillation "leads" angle oscillation by ~90 deg**



At Higher Frequency, Output Amplitude Decreases, Phase Angle Lag Increases

$y_c(t) = \sin(t)$, deg

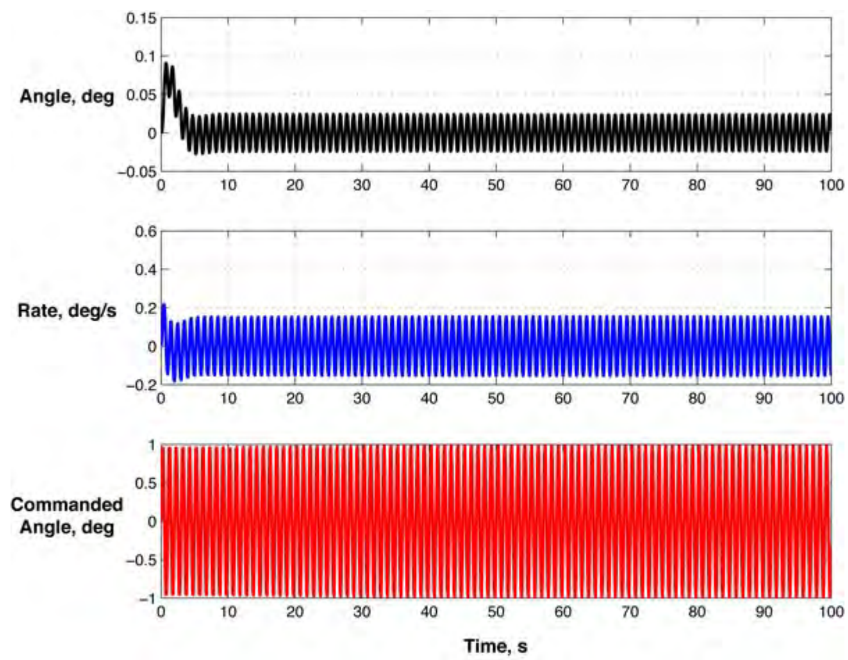
$c_1 / J = 1$; $c_2 / J = 1.414$



At Even Higher Frequency, Amplitude Ratio Decreases

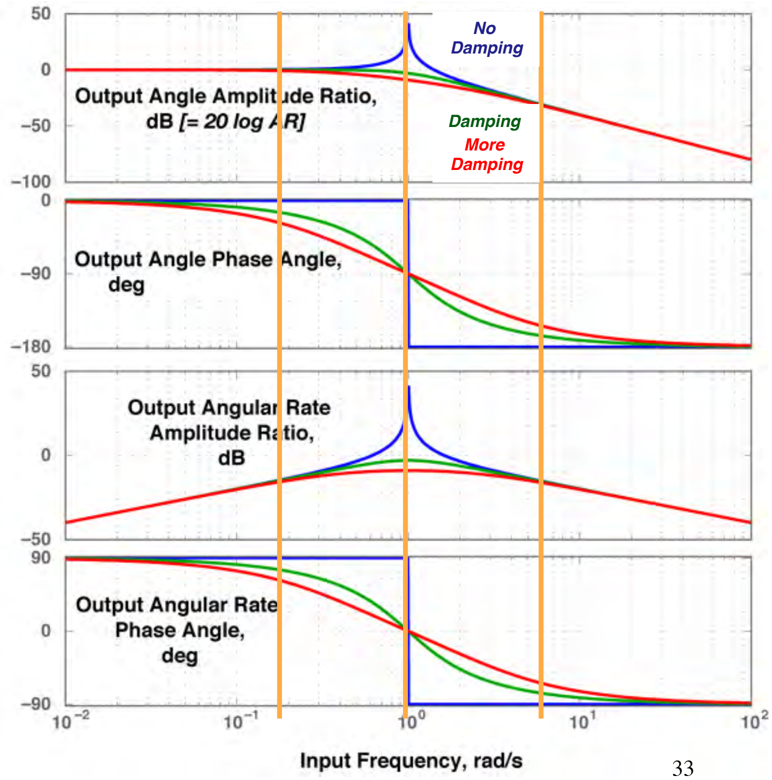
$y_c(t) = \sin(\mathbf{6.28}\, t)$, deg

$c_1 / J = 1$; $c_2 / J = 1.414$



Frequency Response of the DC Motor with Feedback Control

- Long-term response to sinusoidal inputs over range of frequencies
  - Determine experimentally
  - or from the transfer function
- Frequency response depicted by Bode Plot
  - Transfer function
  - Laplace transform of system
  - *TBD*



Next Time: Control Systems