Control Systems

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Robotics and Intelligent Systems MAE 345,
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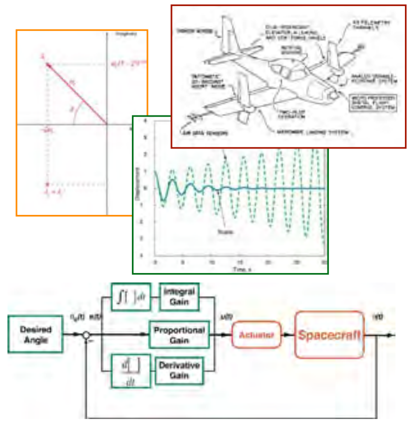
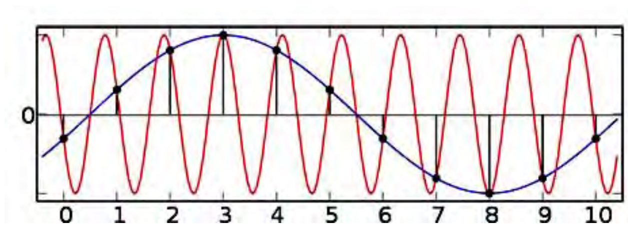
- Analog vs. digital systems
- Continuous- and Discrete-time Dynamic Models
- Frequency Response
- Transfer Functions
- Bode Plots
- Root Locus
- Proportional-Integral-Derivative (PID) Control



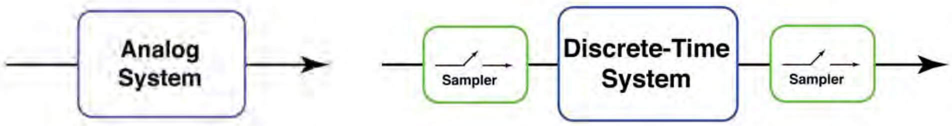
Analog vs. Digital Signals



- <u>Signal</u>: A physical indicator that conveys information, e.g., position, voltage, temperature, or pressure
- <u>Analog signal</u>: A signal that is continuous in time, with <u>infinite precision</u> and <u>infinitesimal time spacing</u> between indication points
- <u>Discrete-time signal</u>: A signal with <u>infinite precision</u> and <u>discrete spacing</u> between indication points
- <u>Digital signal</u>: A signal with <u>finite precision</u> and <u>discrete spacing</u> between indication points

Analog vs. Digital Systems

- **System: Assemblage of parts with structure, connectivity, and behavior that responds to input signals and produces output signals**



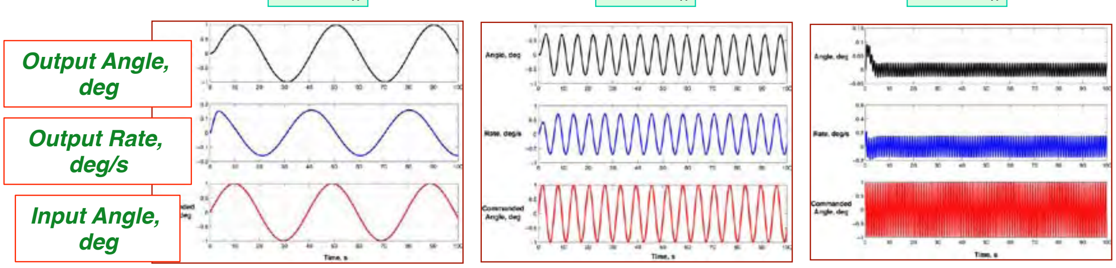
- **Analog system: A system that operates continuously, with infinite precision and infinitesimal time spacing between signaling points**
- **Discrete-time system: A system that operates continuously, with infinite precision and discrete spacing between signaling points**

- **Digital system: A system that operates continuously, with finite precision and discrete spacing between signaling points**

Frequency Response of a Continuous-Time Dynamic System

- **Sinusoidal command input (i.e., Desired rotational angle)**
- **Long-term output of the dynamic system:**
  - **Sinusoid with same frequency as input**
    - **Output/input amplitude ratio dependent on input frequency**
    - **Output/input phase shift dependent on input frequency**
- **Bandwidth: Input frequency below which amplitude and phase angle variations are negligible**

$\omega < \omega_n$ | $\omega = \omega_n$ | $\omega > \omega_n$



ω : Input frequency, rad/s
ω_n : Natural frequency of system, rad/s

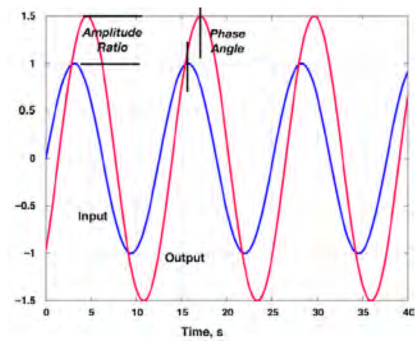
Hz : **Hertz**, frequency, cycles per sec

$$\omega(Hz) = \frac{\omega(rad/s)}{2\pi}$$

Amplitude Ratio and Phase Angle

$$\textbf{Amplitude Ratio}\ (AR) = \frac{\left|y_{Output\ Peak}\right|}{\left|y_{Input\ Peak}\right|}$$

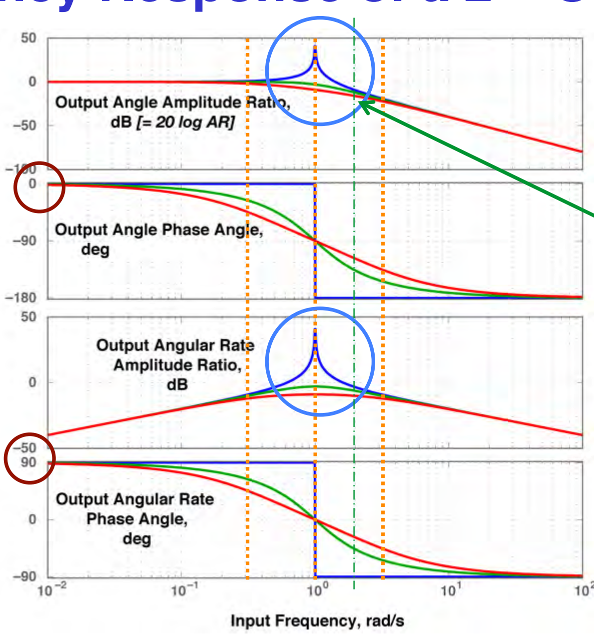
$$\textbf{Phase Angle} = 360\frac{\left(t_{Input\ Peak} - t_{Output\ Peak}\right)}{Period\ of\ Input}, \text{deg}$$



Amplitude-Ratio and Phase-Angle Frequency Response of a 2nd-Order System

Output Angle AR

Output Rate AR



Straight-line asymptotes at high and low frequency

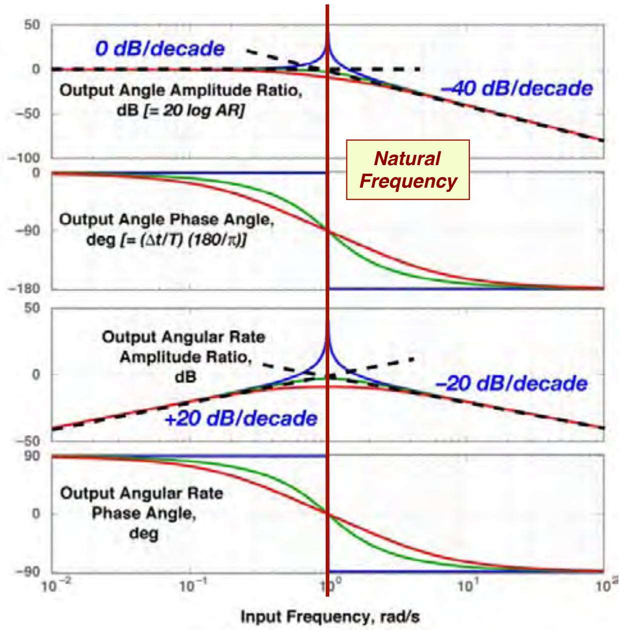
–6 dB Bandwidth, Hz

Straight-line asymptotes at high and low frequency

- Logarithm of transfer function
  - Amplitude ratio and phase angle are orthogonal
  - **Decibels (dB):** 20 log *AR*
  - **Decade:** power of 10

$$20\log AR\ and\ \phi\ vs. \log\omega$$

Asymptotes of 2nd-Order Frequency Response from Bode Plot



$20\ \log_{10}\left(\omega^n\right)$ has slope of $20n$ dB/decade on Bode plot

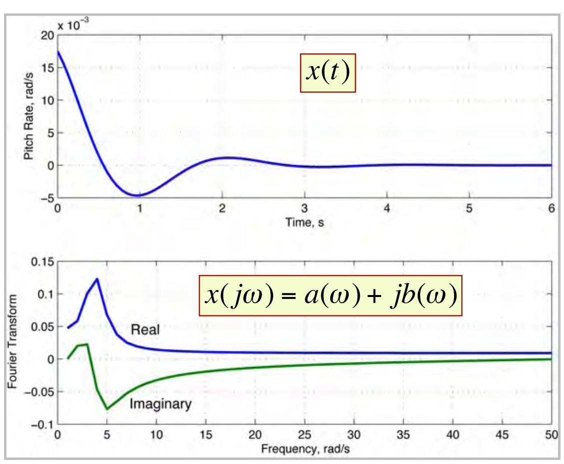
$$
\begin{aligned}
&20\log_{10}\mathcal{H}_{CL_i}(j\omega)\\
&=20\log_{10}\left[AR_i(\omega)\,\mathrm{e}^{j\phi_i(\omega)}\right]\\
&=20\left[\log_{10}AR_i+\log_{10}\mathrm{e}^{j\phi_i(\omega)}\right]\\
&=20\log_{10}AR_i+j\phi_i(\omega)\left(20\log_{10}\mathrm{e}\right)
\end{aligned}
$$

Fourier Transform of a Scalar Variable

$$\mathcal{F}\left[x(t)\right]=x(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$$

$\omega = frequency, rad / s$

$j \triangleq i \triangleq \sqrt{-1}$



$x(t)$: **real** variable

$x(j\omega)$: **complex** variable

$= a(\omega) + jb(\omega)$

$= A(\omega)e^{j\varphi(\omega)}$

A: **amplitude**

φ: **phase angle**

Laplace Transforms of Scalar Variables

- Laplace transform of a scalar variable is a **complex number**
- ***s*** **is the *Laplace operator*, a complex variable**

$$\mathcal{L}\left[x(t)\right] = x(s) = \int_0^{\infty} x(t)e^{-st}\,dt, \quad s = \sigma + j\omega$$

$x(t)$: **real** variable

$x(s)$: **complex** variable
$= a(\omega) + jb(\omega)$
$= A(\omega)e^{j\varphi(\omega)}$

Multiplication by a constant

$$\mathcal{L}\left[a\,x(t)\right] = a\,x(s)$$

Sum of Laplace transforms

$$\mathcal{L}\left[x_1(t) + x_2(t)\right] = x_1(s) + x_2(s)$$

Laplace Transforms of Vectors and Matrices

Laplace transform of a vector variable

$$\mathcal{L}\left[\mathbf{x}(t)\right] = \mathbf{x}(s) = \begin{bmatrix} x_1(s) \\ x_2(s) \\ \dots \end{bmatrix}$$

Laplace transform of a matrix variable

$$\mathcal{L}\left[\mathbf{A}(t)\right] = \mathbf{A}(s) = \begin{bmatrix} a_{11}(s) & a_{12}(s) & \dots \\ a_{21}(s) & a_{22}(s) & \dots \\ \dots & \dots & \dots \end{bmatrix}$$

Laplace transform of a derivative w.r.t. time

$$\mathcal{L}\left[\frac{d\mathbf{x}(t)}{dt}\right] = s\mathbf{x}(s) - \mathbf{x}(0)$$

Laplace transform of an integral over time

$$\mathcal{L}\left[\int \mathbf{x}(\tau)\,d\tau\right] = \frac{\mathbf{x}(s)}{s}$$

Laplace Transforms of the System Equations

Time-Domain System Equations

$$\dot{\mathbf{x}}(t) = \mathbf{F}\,\mathbf{x}(t) + \mathbf{G}\,\mathbf{u}(t)$$ Dynamic Equation

$$\mathbf{y}(t) = \mathbf{H_x}\mathbf{x}(t) + \mathbf{H_u}\mathbf{u}(t)$$ Output Equation

Laplace Transforms of System Equations

$$s\mathbf{x}(s) - \mathbf{x}(0) = \mathbf{F}\,\mathbf{x}(s) + \mathbf{G}\,\mathbf{u}(s)$$ Dynamic Equation

$$\mathbf{y}(s) = \mathbf{H_x}\mathbf{x}(s) + \mathbf{H_u}\mathbf{u}(s)$$ Output Equation

Laplace Transform of State Response to Initial Condition and Control

Rearrange

$$s\mathbf{x}(s) - \mathbf{F}\,\mathbf{x}(s) = \mathbf{x}(0) + \mathbf{G}\,\mathbf{u}(s)$$

$$\left[s\mathbf{I} - \mathbf{F}\right]\mathbf{x}(s) = \mathbf{x}(0) + \mathbf{G}\,\mathbf{u}(s)$$

$$\mathbf{x}(s) = \left[s\mathbf{I} - \mathbf{F}\right]^{-1}\left[\mathbf{x}(0) + \mathbf{G}\,\mathbf{u}(s)\right]$$

Matrix inverse

$$\left[s\mathbf{I} - \mathbf{F}\right]^{-1} = \frac{Adj\left(s\mathbf{I} - \mathbf{F}\right)}{\left|s\mathbf{I} - \mathbf{F}\right|} \quad (n \times n)$$

$Adj\left(s\mathbf{I} - \mathbf{F}\right)$: **Adjoint matrix** $(n \times n)$: **Transpose of matrix of cofactors**

$\left|s\mathbf{I} - \mathbf{F}\right| = \det\left(s\mathbf{I} - \mathbf{F}\right)$: **Determinant** (1×1)

Laplace Transform of System Output

Transfer function relates <u>control input</u> to <u>system output</u>

$$\mathbf{y}(s) = \mathbf{H_x}\left\{\mathbf{x}(s)\right\} + \mathbf{H_u}\mathbf{u}(s)$$
$$= \mathbf{H_x}\left\{(s\mathbf{I} - \mathbf{F})^{-1}\left[\mathbf{G}\,\mathbf{u}(s) + \mathbf{x}(0)\right]\right\} + \mathbf{H_u}\mathbf{u}(s)$$

- **Let:**
  - **$\mathbf{x}(0) = 0$ [initial condition, not control]**
  - **$\mathbf{H_u} = 0$ [control input not contained in the output measurement]**

$$\mathbf{y}(s) = \left[\mathbf{H_x}\left(s\mathbf{I} - \mathbf{F}\right)^{-1}\mathbf{G}\right]\mathbf{u}(s) \triangleq \mathcal{H}(s)\mathbf{u}(s)$$

Transfer Function Matrix, $\mathcal{H}(s)$

[from u(s) to y(s)]

Dimension of $\mathbf{y}(s) = (r \times 1)$
Dimension of $\mathbf{x}(s) = (n \times 1)$
Dimension of $\mathbf{u}(s) = (m \times 1)$

$$\mathcal{H}(s) \triangleq \mathbf{H_x}\left(s\mathbf{I} - \mathbf{F}\right)^{-1}\mathbf{G}$$
$$= \mathbf{H_x}\frac{Adj\left(s\mathbf{I} - \mathbf{F}\right)}{\left(s\mathbf{I} - \mathbf{F}\right)^{-1}}\mathbf{G}$$

Dimension of the Matrix

$$(r \times n)\frac{(n \times n)}{(1 \times 1)}(n \times m) = (r \times m)$$

Characteristic Polynomial of a Dynamic System

Characteristic matrix

$$(s\mathbf{I}-\mathbf{F}) = \begin{pmatrix} (s-f_{11}) & -f_{12} & \dots & -f_{1n} \\ -f_{21} & (s-f_{22}) & \dots & -f_{2n} \\ \dots & \dots & \dots & \dots \\ -f_{n1} & -f_{n2} & \dots & (s-f_{nn}) \end{pmatrix} \quad (n \times n)$$

Characteristic polynomial, $\Delta(s)$

$$\begin{aligned}\Delta(s) &\triangleq |s\mathbf{I}-\mathbf{F}| = \det(s\mathbf{I}-\mathbf{F}) \\ &= s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0\end{aligned}$$

Eigenvalues (or Roots) of the Dynamic System

Characteristic equation

$$\begin{aligned}\Delta(s) &= s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0 = 0 \\ &= (s-\lambda_1)(s-\lambda_2)(\dots)(s-\lambda_n) = 0\end{aligned}$$

λ_i are eigenvalues of $\mathbf{F}$ or roots of the characteristic polynomial, $\Delta(s)$

$\Delta(s)$ is the denominator of the transfer function matrix

Numerator of the Matrix Inverse

Matrix Inverse

$$[s\mathbf{I}-\mathbf{F}]^{-1} = \frac{Adj(s\mathbf{I}-\mathbf{F})}{|s\mathbf{I}-\mathbf{F}|} \quad (n \times n)$$

Adjoint matrix is the transpose of the matrix of cofactors*

$$Adj(s\mathbf{I}-\mathbf{F}) = \mathbf{C}^T \quad (n \times n)$$

2 x 2 example

$$(s\mathbf{I}-\mathbf{F}) = \begin{bmatrix} (s-f_{11}) & -f_{12} \\ -f_{21} & (s-f_{22}) \end{bmatrix}$$

$$Adj(s\mathbf{I}-\mathbf{F}) = \begin{bmatrix} (s-f_{22}) & f_{21} \\ f_{12} & (s-f_{11}) \end{bmatrix}^T = \begin{bmatrix} (s-f_{22}) & f_{12} \\ f_{21} & (s-f_{11}) \end{bmatrix}$$

** Cofactors = signed minor determinants of the matrix*

Example of System Transformation

DC Motor Equations

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} u(t)$$

Dynamic Equation

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Output Equation

Laplace Transforms of Motor Equations

$$\begin{bmatrix} sx_1(s)-x_1(0) \\ sx_2(s)-x_2(0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} u(s)$$

Dynamic Equation

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u(s) = \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix}$$

Output Equation

2 x 2 Eigenvalue Example

Characteristic matrix

$$(s\mathbf{I}-\mathbf{F})=\begin{bmatrix}(s-f_{11}) & -f_{12}\\ -f_{21} & (s-f_{22})\end{bmatrix}$$

Determinant of characteristic matrix

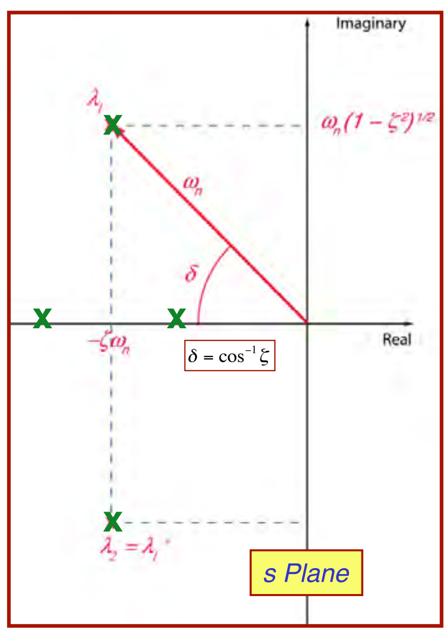
$$|s\mathbf{I}-\mathbf{F}|=\begin{vmatrix}(s-f_{11}) & -f_{12}\\ -f_{21} & (s-f_{22})\end{vmatrix}$$
$$=(s-f_{11})(s-f_{22})-f_{12}f_{21}$$
$$=s^2-(f_{11}+f_{22})s+(f_{11}f_{22}-f_{12}f_{21})$$

Factors of the 2nd-Degree Characteristic Equation

$$\Delta(s)=s^2-(f_{12}+f_{21})s+(f_{11}f_{22}-f_{12}f_{21})$$
$$=0$$

$$\Delta(s)=(s-\lambda_1)(s-\lambda_2)=0$$

- Solutions of the equation are ***eigenvalues*** of the system, either
  - 2 real roots, or
  - Complex-conjugate pair



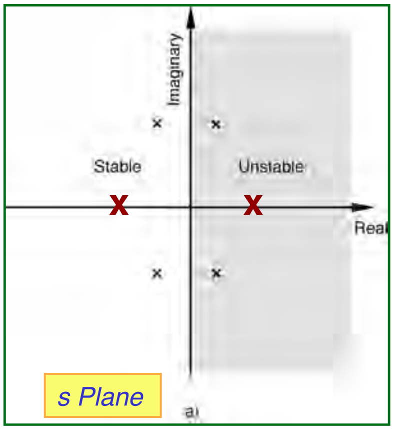
$$\lambda_1=\sigma_1,\quad \lambda_2=\sigma_2$$

$$\lambda_1=\sigma_1+j\omega_1,\quad \lambda_2=\sigma_1-j\omega_1$$

$$\Delta(s)=s^2+2\zeta\omega_n s+\omega_n^2$$

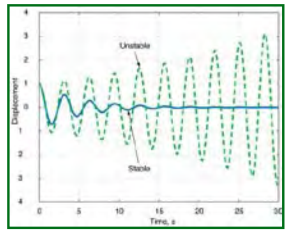
Eigenvalues Determine the Stability of the LTI System

Positive real part represents instability

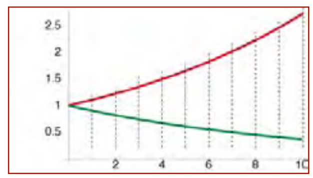


Same criterion for real roots

Envelope of time response converges or diverges



$$\Delta x_1(t) = e^{-\zeta\omega_n t}\cos\left[\omega_n\sqrt{1-\zeta^2}\,t + \varphi_1\right]\Delta x_1(0)$$



$$\left|\Delta x_1(t)\right| = e^{-\zeta\omega_n t}\Delta x_1(0)$$

Open-Loop Transfer Function Matrix for DC Motor

Transfer Function Matrix

$$\mathcal{H}_{OL}(s) = \mathbf{H_x}\left[s\mathbf{I} - \mathbf{F}_{OL}\right]^{-1}\mathbf{G} \quad (r \times m)$$

where

$$\mathbf{F}_{OL} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad \left[s\mathbf{I} - \mathbf{F}_{OL}\right] = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}$$

$$\mathbf{H}_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2; \quad \mathbf{G} = \begin{bmatrix} 0 \\ 1/J \end{bmatrix}$$

Matrix Inverse

$$\left[s\mathbf{I} - \mathbf{F}_{OL}\right]^{-1} = \frac{Adj\left(s\mathbf{I} - \mathbf{F}_{OL}\right)}{\left|s\mathbf{I} - \mathbf{F}_{OL}\right|} = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} = \frac{\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}}{s^2} = \begin{bmatrix} 1/s & 1/s^2 \\ 0 & 1/s \end{bmatrix}$$

Simplify

$$\mathcal{H}_{OL}(s) = \mathbf{H_x}\left[s\mathbf{I} - \mathbf{F}_{OL}\right]^{-1}\mathbf{G} = \mathbf{H_x}\frac{Adj\left(s\mathbf{I} - \mathbf{F}_{OL}\right)}{\left|s\mathbf{I} - \mathbf{F}_{OL}\right|}\mathbf{G}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}}{s^2} \begin{bmatrix} 0 \\ 1/J \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/Js^2 \\ 1/Js \end{bmatrix}$$

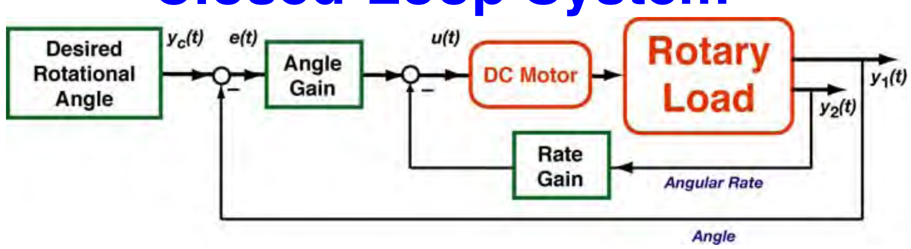
Dimension = 2 x 1

$$\begin{bmatrix} \dfrac{y_1(s)}{u(s)} \\ \dfrac{y_2(s)}{u(s)} \end{bmatrix} \triangleq \begin{bmatrix} \mathcal{H}_1(s) \\ \mathcal{H}_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{Js^2} \\ \dfrac{1}{Js} \end{bmatrix}$$

Angle = Double integral of input torque, u(s)

Angular rate = Integral of input torque, u(s)

Closed-Loop System



Closed-loop dynamic equation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c_1/J & -c_2/J \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ c_1/J \end{bmatrix} y_C$$

$$\triangleq \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} y_C$$

Natural Frequency:

$$\omega_n = \sqrt{c_1/J}$$

Damping Ratio:

$$\zeta = \left(c_2/J\right)/2\omega_n = \left(c_2/J\right)/2\sqrt{c_1/J}$$

Closed-Loop Transfer Function Matrix

$$\mathcal{H}_{CL}(s) = \mathbf{H_x}\left[s\mathbf{I}-\mathbf{F}_{CL}\right]^{-1}\mathbf{G} = \mathbf{H_x}\frac{Adj\left(s\mathbf{I}-\mathbf{F}_{CL}\right)}{\left|s\mathbf{I}-\mathbf{F}_{CL}\right|}\mathbf{G}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} \left(s+c_2/J\right) & 1 \\ -c_1/J & s \end{bmatrix}}{\left(s^2+\frac{c_2}{J}s+\frac{c_1}{J}\right)} \begin{bmatrix} 0 \\ c_1/J \end{bmatrix}$$

Scalar transfer functions differ only in numerators

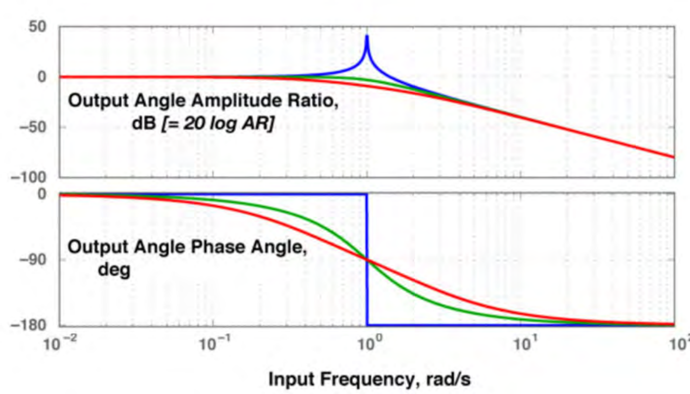
$$\begin{bmatrix} \mathcal{H}_1(s) \\ \mathcal{H}_2(s) \end{bmatrix} = \begin{bmatrix} \frac{y_1(s)}{y_C(s)} \\ \frac{y_2(s)}{y_C(s)} \end{bmatrix} = \frac{\begin{bmatrix} c_1/J \\ s\left(c_1/J\right) \end{bmatrix}}{\left(s^2+\frac{c_2}{J}s+\frac{c_1}{J}\right)} \triangleq \frac{\begin{bmatrix} n_1(s) \\ n_2(s) \end{bmatrix}}{\Delta(s)} \triangleq \frac{\begin{bmatrix} n_1(s) \\ n_2(s) \end{bmatrix}}{d(s)}$$

Angle Frequency Response

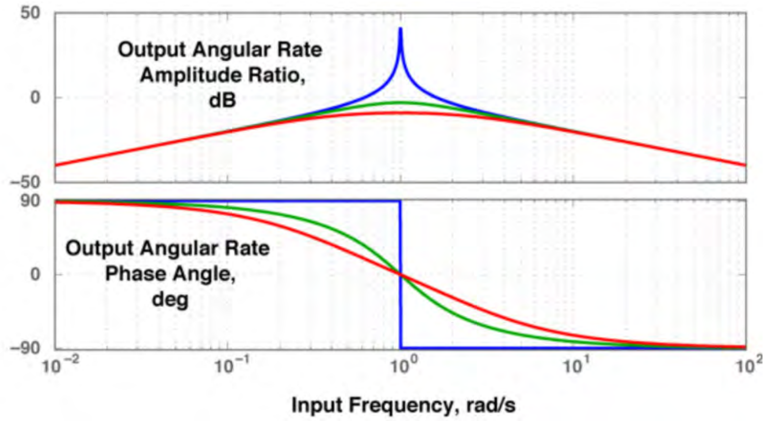
Substitute $s = j\omega$ in transfer functions

$$\mathcal{H}_1(j\omega) = \frac{\textbf{Output Angle}}{\textbf{Commanded Angle}} = \frac{y_1(j\omega)}{y_C(j\omega)} = \frac{c_1/J}{(j\omega)^2+\left(c_2/J\right)(j\omega)+c_1/J}$$

$$= \frac{\omega_n{}^2}{-\omega^2+2\zeta\omega_n(j\omega)+\omega_n{}^2} \triangleq a_1(\omega)+jb_1(\omega) \rightarrow AR_1(\omega)\,e^{j\phi_1(\omega)}$$



Angular Rate Frequency Response



$$\mathcal{H}_2(j\omega) = \frac{\textbf{Output Rate}}{\textbf{Commanded Angle}} = \frac{y_2(j\omega)}{y_C(j\omega)} = \frac{(j\omega)\,c_1/J}{(j\omega)^2 + (c_2/J)(j\omega) + c_1/J}$$

$$= \frac{(j\omega)\omega_n^{\;2}}{-\omega^2 + 2\zeta\omega_n(j\omega) + \omega_n^{\;2}} \rightarrow AR_2(\omega)\, e^{j\phi_2(\omega)}$$

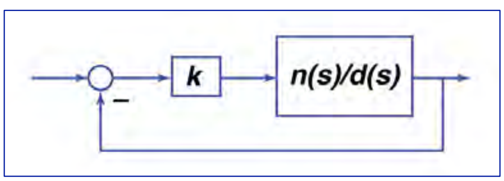
Root (Eigenvalue) Locus

$$\begin{bmatrix} \dfrac{y_1(s)}{y_C(s)} \\ \dfrac{y_2(s)}{y_C(s)} \end{bmatrix} = \frac{\begin{bmatrix} c_1/J \\ (c_1/J)s \end{bmatrix}}{\left(s^2 + \dfrac{c_2}{J}s + \dfrac{c_1}{J}\right)} = \frac{\begin{bmatrix} \omega_n^2 \\ \omega_n^2 s \end{bmatrix}}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} = \frac{\begin{bmatrix} 1 \\ s \end{bmatrix}}{\left(s^2 + 1.414\omega_n s + 1\right)}$$

- **Variation of roots as <u>scalar</u> gain, *k*, goes from 0 to ∞**
- **With nominal gains, c_1 and c_2,**

$$\omega_n = \sqrt{\frac{c_1}{J}} = 1\; rad/s, \text{ Natural frequency}$$

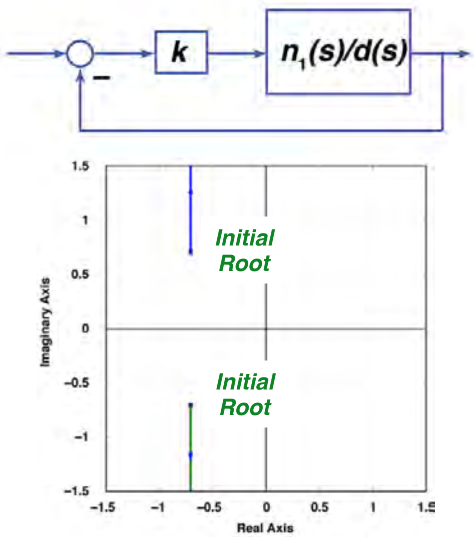
$$\zeta = \frac{c_2/J}{2\omega_n} = 0.707, \quad \text{Damping ratio}$$

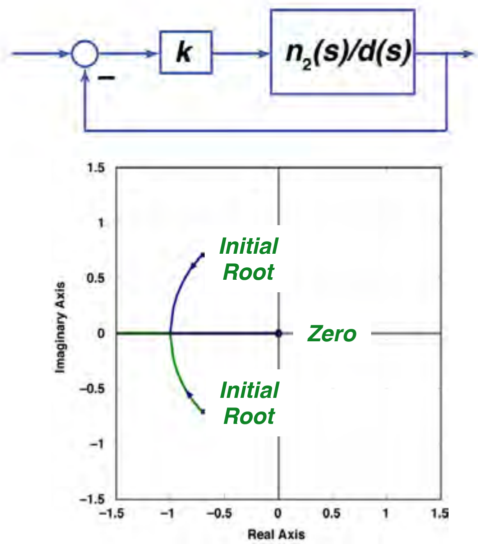



```
%   Root Locus of DC Motor Angle Control

F   =  [0 1;-1 -1.414];
G   =  [0;1];

Hx1 =  [1 0];  %  Angle Output
Hx2 =  [0 1];  %  Angular Rate Output

Sys1  =  ss(F,G,Hx1,0);
Sys2  =  ss(F,G,Hx2,0);

rlocus(Sys1), grid
figure
rlocus(Sys2), grid
```

Root (Eigenvalue) Locus

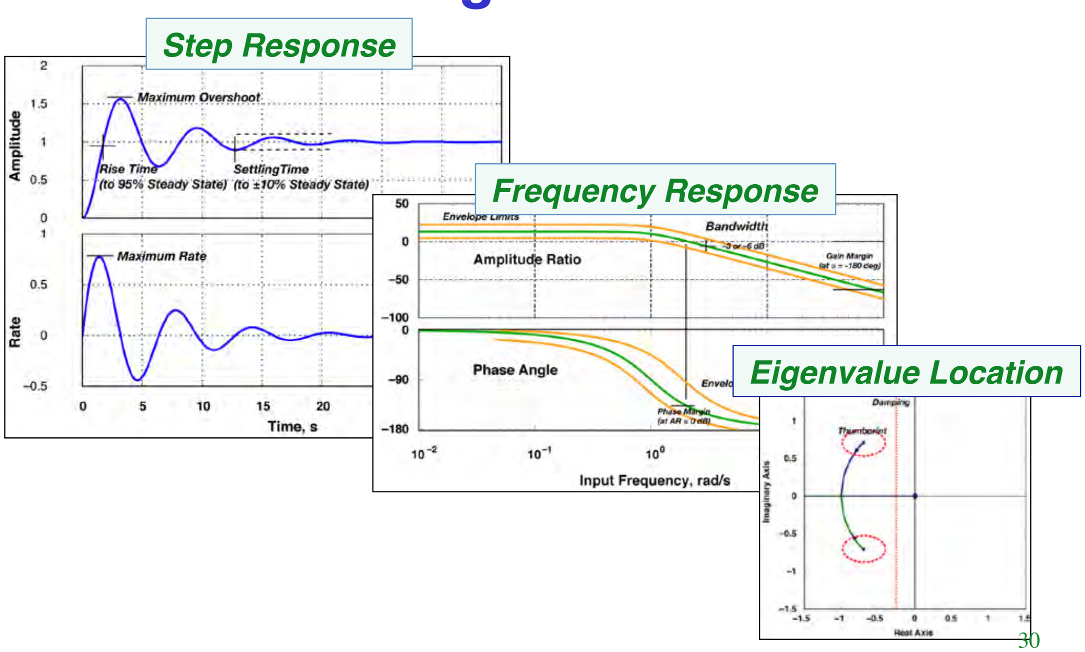
Increase Angle Feedback Gain, *Sys1* **Increase Rate Feedback Gain, *Sys2***

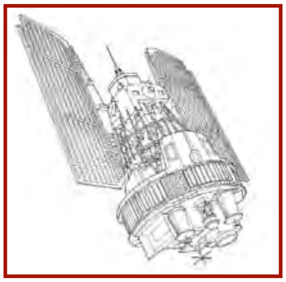




See http://www.princeton.edu/~stengel/MAE331Lecture16.pdf
for rules of root locus construction

Classical Control System Design Criteria





Single-Axis Angular Control of Non-Spinning Spacecraft

Pitching motion (about the *y* axis) is to be controlled

$$\begin{bmatrix} \dot{\theta}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \theta(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/I_{yy} \end{bmatrix} M_y(t)$$

$$\begin{bmatrix} \theta(t) \\ q(t) \end{bmatrix} = \begin{bmatrix} Pitch\ Angle \\ Pitch\ Rate \end{bmatrix}$$

Identical to the DC Motor control problem

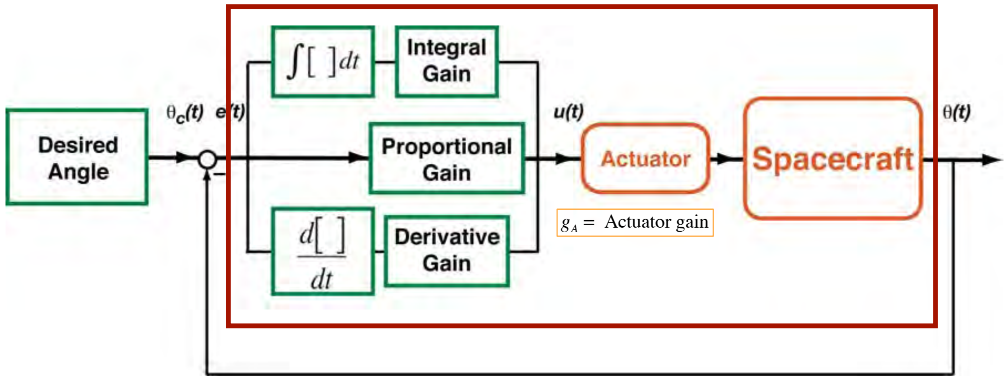
Proportional-Integral-Derivative (PID) Controller

Control Error	$e(s) = \theta_C(s) - \theta(s)$
Control Command	$u(s) = c_P e(s) + c_I \dfrac{e(s)}{s} + c_D s e(s)$
Control Law Transfer Function (w/common denominator)	$\dfrac{u(s)}{e(s)} = \dfrac{c_P s + c_I + c_D s^2}{s}$

- **Proportional term weights control error directly**
- **Integrator compensates for persistent (bias) disturbance**
- **Differentiator produces rate term for damping**

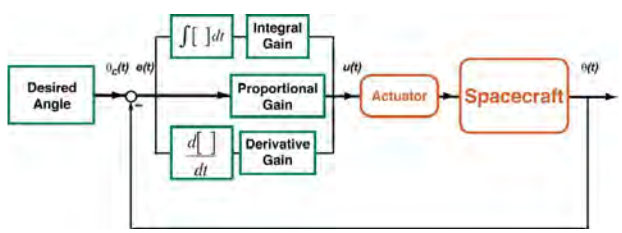
Ziegler-Nichols PID Tuning Method
http://en.wikipedia.org/wiki/Ziegler–Nichols_method

Open-Loop Angle Transfer Function



$$\mathcal{H}_{OL}(s) = \frac{\theta(s)}{e(s)} = \left[\frac{c_I + c_P s + c_D s^2}{s}\right]\left[\frac{g_A}{I_{yy} s^2}\right]$$

Closed-Loop Spacecraft Control Transfer Function w/PID Control



$$\theta(s) = \mathcal{H}_{OL}(s)e(s) = \mathcal{H}_{OL}(s)\left[\theta_c(s) - \theta(s)\right]$$

$$\left[\mathcal{H}_{OL}(s) + 1\right]\theta(s) = \mathcal{H}_{OL}(s)\theta_c(s)$$

$$\mathcal{H}_{CL}(s) = \frac{\theta(s)}{\theta_c(s)} = \frac{\mathcal{H}_{OL}(s)}{1 + \mathcal{H}_{OL}(s)} = \frac{\dfrac{\theta(s)}{e(s)}}{1 + \dfrac{\theta(s)}{e(s)}}$$

Closed-Loop Frequency Response w/PID Control

$$\mathcal{H}_{CL}(s)=\frac{\left[\dfrac{c_I+c_Ps+c_Ds^2}{s}\left(\dfrac{g_A}{I_{yy}s^2}\right)\right]}{1+\left[\dfrac{c_I+c_Ps+c_Ds^2}{s}\left(\dfrac{g_A}{I_{yy}s^2}\right)\right]}=\frac{c_Ds^2+c_Ps+c_I}{I_{yy}s^3/g_A+c_Ds^2+c_Ps+c_I}$$

Let $s = j\omega$. As $\omega \to 0$

$$\mathcal{H}_{CL}(j\omega)=\frac{\theta(j\omega)}{\theta_c(j\omega)}\to\frac{c_I}{c_I}=1$$

Steady-state output = desired steady-state input

As $\omega \to \infty$

$$\mathcal{H}_{CL}(j\omega)=\frac{\theta(j\omega)}{\theta_c(j\omega)}\to\frac{-c_D\omega^2}{-jI_{yy}\omega^3}g_A=\frac{c_D}{jI_{yy}\omega}g_A=-\frac{jc_D}{I_{yy}\omega}g_A$$

$$AR\to\frac{c_D}{I_{yy}\omega}g_A;\quad \varphi\to-90\text{ deg}$$

High-frequency response "rolls off" and lags input

Digital Control

Stengel, *Optimal Control and Estimation*, 1994, pp. 79-84, E-Reserve

From Continuous- to Discrete-Time Systems

- **Continuous-time systems** are described by <u>differential</u> equations, e.g.,

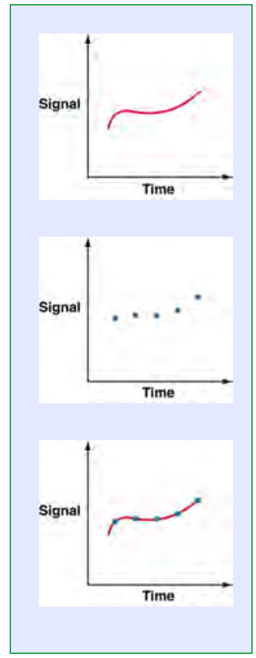
$$\Delta\dot{\mathbf{x}}(t) = \mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t)$$

- **Discrete-time systems** are described by <u>difference</u> equations, e.g.,

$$\Delta\mathbf{x}(t_{k+1}) = \mathbf{\Phi}\Delta\mathbf{x}(t_k) + \mathbf{\Gamma}\Delta\mathbf{u}(t_k)$$

- **Discrete-time systems that are meant to describe continuous-time systems are called sampled-data systems**

$$\mathbf{\Phi} = fcn\left(\mathbf{F}\right); \quad \mathbf{\Gamma} = fcn\left(\mathbf{F},\mathbf{G}\right)$$



Integration of Linear, Time-Invariant (LTI) Systems

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t)$$

$$\Delta\mathbf{x}(t) = \Delta\mathbf{x}(0) + \int_0^t \Delta\dot{\mathbf{x}}(\tau)d\tau = \Delta\mathbf{x}(0) + \int_0^t \left[\mathbf{F}\Delta\mathbf{x}(\tau) + \mathbf{G}\Delta\mathbf{u}(\tau)\right]d\tau$$

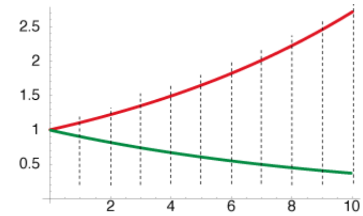
Homogeneous solution (no input), Δu = 0

$$\Delta\mathbf{x}(t) = \Delta\mathbf{x}(0) + \int_0^t \left[\mathbf{F}\Delta\mathbf{x}(\tau)\right]d\tau = e^{\mathbf{F}(t-0)}\Delta\mathbf{x}(0)$$
$$= \mathbf{\Phi}\left(t,0\right)\Delta\mathbf{x}(0)$$

$$\mathbf{\Phi}\left(t,0\right) = e^{\mathbf{F}(t-0)} = e^{\mathbf{F}t} = \textbf{State transition matrix } \textit{from } 0 \textit{ to } t$$

Equivalence of 1st-Order Continuous- and Discrete-Time Systems

Continuous- and discrete-time values are identical at time instants, *k*Δ*t*

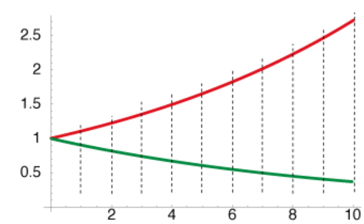


$$\dot{x}(t) = ax(t)\ , x(0) \text{ given}$$

$$x(\Delta t) = x(0) + \int_0^{\Delta t} \dot{x}(t)\,dt = e^{a\Delta t} x(0) \triangleq \phi(\Delta t) x(0)$$

Choose small time interval, Δ*t*
Propagate state to time, *t* + *n*Δ*t*

Equivalence of 1st-Order Continuous- and Discrete-Time Systems



$$x_0 = x(0)$$

$$x_1 = x(\Delta t) = e^{a\Delta t} x_0 = \phi(\Delta t) x_0$$

$$x_2 = x(2\Delta t) = e^{2a\Delta t} x_0 = e^{a\Delta t} x_1 = \phi(\Delta t) x_1$$

$$\ldots\ldots.$$

$$x_k = x(k\Delta t) = e^{ka\Delta t} x_0 = e^{a\Delta t} x_{k-1} = \phi(\Delta t) x_{k-1}$$

$$\ldots\ldots.$$

$$x_n = x(n\Delta t) = x(t) = e^{na\Delta t} x_0 = \phi(\Delta t) x_{n-1}$$

Input Response of LTI System

Inhomogeneous (forced) solution

$$\Delta \mathbf{x}(t) = e^{\mathbf{F}t}\Delta \mathbf{x}(0) + \int_0^t \left[e^{\mathbf{F}(t-\tau)}\mathbf{G}\Delta \mathbf{u}(\tau) \right] d\tau$$

$$= \mathbf{\Phi}(t-0)\Delta \mathbf{x}(0) + \mathbf{\Phi}(t-0)\int_0^t \left[\mathbf{\Phi}(t-\tau)\mathbf{G}\Delta \mathbf{u}(\tau) \right] d\tau$$

Sampled-Data System

(Stepwise application of prior results)

Assume ($t_k - t_{k+1}$) = Δt = constant

$$\mathbf{\Phi}\left(t_{k+1} - t_k\right) = e^{\mathbf{F}(t_{k+1}-t_k)} = e^{\mathbf{F}(\Delta t)} = \mathbf{\Phi}(\Delta t)$$

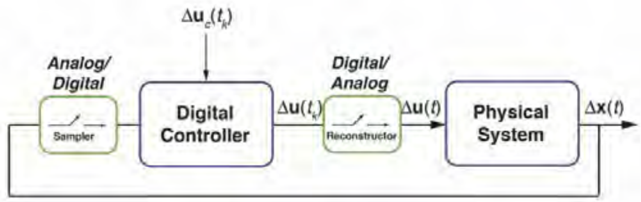
Assume Δu = constant from t_k to t_{k+1}

$$\Delta \mathbf{x}(t_{k+1}) = \mathbf{\Phi}(\Delta t)\Delta \mathbf{x}(t_k) + \mathbf{\Phi}(\Delta t)\int_0^{\Delta t} \left[\mathbf{\Phi}(\Delta t - \tau)\mathbf{G} \right] d\tau \; \Delta \mathbf{u}_k$$

Evaluate the integral

$$\Delta \mathbf{x}(t_{k+1}) = \mathbf{\Phi}(\Delta t)\Delta \mathbf{x}(t_k) + \mathbf{\Phi}(\Delta t)\left[\mathbf{I} - \mathbf{\Phi}^{-1}(\Delta t) \right]\mathbf{F}^{-1}\mathbf{G}\Delta \mathbf{u}(t_k)$$
$$= \mathbf{\Phi}(\Delta t)\Delta \mathbf{x}(t_k) + \mathbf{\Gamma}(\Delta t)\Delta \mathbf{u}(t_k)$$

Digital Feedback Control System



$$\Delta \mathbf{x}(t_{k+1}) = \mathbf{\Phi}(\Delta t)\Delta \mathbf{x}(t_k) + \mathbf{\Gamma}(\Delta t)\Delta \mathbf{u}(t_k)$$

Linear feedback control law with command input

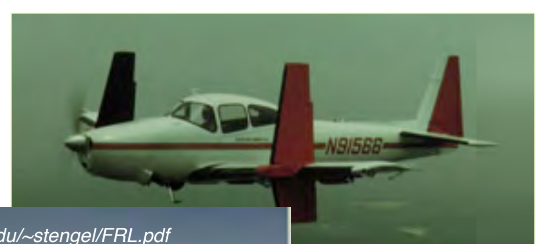
$$\Delta \mathbf{u}(t_k) = \Delta \mathbf{u}_c(t_k) - \mathbf{C}_k \Delta \mathbf{x}(t_k)$$

Closed-loop system

$$\begin{aligned}\Delta \mathbf{x}(t_{k+1}) &= \mathbf{\Phi}(\Delta t)\Delta \mathbf{x}(t_k) + \mathbf{\Gamma}(\Delta t)\left[\Delta \mathbf{u}_c(t_k) - \mathbf{C}_k \Delta \mathbf{x}(t_k)\right] \\ &= \left[\mathbf{\Phi}(\Delta t) - \mathbf{\Gamma}(\Delta t)\mathbf{C}_k\right]\Delta \mathbf{x}(t_k) + \mathbf{\Gamma}(\Delta t)\Delta \mathbf{u}_c(t_k)\end{aligned}$$

Digital Flight Control

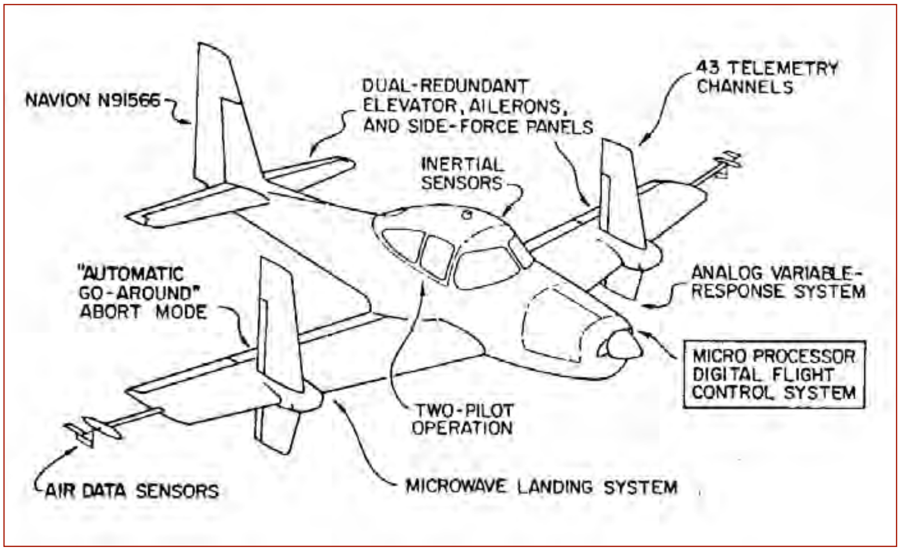
- **Historical notes:**
  - *NASA Fly-By-Wire F-8C (Apollo GNC system, 1972)*
    - 1st conventional aircraft with digital flight control
    - NASA Armstrong (Dryden) Research Center
  - *Princeton's Variable-Response Research Aircraft (Z-80 microprocessor, Spring, 1978)*
    - 2nd conventional aircraft with digital flight control
    - Princeton University's Flight Research Laboratory



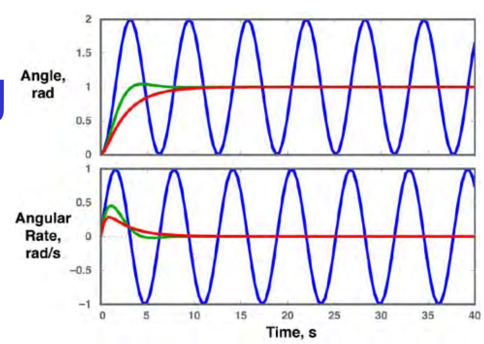
Next Time:
Sensors and Actuators

SUPPLEMENTAL MATERIAL

Princeton's Variable-Response Research Aircraft



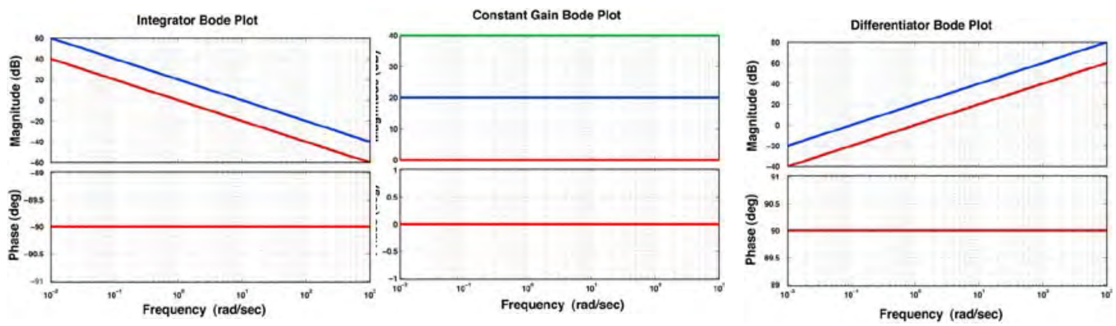
Eigenvalues, Damping Ratio, and Natural Frequency




```
%   Eigenvalues, Damping Ratio,
and Natural Frequency of Angle Control

   F1  =  [0 1;-1 0];
   G1  =   [0;1];

   F1a =  [0 1;-1 -1.414];
   F1b =   [0 1;-1 -2.828];

   Hx  =  [1 0;0 1];

   Sys1   =  ss(F1,G1,Hx,0);
   Sys2   =  ss(F1a,G1,Hx,0);
   Sys3   =  ss(F1b,G1,Hx,0);

   eig(F1)
   eig(F1a)
   eig(F1b)

   damp(Sys1)
   damp(Sys2)
```

Eigenvalues λ_1, λ_2	Damping Ratio ζ	Natural Frequency ω_n
0 + 1i 0 - 1i	0	1
-0.707 + 0.707i -0.707 - 0.707i	0.707	1
-0.414 -2.414	*Overdamped*	

Bode Plots of Proportional, Integral, and Derivative Compensation



$$H(j\omega) = 1$$

$$H(j\omega) = 10$$

$$H(j\omega) = 100$$

- **Amplitude ratio, *AR(dB)* = constant**

Phase Angle, $\phi(\text{deg}) = 0$

$$H(j\omega) = \frac{1}{j\omega}$$

$$H(j\omega) = \frac{10}{j\omega}$$

- ***Slope of AR(dB)* = –20 dB/decade**

Phase Angle, $\phi = -90$ deg

$$H(j\omega) = j\omega$$

$$H(j\omega) = 10\,j\omega$$

- ***Slope of AR(dB)* = +20 dB/decade**

Phase Angle, $\phi = +90$ deg

Equilibrium Response of Linear Discrete-Time Model

- **Dynamic model**

$$\Delta\mathbf{x}_{k+1} = \mathbf{\Phi}\Delta\mathbf{x}_k + \mathbf{\Gamma}\Delta\mathbf{u}_k + \mathbf{\Lambda}\Delta\mathbf{w}_k$$

- **At equilibrium**

$$\Delta\mathbf{x}_{k+1} = \Delta\mathbf{x}_k = \Delta\mathbf{x}^* = \text{constant}$$

- **Equilibrium response**

$$(\mathbf{I} - \mathbf{\Phi})\Delta\mathbf{x}^* = \mathbf{\Gamma}\Delta\mathbf{u}^* + \mathbf{\Lambda}\Delta\mathbf{w}^*$$

$$\Delta\mathbf{x}^* = (\mathbf{I} - \mathbf{\Phi})^{-1}(\mathbf{\Gamma}\Delta\mathbf{u}^* + \mathbf{\Lambda}\Delta\mathbf{w}^*)$$

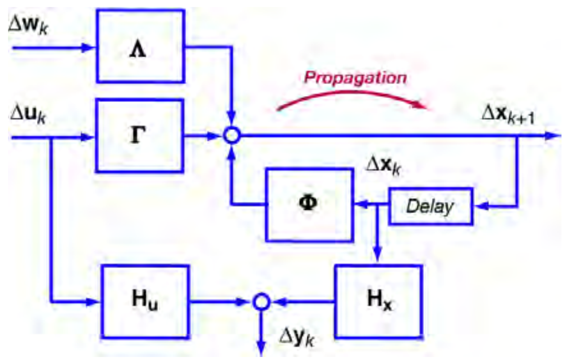
(.) = constant*

Discrete-Time and Sampled-Data Models

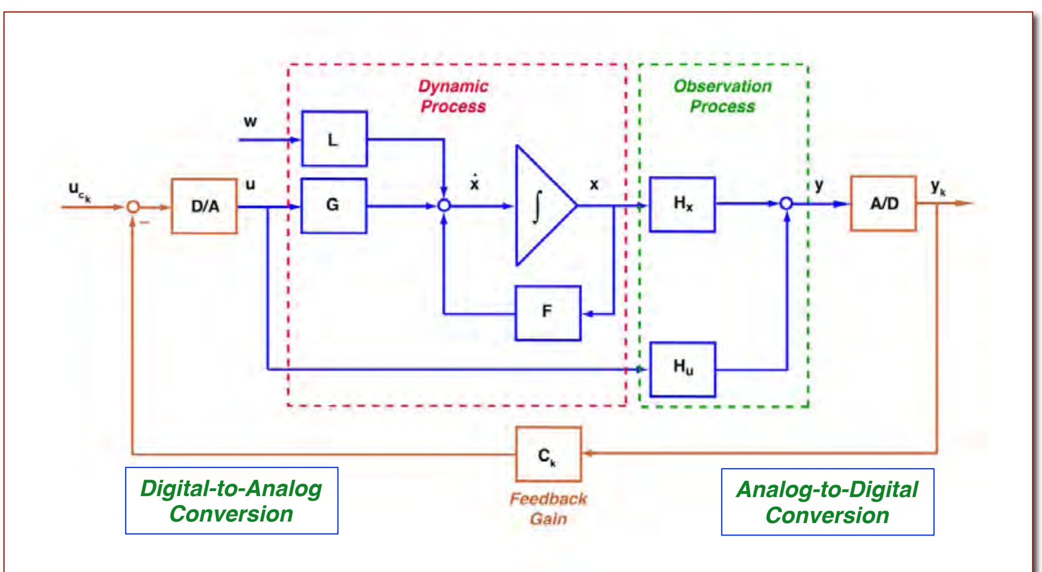
Linear, time-invariant, discrete-time model

$$\Delta \mathbf{x}_{k+1} = \boldsymbol{\Phi} \Delta \mathbf{x}_k + \boldsymbol{\Gamma} \Delta \mathbf{u}_k + \boldsymbol{\Lambda} \Delta \mathbf{w}_k$$

Discrete-time model is a sampled-data model if it represents a continuous-time system

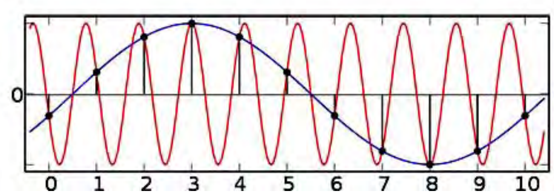


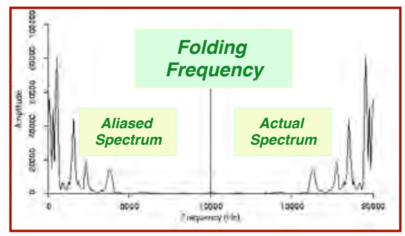
Sampled-Data (*Digital*) Feedback Control



Sampling Effect on Continuous Signal

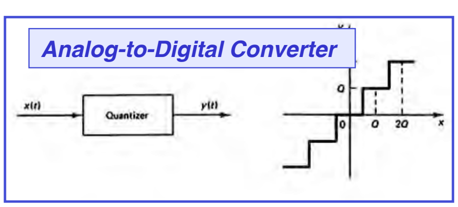
- Two waves, different frequencies, indistinguishable in periodically sampled data
- Frequencies above the sampling frequency *aliased* to appear at lower frequency *(frequency folding)*
- Solutions: Either
  - *Sampling at higher rate* or
  - *Analog low-pass filtering*

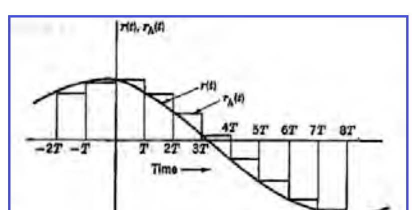




Quantization and Delay Effects

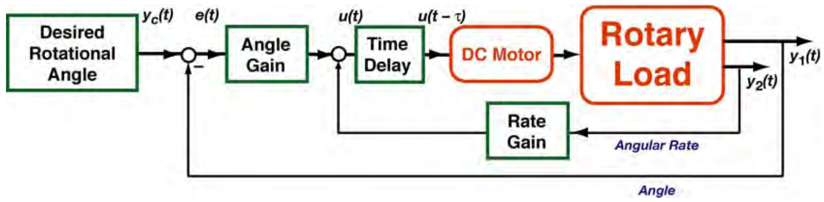
- Continuous signal sampled with finite precision (*quantized*)
  - Solution: *More bits* in A/D and D/A conversion
- *Effective delay* of sampled signal
  - Described as *phase shift* or *lag*
  - Solution:
    - *Higher sampling rate* or
    - *Lead compensation*





r_h illustrates zero-order hold effect

Time Delay Effect



- Control command delayed by τ sec
- Laplace transform of pure time delay

$$\mathcal{L}\left[u(t-\tau)\right] = e^{-\tau s}u(s)$$

- Delay introduces frequency-dependent phase lag with no change in amplitude

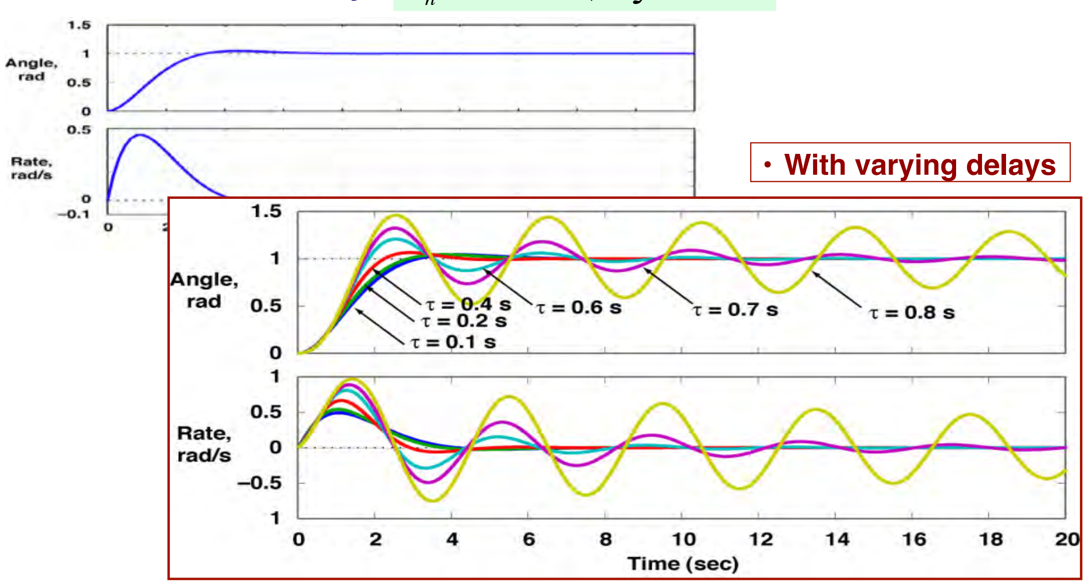
$$AR\left(e^{-j\tau\omega}\right) = 1$$
$$\phi\left(e^{-j\tau\omega}\right) = -\omega$$

- Phase lag reduces closed-loop stability

Effect of Closed-Loop Time Delay on Step Response

- With no delay $\omega_n = 1\,rad/s, \quad \zeta = 0.707$

- With varying delays



Discrete-Time Frequency Response can be Evaluated Using the z Transform

$$\Delta \mathbf{x}_{k+1} = \mathbf{\Phi}\,\Delta \mathbf{x}_k + \mathbf{\Gamma}\,\Delta \mathbf{u}_k$$

z transform is the Laplace transform of a periodic sequence

System *z* transform is

$$z\Delta \mathbf{x}(z) - \Delta \mathbf{x}(0) = \mathbf{\Phi}\,\Delta \mathbf{x}(z) + \mathbf{\Gamma}\Delta\,\mathbf{u}(z)$$

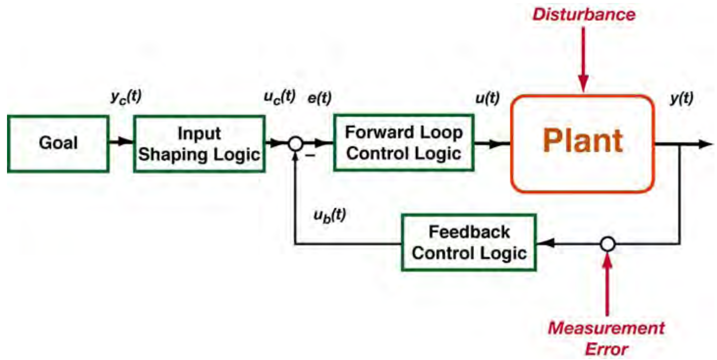
which leads to

$$\Delta \mathbf{x}(z) = \left(z\mathbf{I} - \mathbf{\Phi}\right)^{-1}\left[\Delta \mathbf{x}(0) + \mathbf{\Gamma}\Delta\,\mathbf{u}(z)\right]$$

Further details are beyond the present scope

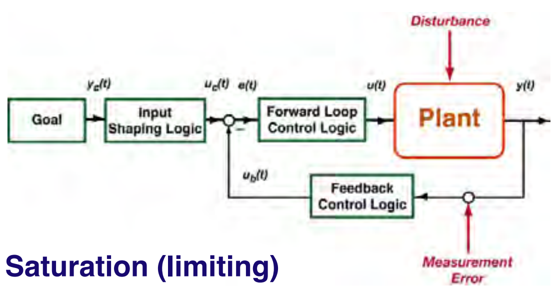
Factors That Complicate Precise Control

Error, Uncertainty, and Incompleteness May Have Significant Effects

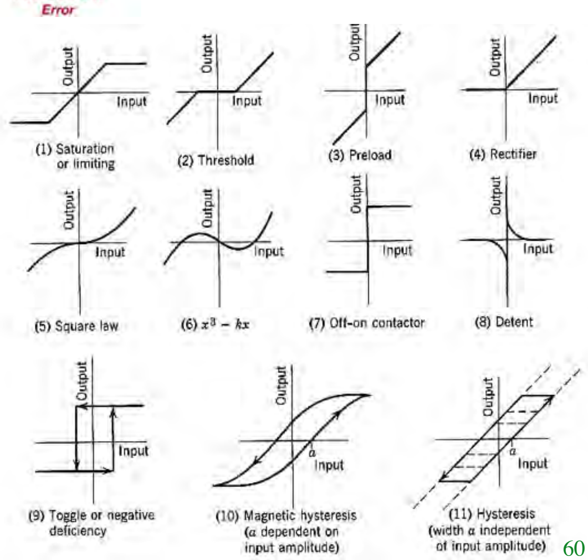


- **Scale factors and biases:** $y = kx + b$
- **Disturbances:** Constant/variable forces
- **Measurement error:** Noise, bias
- **Modeling error**
  - Parameters: e.g., Inertia
  - Structure: e.g., Higher-order modes

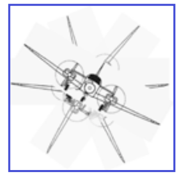
Nonlinearity Complicates Response



- **Saturation (limiting)**
  - Displacement limit, maximum force or rate
- **Threshold**
  - Dead zone, slop
- **Preload**
  - Breakout force
- **Friction**
  - Sliding surface resistance
- **Rectifier**
  - Hard constraint, absolute value (double)
- **Hysteresis**
  - Backlash, slop
- **Higher-degree terms**
  - Cubic spring, separation



Second-Order Example: Airplane Roll Motion



$$\begin{bmatrix} \Delta p \\ \Delta \phi \end{bmatrix} = \begin{bmatrix} \text{Roll rate, rad/s} \\ \text{Roll angle, rad} \end{bmatrix}$$

$\Delta\delta A = \text{Aileron deflection, rad}$

L_p : *roll damping coefficient*

$L_{\delta A}$: *aileron control effect*

Rolling motion of an airplane, continuous-time

$$\begin{bmatrix} \Delta \dot{p} \\ \Delta \dot{\phi} \end{bmatrix} = \begin{bmatrix} L_p & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta \phi \end{bmatrix} + \begin{bmatrix} L_{\delta A} \\ 0 \end{bmatrix} \Delta\delta A$$

Rolling motion of an airplane, discrete-time

$$\begin{bmatrix} \Delta p_k \\ \Delta \phi_k \end{bmatrix} = \begin{bmatrix} e^{L_p T} & 0 \\ \dfrac{\left(e^{L_p T} - 1\right)}{L_p} & 1 \end{bmatrix} \begin{bmatrix} \Delta p_{k-1} \\ \Delta \phi_{k-1} \end{bmatrix} + \begin{bmatrix} \sim L_{\delta A} T \\ 0 \end{bmatrix} \Delta\delta A_{k-1}$$

T = sampling interval, s