Credibility of Confidence Sets in Nonstandard Econometric Problems

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Motivation

- Confidence sets are most common way of describing parameter uncertainty in econometrics

- Definition of confidence sets does not ensure that they do so in a "reasonable" fashion:

  – Observe $X \sim \mathcal{N}(\theta, 1)$, $\theta > 0$

  – Use CS$= [X - 1.96, X + 1.96] \cap (0, \infty)$ (inversion of UMPU tests)

  – Empty with positive probability

  – "Too short" if $X \approx -1.95$

Obviously "Unreasonable" Non-Empty CS

- Observe $X = (Y, S)$, where

$$\begin{aligned} P(S &= 1) = P(S = 5) = 1/2 \\ Y|S &\sim \mathcal{N}(\theta, S^2) \end{aligned}$$

- Standard 95% CS_0 is $[Y \pm 1.96S]$. But

$$\text{CS}(X) = \begin{cases} [Y \pm 2.58S] & \text{if } S = 1 \\ [Y \pm 1.70S] & \text{if } S = 5 \end{cases}$$

  has shorter expected length

- CS is "unreasonably" short whenever $S = 5$ (failure to condition on the ancillary statistic S?)

Formalization of "Reasonable" CS

- After realization $X = x$, inspector can choose to bet against the event that $\theta \in$CS(x), winning 1 if right, and losing $\alpha/(1-\alpha)$ if wrong

- Can inspector generate positive average winnings from repeated draws, for all possible values of true parameter?
  - Surely yes if CS is empty with positive probability
  - Also if CS overstates the accuracy, as in conditional normal example

- Central idea of this paper: Use this betting scheme do study reasonableness of confidence sets in non-standard problems of recent interest
  1. Magnitude of maximal expected winnings is quantitative measure of "unreasonableness" of previously suggested CS
  2. Construction of attractive "reasonable" (=bet-proof) CS

Literature

- Previous results on betting and conditioning
  - Relevant/recognizable subsets and betting: Fisher (1956), Buehler (1959), Wallace (1959), Cornfield (1969), Pierce (1973), Robinson (1977), ...
  - Condition on ancillary statistics: Fisher (1956), Basu (1959), Basu (1964), ...
- Interesting to revisit these ideas, as issues are potentially important in nonstandard problems of applied interest
- We make (analytical and numerical) progress in construction of bet-proof CS

Remainder of Talk

1. Quantification of unreasonableness of popular CS in nonstandard problems

   (a) AR(1) coefficient near unity; (b) weak instruments

2. Characterization of bet-proof sets

3. Construction of attractive bet-proof sets: Theory

4. Construction of attractive bet-proof sets: Applications

   (a) AR(1) coefficient near unity; (b) weak instruments

5. Conclusion

Quantifying Unreasonableness

- For any weighting function π on the parameter space, one can characterize the betting strategy that maximizes weighted average expected winnings, subject to non-negative expected winnings for each θ

- Also investigate magnitude of maximal expected winnings under less favorable payoffs: inspector still wins 1 if right, but now loses $\alpha'/(1-\alpha')$ for wrong objections, where $\alpha' > 0.05$

- Upper bound on expected winnings for all problems and $1-\alpha$ CS is $\alpha = 0.05$

AR(1) Coefficient Near Unity

- Consider AR(1)

$$\begin{aligned} y_t &= \mu + u_t \\ u_t &= \rho u_{t-1} + \varepsilon_t \end{aligned}$$

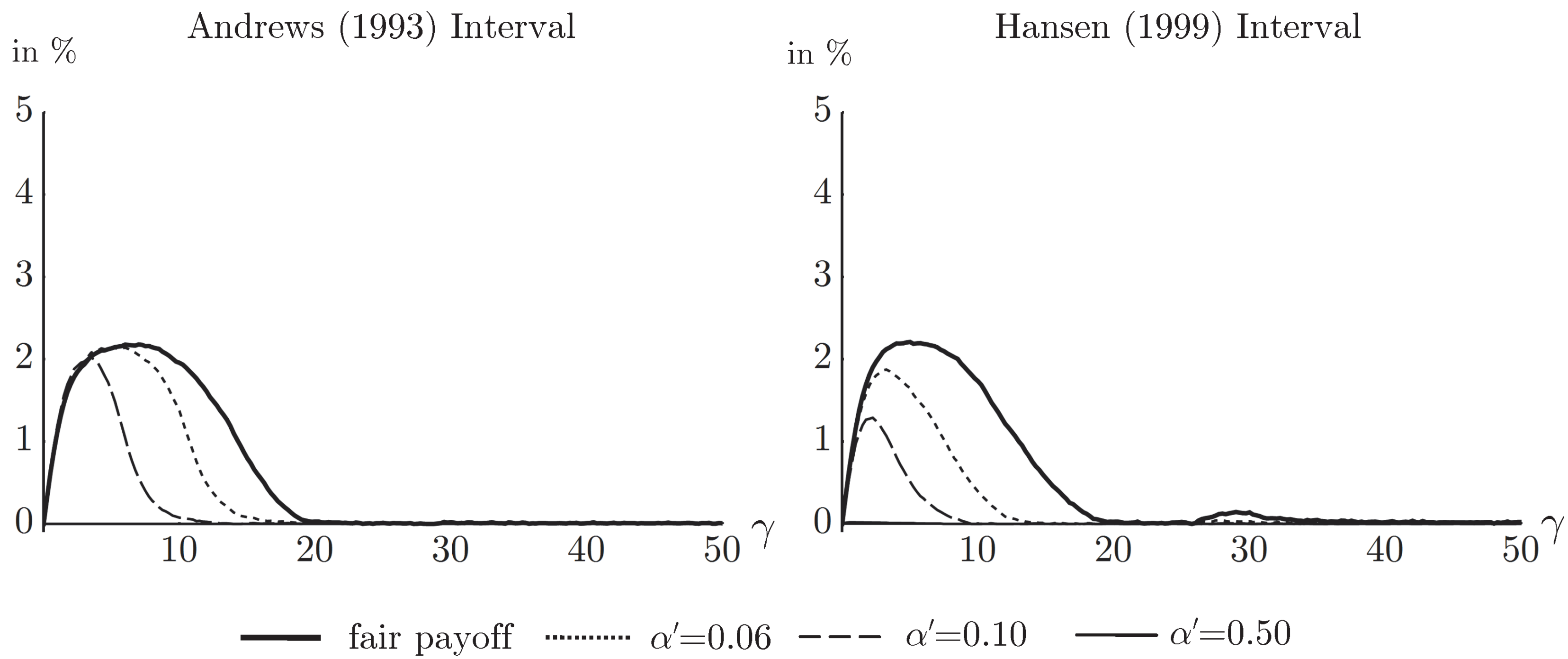
where $\varepsilon_t \sim iid\mathcal{N}(0, \sigma^2)$ and $u_0 \sim \mathcal{N}(0, \sigma^2/(1-\rho^2))$

- Analyze inference under local-to-unity asymptotics $\rho = \rho_T = 1 - \gamma/T$

- Two 95% CS for γ by inverting equal-tailed tests based on
  - OLS estimator $\hat{\rho}$ (Andrews (1993))
  - OLS t-statistic based on $\hat{\rho}$ (Hansen (1999))

- Specify $\pi(\gamma) \propto (100+\gamma)^{-1.1}\mathbf{1}[\gamma \geq 0]$

AR(1) Coefficient Near Unity



Weak Instruments

- In Chamberlain's (2005) reparamterization, weak instrument inference with one endogenous variable and one instrument is identical to inference about $\tan(\phi)$ when observing

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \rho \sin\phi \\ \rho \cos\phi \end{pmatrix}, I_2\right)$$

  where $\phi \in [0, 2\pi]$ and $\rho \geq 0$.

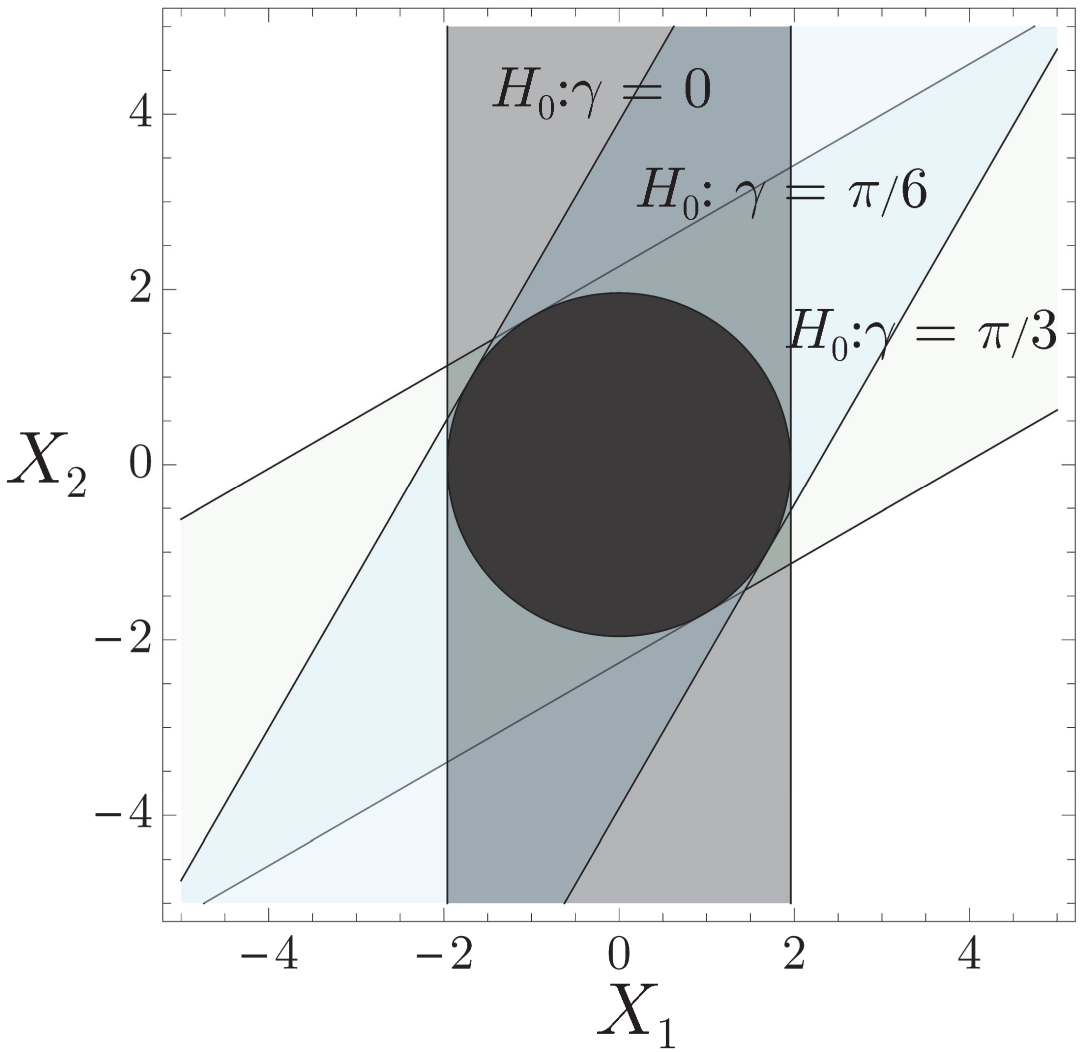
- $\tan(\phi)$ is a one-to-one function of $\gamma = (\phi \bmod \pi) \in [0, \pi]$

- 5% level Anderson-Rubin test of $H_0 : \gamma = \gamma_0$ rejects if $|X_1 \cos\gamma_0 - X_2 \sin\gamma_0| > 1.96$

  $\Rightarrow$ Inverting the AR test yields a similar 95% CS

  $\Rightarrow$ If $||X|| < 1.96$, then AR-CS$(X) = [0, \pi]$ (no value is excluded)

Acceptance Regions



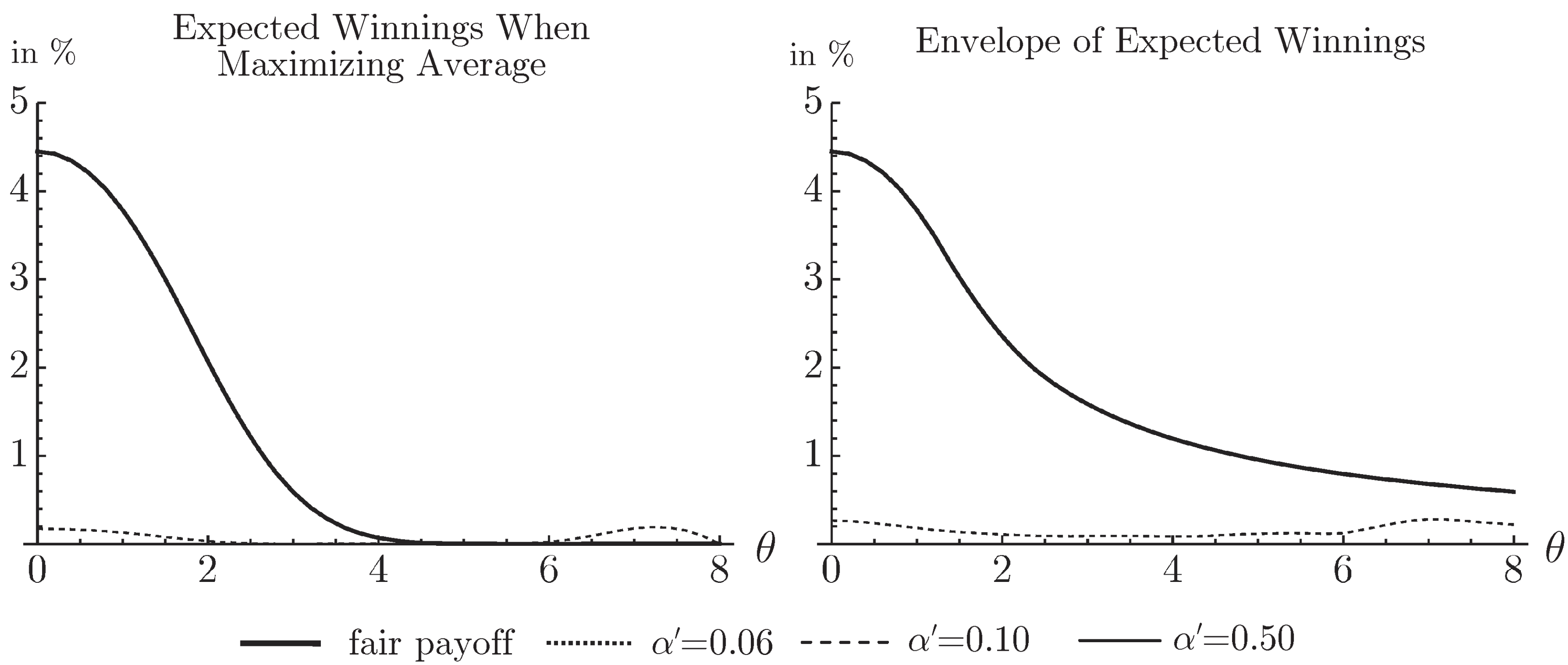
General Result

- If a *similar* CS is equal to "all parameter values" with positive probability, then betting whenever "CS$\neq$all values" yields uniformly positive expected winnings

- Intuition:
  - Probability of coverage conditional on "CS$=$all values" is unity
  - Similarity forces that average probability of coverage is equal to $1 - \alpha$

$\Rightarrow$ Similar CS must undercover conditional on "CS$\neq$all values"

Weak Instruments



Characterization of Bet-Proof CS

- Previously known Theorem: Under weak technical conditions, a level $(1-\alpha)$ CS is bet-proof (there is no betting strategy with positive average winnings uniformly over the parameter space) if and only if CS is a superset of a $(1-\alpha)$ credible set for some prior.

$\Rightarrow$ Asymptotic equivalence of frequentist CS and Bayesian HPDs in standard parametric models implies that frequentist CS are nearly bet-proof

Construction of Bet-Proof Sets

- By previous result: any bet-proof set must be a superset of a credible set relative to some prior

- One approach: Can one find a prior that turns a specific level $1-\alpha$ credible set (such as, say, HPD) into a confidence set?

  - Yes, see Müller and Norets (2015, forthcoming JASA)

  - But might be conservative in presence of nuisance parameters

- Alternative pursued here: Guaruantee bet-proofness by always including (some specific) credible set relative to exogenous prior

Construction of Bet-Proof Sets, Ctd

- Given the exogenous prior π^0, construct credible set $S_0(x)$ from data x

- Enlarge S_0 to ensure coverage by a minimal amount in a weighted average length sense

$$\min_S \int E_\theta[\mathrm{vol}(S(X))]d\pi^0(\theta)$$
$$\text{s.t. } P_\theta(\theta \in S(X)) \geq 1 - \alpha$$
$$\text{and } \; S(x) \supset S_0(x) \; \text{ for all } x$$

$\Rightarrow S$ inverts weighted average power maxiziming tests (Pratt 1961), subject to not rejecting if $\theta_0 \in S_0(x)$

- Paper contains details on extension to invariant problems, and problems with nuisance parameters

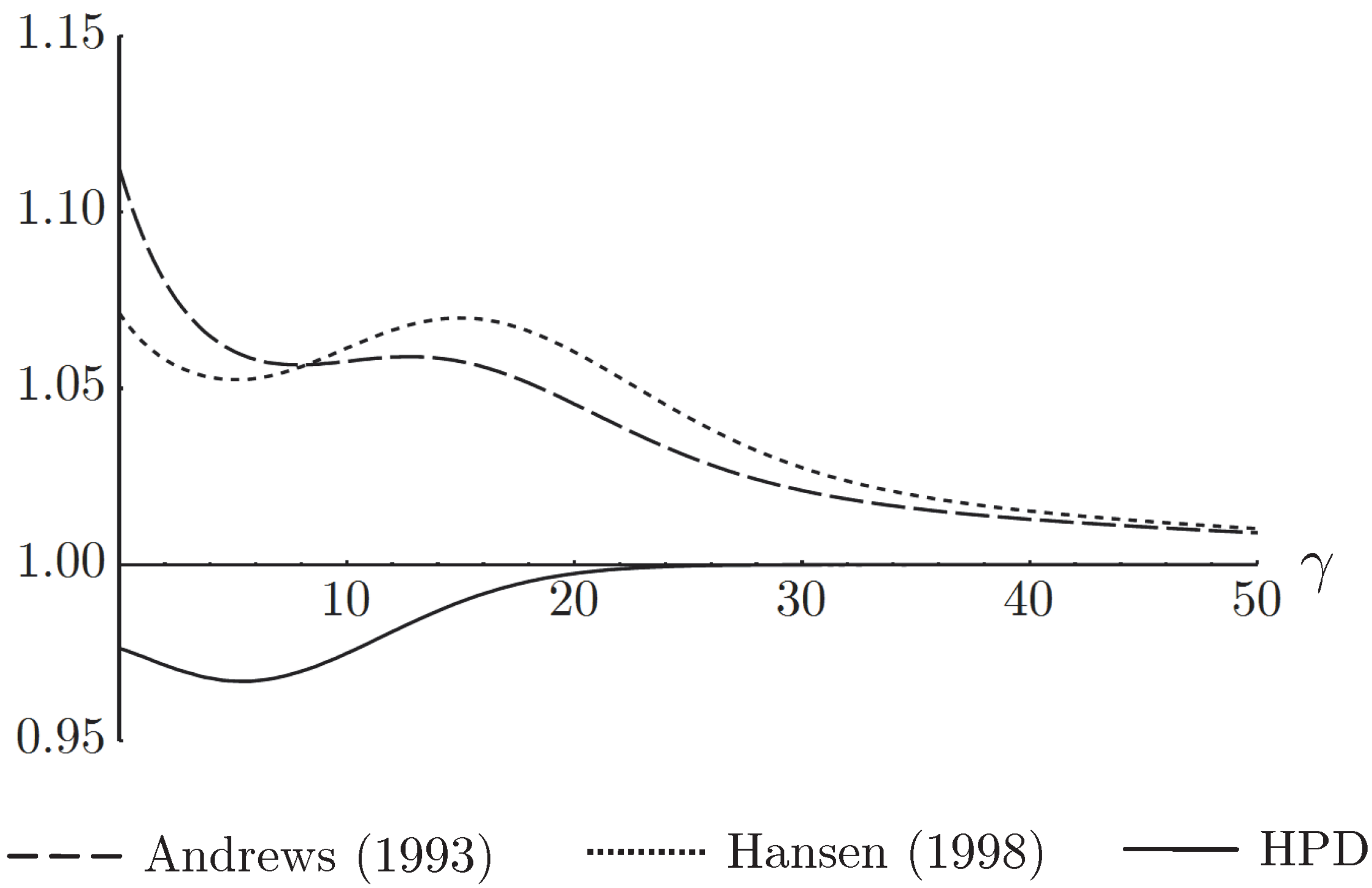
AR(1) Coefficient Near Unity

- Impose tranlation invariance

- Use same weighting function/prior as before:

$$\pi^0(\gamma) \propto (100 + \gamma)^{-1.1} \mathbf{1}[\gamma \geq 0]$$

AR(1) Expected Lengths Relative to S

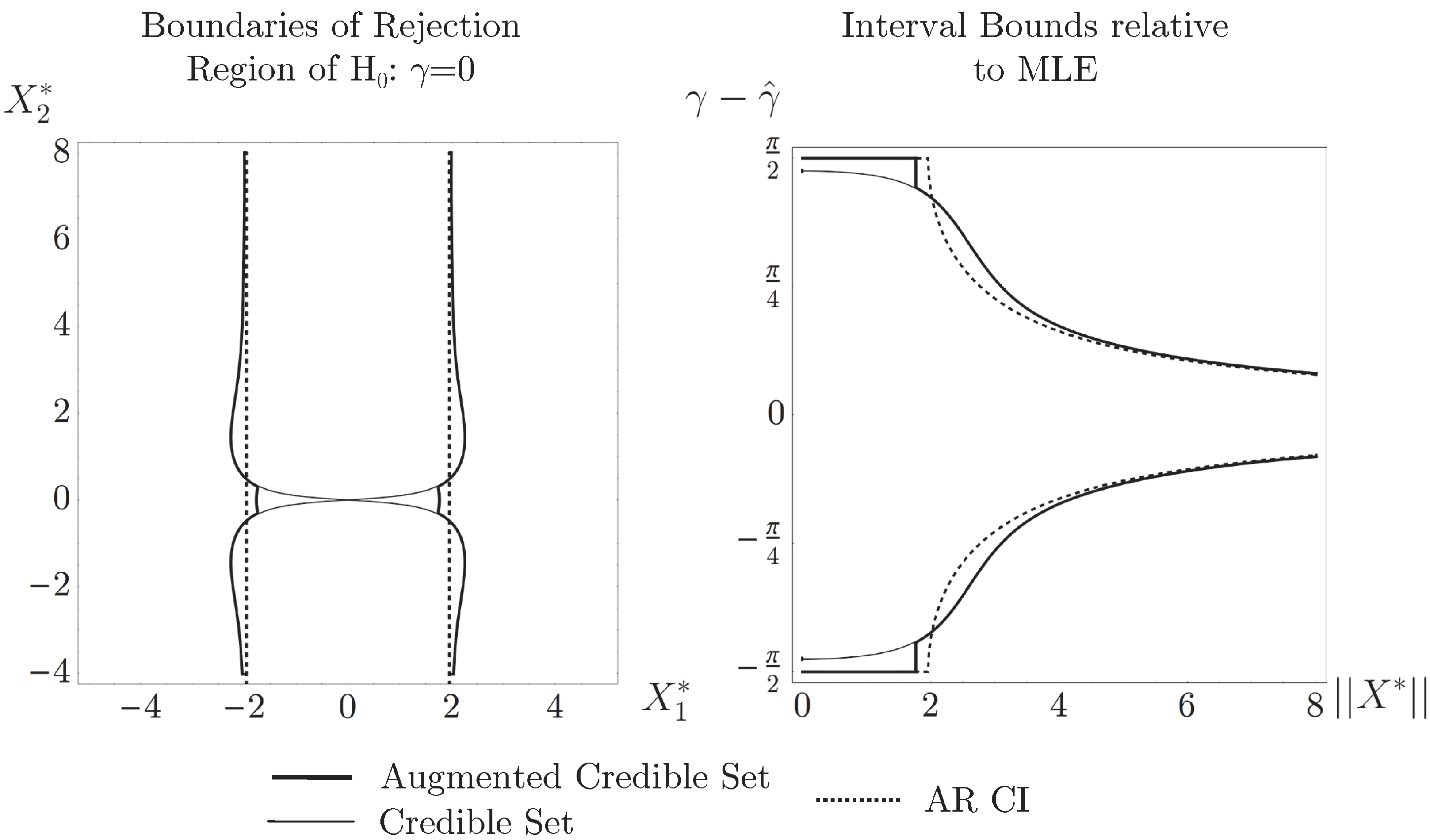


Weak Instruments

- Impose rotational invariance

- Specify prior $\pi^0(\rho) \propto (100 + \rho)^{-1.1}\mathbf{1}[\rho \geq 0]$

Weak Instruments



Concluding Remarks

- Alternative perspective on confidence sets in non-standard problems

- Also applicable to construction of predictive sets in forecasting problem under parameter uncertainty