Econometrica Supplementary Material

SUPPLEMENT TO "CREDIBILITY OF CONFIDENCE SETS IN NONSTANDARD ECONOMETRIC PROBLEMS": ADDITIONAL APPENDICES

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This is supplementary material for Müller and Norets (2016). Section S1 presents Chamberlain's (2007) reparameterization of the weak instrument problem. Section S2 contains implementation details. Section S3 includes additional figures.

S1. CHAMBERLAIN'S (2007) REPARAMETERIZATION OF THE WEAK INSTRUMENT PROBLEM

THE STRUCTURAL AND REDUCED form equations are

$$y_{1,t} = y_{2,t}\beta + u_{t,1},$$
$$y_{2,t} = z_t\gamma + v_{t,2},$$

with β the parameter of interest, and the reduced form for $y_{1,t}$ is given by

$$y_{1,t} = z_t\gamma\beta + v_{t,1}.$$

For nonstochastic z_t and $v_t = (v_{1,t}, v_{2,t})' \sim$ i.i.d. $\mathcal{N}(0, \Omega)$ with Ω known, by sufficiency, the relevant data are effectively two-dimensional,

$$W = \sum_{t=1}^{T} \begin{pmatrix} z_t y_{1,t} \\ z_t y_{2,t} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} S_z\gamma\beta \\ S_z\gamma \end{pmatrix}, \Omega S_z\right), \quad S_z = \sum_{t=1}^{T} z_t^2.$$

The reparameterization is $X^* = S_z^{-1/2}\Omega^{-1/2}W$ and $S_z^{1/2}\Omega^{-1/2}(\gamma\beta, \gamma)' = \rho(\sin\phi, \cos\phi)'$. Inference about β based on W, with Ω and S_z known and γ a nuisance parameter, is then transformed into inference about $\mathrm{mod}(\phi, \pi)$ in (17) in Müller and Norets (2016). For $\gamma \neq 0$ (or, equivalently, $\rho \neq 0$),

$$\beta = \frac{[\Omega^{1/2}(\sin\phi, \cos\phi)']_1}{[\Omega^{1/2}(\sin\phi, \cos\phi)']_2},$$

where $[a]_i$ stands for ith coordinate of the vector a.

S2. IMPLEMENTATION DETAILS

S2.1. *Quantifying Violations of Bet-Proofness*

For all except the autoregressive root near unity problem, the maximal expected winnings are computed via linear programming. Specifically, the betting

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strategy space is discretized via disjoint sets $\mathcal{X}_j \subset \mathcal{X}$, so that the only possible $b(x)$ are of the form $b(x) = \sum_{j=1}^{n} b_j \mathbf{1}[x \in \mathcal{X}_j]$ with $b_j \in [0, 1]$. The expected winnings of this betting strategy for a given θ and α' are ((2) in Müller and Norets (2016))

$$\frac{1}{1-\alpha'} \int \left[\varphi\big(f(\theta), x\big) - \alpha'\right] b(x) p(x|\theta)\, d\nu(x)$$
$$= \frac{1}{1-\alpha'} \sum_{j=1}^{n} b_j \int_{\mathcal{X}_j} \left[\varphi\big(f(\theta), x\big) - \alpha'\right] p(x|\theta)\, d\nu(x).$$

The integrals $A_j = \int_{\mathcal{X}_j} [\varphi(f(\theta), x) - \alpha'] p(x|\theta)\, d\nu(x)$ are computed analytically or numerically, depending on the problem.

For the weak instrument problem, define (ρ_X, ϕ_X) by $(X_1^*, X_2^*) = (\rho_X \sin\phi_X, \rho_X \cos\phi_X)$. Lemma 3 in Müller and Norets (2016) implies

$$\varphi\big(f(\theta), X\big) = E_\theta\left[\varphi^*\big(f(\theta), g\big(U\big(X^*\big), X\big)\big)|X\right]$$
$$= E_\rho\left[\varphi^*\big(0, (\rho_X, \phi_X)\big)|\rho_X\right].$$

The Jacobian determinant of the transformation $(\rho_X, \phi_X) \to (\rho_X \sin\phi_X, \rho_X \cos\phi_X) = X^{*\prime}$ is equal to $-\rho_X$. Thus,

$$p\big((\rho_X, \phi_X)|\theta\big) \propto |\rho_X| \exp\left[\rho\rho_X \cos\phi_X - \frac{1}{2}\rho_X^2\right]$$

so that

$$p(\phi_X|\rho_X, \theta) \propto \exp[\rho\rho_X \cos\phi_X].$$

Also note that the AR interval can be written as follows:

$$\varphi^*\big(0, (\rho_X, \phi_X)\big) = \mathbf{1}\left[\phi_X \in [\psi, \pi - \psi] \cup [\pi + \psi, 2\pi - \psi]\right],$$

where $\psi = \arcsin\min(1, z_\alpha/\rho_X)$. Thus

$$\varphi(\rho, \rho_X) = \frac{2\int_\psi^{\pi-\psi} \exp\{\rho\rho_X \cos\phi_X\}\, d\phi_X}{\int_0^{2\pi} \exp\{\rho\rho_X \cos\phi_X\}\, d\phi_X},$$

where the denominator is equal to 2π times the modified Bessel function of the first kind, $I_0(\rho\rho_X)$, which can be evaluated by standard software, and the numerator can be computed numerically. The integrals A_j are computed numerically on the sets $\mathcal{X}_j \in \{[0, 0.2), [0.2, 0.4), \ldots, [12.8, 13), [13, \infty)\}$.

In the Imbens–Manski problem, the Stoye and EMW intervals are invariant and can be written as $\varphi^*(\gamma, x^*) = 1 - \mathbf{1}[x_L^* + l(x) \leq \gamma \leq x_L^* + u(x)]$ with $x = x_U^* - x_L^*$. The corresponding φ in (11) in Müller and Norets (2016) thus becomes

$$\begin{aligned}\varphi\big(f(\theta), x\big) &= E_\theta\big[\big(1 - \mathbf{1}\big[X_L^* + l(X) \leq \lambda\Delta \leq X_L^* + u(X)\big]\big)|X = x\big] \\ &= \Phi\left(\frac{l(x) - \lambda\Delta - \frac{1}{2}(x - \Delta)}{2^{-1/2}}\right) + 1 \\ &\quad - \Phi\left(\frac{u(x) - \lambda\Delta - \frac{1}{2}(x - \Delta)}{2^{-1/2}}\right),\end{aligned}$$

for Φ the c.d.f. of a standard normal, since $X_L^*|X = x \sim N(-\frac{1}{2}(x-\Delta), \frac{1}{2})$. From this expression, the integrals A_j on the sets $\mathcal{X}_j \in \{[-4, -3.8), [-3.8, -3.6), \dots, [13.8, 14)\}$ are computed by numerical integration. As one might intuitively guess, the measure K in Lemma 1 in Müller and Norets (2016) puts all mass on values with $\lambda = 1/2$, where the inspector's expected winnings are smallest.

In the autoregressive root near unity problem, discretization of the sample space with a four-dimensional sufficient statistic is computationally demanding. We thus apply Lemma 1 in Müller and Norets (2016) directly and numerically approximate K as a discrete measure on the grid $\theta = (\gamma, 0)$ with $\gamma \in \{0, 0.25, \dots, 200\}$ by iteratively adjusting the weight K_j at θ_j as a function of whether or not expected winnings at θ_j are positive or negative under the optimal betting strategy based on the previous value of K. In this computation, the expected winnings are approximated by Monte Carlo integration using importance sampling over 200,000 draws of a stationary Gaussian AR(1) with 2,500 observations and γ drawn from the grid $\gamma \in \{0, 0.25, \dots, 200\}$. For a similar numerical approach, see Elliott, Müller, and Watson (2015).

S2.2. *Bet-Proof Confidence Set*

In the near unit root example, the values cv_{γ_0} in Theorem 2 in Müller and Norets (2016) are the 95% percentiles of the statistic $\mathbf{1}[\gamma_0 \notin S^0(X)] \times \int p(X|\theta)\, dF(\theta)/p(X|\theta_0)$ with $\theta = (\gamma, 0)$ and $\theta_0 = (\gamma_0, 0)$ under θ_0, which we numerically approximate using the same Monte Carlo approximations scheme as in the computation of maximal expected winnings. For the determination of $\tilde{\Lambda} = \mathrm{cv}\,\Lambda$ of Theorem 3 in Müller and Norets (2016) in the weak instrument and Imbens–Manski examples, note that the coverage of φ_0 under $\tilde{\Lambda}$ amounts to $\mathrm{RP}_{\tilde{\Lambda}}(\theta) = E_\theta[\varphi_0(\hat{g}(U(X^*)^{-1}, \gamma), T(X^*))] \leq \alpha$ for all θ. For given $\tilde{\Lambda}$, $\mathrm{RP}_{\tilde{\Lambda}}(\theta)$ can be approximated by Monte Carlo integration over X^*. Furthermore, to approximate a $\tilde{\Lambda}$ satisfying $\int \mathrm{RP}_{\tilde{\Lambda}}(\theta)\, d\tilde{\Lambda}(\theta) = \alpha$, we posit a discrete grid Θ_g

on θ, and employ fixed-point iterations to adjust the mass points of a candidate $\tilde{\Lambda}_c$ on Θ_g as a function of whether $\text{RP}_{\tilde{\Lambda}_c}(\theta) < \alpha$ or $\text{RP}_{\tilde{\Lambda}_c}(\theta) > \alpha$, analogous to the algorithm suggested by Elliott, Müller, and Watson (2015). Specifically, Θ_g in the weak instrument example is equal to $\theta = (0, \rho)'$ with $\rho_j \in \{0, 0.05, 0.01, \dots, 10\}$, and it is equal to $\theta = (0, \Delta, \lambda)'$ with $\lambda \in \{0, 1\}$ and $\Delta \in \{0, 0.05, 0.01, \dots, 15\}$ in the Imbens–Manski example.

S3. CONDITIONAL NONCOVERAGE AND BETTING PROBABILITY

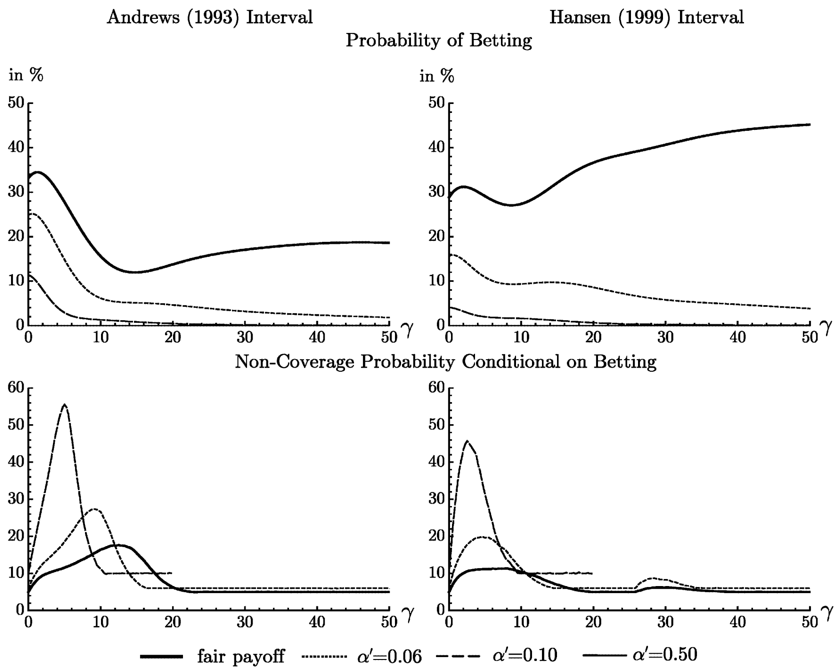

FIGURE S1.—Autoregressive coefficient near unity. (The dashed line in the bottom panels is not shown for $\gamma > 20$ because the betting probability in the denominator becomes very small and the corresponding conditional probability is numerically unstable.)

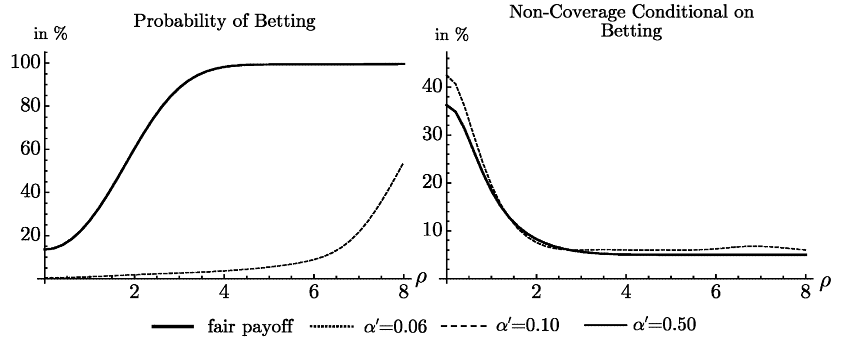

FIGURE S2.—Weak instruments.

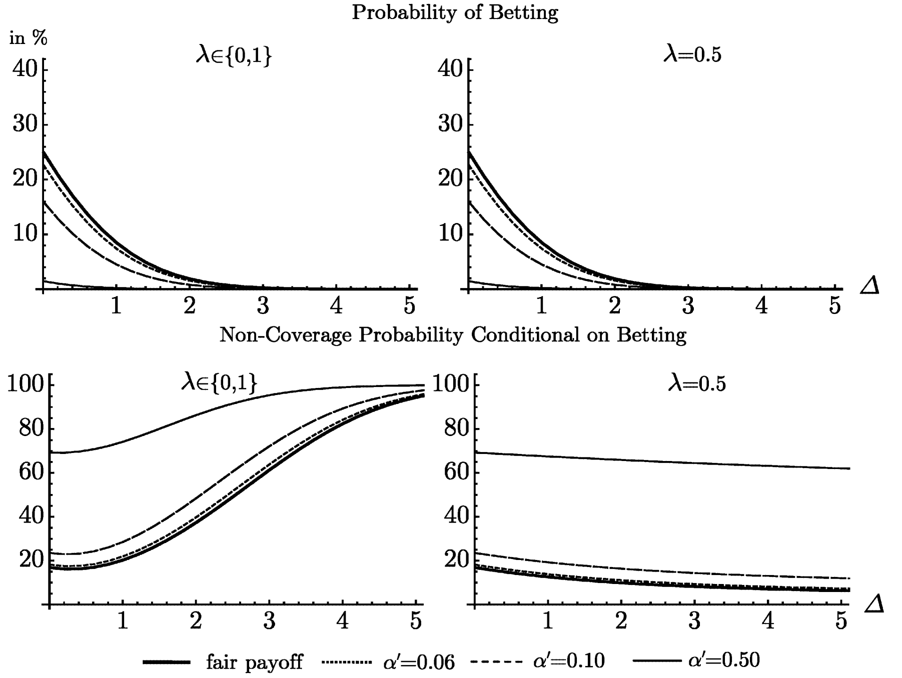

FIGURE S3.—Imbens–Manski problem, Stoye's interval.

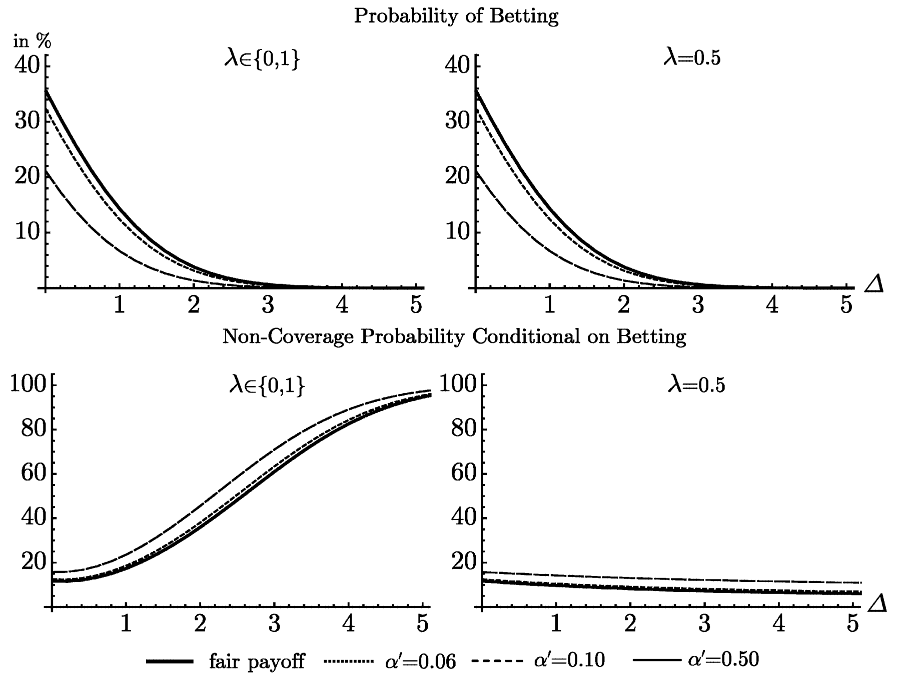

FIGURE S4.—Imbens–Manski problem, EMW's interval.

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