Solution to text problem 3.2 by Ankit Gupta

Statement of the problem
Consider the binary hypothesis testing problem between a standard Cauchy distribution and a zero mean Gaussian distribution with variance equal to 2 as shown in figure 1.

(a) Find the minimum probability of error $P$ if both hypothesis are equiprobable
(b) Find the test that minimizes the maximum of the probability of deciding $H_1$ when $H_2$ is true and the probability of deciding $H_2$ when $H_1$ is true.

![Figure 1: Gaussian and Cauchy hypothesis](image)

Solution to part A

Let the decision regions be $\Omega_1$ and $\Omega_2$ i.e. we declare hypothesis $H_1$ if the observable falls in $\Omega_1$ and hypothesis $H_1$ if the observable falls in $\Omega_2$. The the regions that minimize error probability are given by Proposition 3.1 in the textbook. Note that $H_1$ is the cauchy distribution and $H_2$ is the gaussian distribution.

$$\Omega_i = \left\{ z : f_{z|j}(z) = \max_{j=1,\ldots,n} f_{z|j}(z) \right\} - \bigcup_{j=1}^{i-1} \Omega_j$$

For the given distributions the regions are derived in the textbook using 1.

$$\Omega_1 = (-\infty, -2.9220) \cup (-0.4248, 0.4248) \cup (2.9220, \infty)$$
$$\Omega_2 = (-2.9220, -0.4248) \cup (0.4248, 2.9220)$$
Thus the probability of error is computed to be equal to

\[ P = 1 - \frac{1}{2} \sum_{i=1}^{2} \int_{\Omega_i} f_{z|i}(z)dz \]  

(4)

Using \( \Omega_1 \) and \( \Omega_2 \) from equations 2 and 3 respectively and substituting in equation 4

\[ P = 1 - \frac{1}{2} (0.4632 + 0.7275) = 0.4046 \]  

(5)

**Solution to part B**

Suppose as before the decision regions are \( \Omega_1 \) and \( \Omega_2 \). Then

\[ \Omega_1 + \Omega_2 = \Omega \]  

(6)

Then we have to find \( \Omega_1 \) and \( \Omega_2 \) such as to optimize the following equations. Where equivalence indicates that the optimal \( \Omega_1 \) and \( \Omega_2 \) are the same for both equations.

\[
\text{Min}(\text{Max}(P((i \in \Omega_1)|H_2), P((i \in \Omega_1)|H_2))) \equiv \text{Min}(\text{Max}(1 - P((i \in \Omega_1)|H_1), 1 - P((i \in \Omega_2)|H_2))) \\
\equiv \text{Min}(\text{Max}(-P(i \in \Omega_1|H_1), -P(i \in \Omega_2|H_2))) \\
\equiv \text{Max}(\text{Min}(P(i \in \Omega_1|H_1), P(i \in \Omega_2|H_2))) 
\]  

(7)

The last step is justified due to the negative sign that causes maxima to change to minima and vice versa. Consider the following two propositions for finding the optimal \( \Omega_1 \) and \( \Omega_2 \)

**Proposition 1**

The optimal \( \Omega_1 \) and \( \Omega_2 \) are such that.

\[
\int_{\Omega_1} f_{z|1} = \int_{\Omega_2} f_{z|2} 
\]  

(8)

**Proof by contradiction**

Suppose 8 is not true at the optimum value of \( \Omega_1 \) and \( \Omega_2 \) satisfying 7, then the smaller of the R.H.S and L.H.S in equation 8 contributes to the final optimum expression. Now since both the PDFs are greater than zero for all value in \( \Omega \) we can add sample points from the bigger side of the inequality to the smaller side thus making the smaller value large and shrinking the larger value and thus obtaining a value which is greater than the original hence the original values are not the optimal decision regions. Therefore the optimal \( \Omega_1 \) and \( \Omega_2 \) satisfy the equation 8
Proposition 2
The optimal $\Omega_1$ and $\Omega_2$ are given by.

$$\begin{align*}
\Omega_1 &= \{ z : \frac{f_1(z)}{f_2(z)} \geq k \} \\
\Omega_2 &= \Omega - \Omega_2
\end{align*}$$

(9) (10)

Where $k$ is chosen to satisfy 8

Proof by contradiction
Suppose that 9 is not true at optimality. But 8 still holds at optimality as shown in Proposition 1. This implies that $\Omega_1$ contains some sample space points apart from those for which 9 holds. But since $k$ was chosen to satisfy 8 therefore $\Omega_2$ contains sample space points where the equation 9 holds. If we choose an infinitesimal subspace $\Omega_2'$ in $\Omega_2$ where 9 is true i.e. $\frac{f_1(z)}{f_2(z)} \geq k$. Such that

$$\int_{\Omega_2'} f_2 = \epsilon$$

(11)

Then

$$\int_{\Omega_2'} f_1 = k' \epsilon$$

(12)

$$k' > k$$

(13)

$$k' = k + \delta$$

(14)

And we choose another infinitesimal subspace $\Omega_1'$ in $\Omega_1$ where 9 does not hold. And let

$$\int_{\Omega_1'} f_1 = k' \epsilon$$

(15)

Then

$$\int_{\Omega_1'} f_2 = \frac{k}{k'} \epsilon$$

(16)

$$k'' < k$$

(17)

$$\int_{\Omega_1'} f_2 = \epsilon + \gamma$$

(18)

$$\gamma > 0$$

(19)

If we exchange the subspaces $\Omega_2'$ and $\Omega_1'$ amongst $\Omega_1$ and $\Omega_2$. Then with such an exchange call the new decision regions $\Omega_1'$ and $\Omega_2'$. Then

$$\int_{\Omega_1'} f_{z|1} = \int_{\Omega_1} f_{z|1} + \delta \epsilon$$

(20)

$$\int_{\Omega_2'} f_{z|2} = \int_{\Omega_2} f_{z|2} + \gamma$$

(21)
Instead of $\Omega_1$ and $\Omega_2$ if we use $\Omega'_1$ and $\Omega'_2$ in equation 7 we get a greater value. Therefore the assumption of optimality was false to begin with hence the optimal values satisfy the equation 9.

**Final Solution**

For the problem at hand and using Proposition 1 and Proposition 2 we get the

![Figure 2: Ratio of Cauchy and gaussian hypothesis](image)

optimal decision regions using Matlab as,

\[
\begin{align*}
\Omega_2 & = (2.7567, 0.6585) \cup (0.6585, 2.7567) \\
\Omega_1 & = \Omega - \Omega_2 \\
k & = 1.14
\end{align*}
\]

The value for the minimax error probability is 0.41.