Problem 3.22
Consider the composite hypothesis testing problem in Proposition 3.4 with \( \|x_1\| = \cdots = \|x_m\| \). Consider the test that selects the most likely hypothesis among the \( m \) hypotheses and then checks whether that hypotheses belongs to \( J \).

a) Show that this test is not optimal by means of an example.
b) Show that the decisions regions of this test converge to the optimal ones as \( \sigma \to 0 \).

Solution by Jimmy Chui
a) We consider the following 4-user case. Take \( s_1(t) \) and \( s_2(t) \) to be orthonormal signals defined on the interval \( t \in \mathcal{T} \), and let:

\[
\begin{align*}
    x_1(t) &= A s_1(t) \\
    x_2(t) &= A s_2(t) \\
    x_3(t) &= (-A s_1(t) + A s_2(t))/\sqrt{2} \\
    x_4(t) &= (A s_1(t) - A s_2(t))/\sqrt{2}
\end{align*}
\]

Furthermore, let \( \mathcal{H}_J = \{H_1, H_2\} \) and \( \bar{\mathcal{H}}_J = \{H_3, H_4\} \). We plot these vectors on \( s_1s_2 \)-axes in Figure 1.

![Figure 1: \( x_1, x_2, x_3, x_4 \) plotted on \( s_1s_2 \)-axes, with \( A = 1 \)](image-url)
According to Proposition 3.2, if we were to select the most likely hypothesis among the 4 hypotheses, the decision regions that minimize the error of probability are exactly the points minimizing the mean-square distance $\int_{[a,b]} |y(t) - x_i(t)|^2 dt$. This corresponds to the regions noted in Figure 2, where we have divided the decision regions of $H_1$, $H_2$, and $H_3$, $H_4$ by a solid black line.

We obtain these decision regions by taking perpendicular bisectors between pairwise points. Specifically, for this method, the boundary dividing $\mathcal{H}_J$ and $\bar{\mathcal{H}}_J$ are two semi-infinite lines. The equations of these lines, in polar coordinates, are:

$$\theta = \frac{5\pi}{8} \quad (r \geq 0) \quad (5)$$

$$\theta = -\frac{\pi}{8} \quad (r \geq 0) \quad (6)$$

Figure 2: Optimal decision regions for $H_1$, $H_2$, $H_3$, and $H_4$. ($A = 1$)
Since $\|x_1\| = \|x_2\| = \|x_3\| = \|x_4\|$, by Proposition 3.4, the optimal decision rule for the compound hypothesis testing problem is given by

$$
\sum_{i \in J} \exp \left( \frac{1}{\sigma^2} \int_I y(t)x_i(t) \right) dt \geq \sum_{i \notin J} \exp \left( \frac{1}{\sigma^2} \int_I y(t)x_i(t) \right) dt \tag{7}
$$

Let the projection of $y(t)$ onto $s_1(t)$ and $s_2(t)$ be $\alpha_1 A$ and $\alpha_2 A$, so that

$$
y(t) = \alpha_1 As_1(t) + \alpha_2 As_2(t) + z(t) \tag{8}
$$

where $z(t)$ is orthogonal to both $s_1(t)$ and $s_2(t)$. We also see that

$$
\int_I y(t)(\beta_1 As_1(t) + \beta_2 As_2(t)) dt \\
= A^2 \int_I (\alpha_1 s_1(t) + \alpha_2 s_2(t) + z(t)) (\beta_1 s_1(t) + \beta_2 s_2(t)) dt \tag{9}
$$

$$
= A^2 (\alpha_1 \beta_1 + \alpha_2 \beta_2) \tag{10}
$$

From (10), and using (1) to (4), we can deduce that (7) is equivalent to

$$
\exp \left( \frac{A^2}{\sigma^2} \alpha_1 \right) + \exp \left( \frac{A^2}{\sigma^2} \alpha_2 \right) \\
\geq \exp \left( \frac{A^2}{\sigma^2} \left( -\frac{1}{\sqrt{2}} \alpha_1 + \frac{1}{\sqrt{2}} \alpha_2 \right) \right) \\
+ \exp \left( \frac{A^2}{\sigma^2} \left( \frac{1}{\sqrt{2}} \alpha_1 - \frac{1}{\sqrt{2}} \alpha_2 \right) \right) \tag{11}
$$

We plot this curve with a solid black line in Figure 3, with the settings $A = 1$ and $\sigma = 1$, with $\mathcal{H}_J$ being the upper right region, and $\bar{\mathcal{H}}_J$ being the lower left region. The dotted line represents the boundary of the suboptimal decision regions calculated earlier.
b) We use the dot product notation $\langle y, x_i \rangle = \int_I y(t)x_i(t)dt$.

The optimal decision rule is

$$\sum_{i \in J} \exp \left( \frac{1}{\sigma^2} \langle y, x_i \rangle \right) dt \gtrless \sum_{i \notin J} \exp \left( \frac{1}{\sigma^2} \langle y, x_i \rangle \right) dt$$

(12)

Note that, as $\sigma \to 0$, each sum grows asymptotically as fast as the term with the maximum dot product. That is, the decision rule becomes equivalent to:

$$\max_{i \in J} \exp \left( \frac{1}{\sigma^2} \langle y, x_i \rangle \right) dt \gtrless \max_{i \notin J} \exp \left( \frac{1}{\sigma^2} \langle y, x_i \rangle \right) dt \text{ when } \sigma \to 0$$

(13)

This is precisely identical to the process of evaluating all the $m$ dot products, choosing the index corresponding to the largest dot product, and determining whether it lies within $J$ or not.

(We take note that it is possible for two or more maximum dot products to occur which makes (13) invalid; however, this situation occurs only at the border of the decision regions, has measure zero, and hence can be ignored.)

Hence, as $\sigma \to 0$, the proposed decision rule approaches the optimal decision rule (almost surely).