China's Model of Managing the Financial System

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Internet Appendix

This internet appendix present proofs of the propositions in the main paper.

Proof of Proposition A1

We derive the perfect information equilibrium with trading by the government. We first conjecture that, when v_{t+1} and N_t are observable to the government and investors, the stock price takes the linear form

$$P_t = p_v v_{t+1} + p_N N_t + p_g G_t.$$

Given that dividends are $D_t = v_t + \sigma_D \varepsilon_t^D$, the stock price must react to a deterministic unit shift in v_{t+1} by the present value of dividends deriving from that shock, $\frac{1}{R^f - \rho_v}$, it follows that $p_v = \frac{1}{R^f - \rho_v}$. The innovations to v_{t+1} and N_t are the only source of risk and, from the perspective of all economic agents, the conditional expectation and variance of R_{t+1} are

$$\begin{aligned} E\left[R_{t+1} \mid \mathcal{F}_t\right] &= p_N \left(\rho_N - R^f\right) N_t - R^f p_g G_t, \\ Var\left[R_{t+1} \mid \mathcal{F}_t\right] &= \sigma_D^2 + \left(\frac{1}{R^f - \rho_v}\right)^2 \sigma_v^2 + p_N^2 \sigma_N^2 + p_g^2 \sigma_G^2. \end{aligned}$$

Since all investors are identical when v_t and N_t are observable, it follows that in the CARA-Normal environment all investors have an identical mean-variance demand for the risky asset:

$$X_t^S = \frac{1}{\gamma} \frac{E\left[R_{t+1} \mid \mathcal{F}_t\right]}{Var\left[R_{t+1} \mid \mathcal{F}_t\right]} = \frac{1}{\gamma} \frac{p_N \left(\rho_N - R^f\right) N_t - R^f p_g G_t}{\sigma_D^2 + \left(\frac{1}{R^f - \rho_v}\right)^2 \sigma_v^2 + p_N^2 \sigma_N^2 + p_g^2 \sigma_G^2}.$$

In the government's intervention rule

$$X_t^G = \vartheta_N N_t + \vartheta_N \sigma_N G_t,$$

ϑ_N is determined by

$$U^G = \sup_{\vartheta} -\gamma_\sigma \left(\sigma_D^2 + \left(\frac{1}{R^f - \rho_v}\right)^2 \sigma_v^2 + p_N^2 \sigma_N^2 + p_g^2 \sigma_G^2\right),$$

Finally, by imposing market-clearing, we arrive at

$$\begin{aligned} N &= \frac{1}{\gamma}\frac{p_N\left(\rho_N - r\right)N}{\sigma_D^2 + \left(\frac{1}{R^f-\rho_v}\right)^2\sigma_v^2 + p_N^2\sigma_N^2} + \vartheta_N N, \\ \vartheta_N\sigma_N^2 G_t &= \frac{1}{\gamma}\frac{-R^f p_g}{\sigma_D^2 + \left(\frac{1}{R^f-\rho_v}\right)^2\sigma_v^2 + p_N^2\sigma_N^2 + p_g^2\sigma_G^2} G_t \end{aligned}$$

which, by matching coefficients, reveals that

$$\begin{aligned} \frac{1}{\gamma}\frac{p_N\left(\rho_N - R^f\right)}{\sigma_D^2 + \left(\frac{1}{R^f-\rho_v}\right)^2\sigma_v^2 + p_N^2\sigma_N^2 + p_g^2\sigma_G^2} + \vartheta_N &= 1, \\ -\frac{p_N\left(\rho_N - R^f\right)}{R^f}\frac{\vartheta_N}{1-\vartheta_N}\sigma_N &= p_g. \end{aligned}$$

This confirms the conjectured equilibrium.

Rearranging this equation for p_N, and substituting for p_g, we arrive at the quadratic equation for p_N

$$\begin{aligned} &\left(1 + \left(\frac{\rho_N - R^f}{R^f}\right)^2\left(\frac{\vartheta_N}{1-\vartheta_N}\right)^2\sigma_G^2\right)p_N^2 \\ &+\frac{R^f - \rho_N}{\gamma\sigma_N^2\left(1-\vartheta_N\right)}p_N + \frac{\sigma_D^2}{\sigma_N^2} + \left(\frac{1}{R^f-\rho_v}\right)^2\frac{\sigma_v^2}{\sigma_N^2} = 0, \end{aligned} \tag{IA.1}$$

from which follows that p_N has two roots

$$p_N\left(\vartheta_N\right) = \frac{1}{2}\frac{\frac{\rho_N - R^f}{\gamma\sigma_N^2(1-\vartheta_N)} \pm \sqrt{\left(\frac{R^f-\rho_N}{\gamma\sigma_N^2(1-\vartheta_N)}\right)^2 - 4\left(1 + \left(\frac{\rho_N-R^f}{R^f}\right)^2\left(\frac{\vartheta_N}{1-\vartheta_N}\right)^2\sigma_G^2\right)\left(\frac{\sigma_D^2}{\sigma_N^2} + \left(\frac{1}{R^f-\rho_v}\right)^2\frac{\sigma_v^2}{\sigma_N^2}\right)}}{1 + \left(\frac{\rho_N-R^f}{R^f}\right)^2\left(\frac{\vartheta_N}{1-\vartheta_N}\right)^2\sigma_G^2}.$$

Recognizing that two negative solutions for P_N exist if the expression under the square root is nonnegative, it follows that the market breaks down occurs whenever

$$R^f < \rho_N + 2\left(1-\vartheta_N\right)\gamma\sqrt{\left(1 + \left(\frac{\rho_N - R^f}{R^f}\right)^2\left(\frac{\vartheta_N}{1-\vartheta_N}\right)^2\sigma_G^2\right)\left(\sigma_D^2\sigma_N^2 + \left(\frac{1}{R^f-\rho_v}\right)^2\sigma_v^2\sigma_N^2\right)}.$$

Given that $Var\left(\Delta P_t \mid \mathcal{F}_{t-1}\right) = \sigma_D^2 + \left(\frac{1}{R^f-\rho_v}\right)^2\sigma_v^2 + p_N^2\sigma_N^2 + p_g^2\sigma_G^2$, substituting for p_g, the government's optimization problem consequently reduces to

$$U^G = \sup_{\vartheta_N} -\left(\gamma_v + \gamma_\sigma\right)\left(1 + \left(\frac{\rho_N - R^f}{R^f}\right)^2\left(\frac{\vartheta_N}{1-\vartheta_N}\right)^2\sigma_G^2\right)p_N^2\sigma_N^2,$$

and, from the two market-clearing condition restrictions on the coefficients p_N and p_g, that ϑ_N is determined by

$$\vartheta_N = 1 - \frac{1}{\gamma} \frac{p_N \left(\rho_N - R^f\right)}{\sigma_D^2 + \left(\frac{1}{R^f - \rho_v}\right)^2 \sigma_v^2 + \left(1 + \left(\frac{\rho_N - R^f}{R^f}\right)^2 \left(\frac{\vartheta_N}{1-\vartheta_N}\right)^2 \sigma_G^2\right) p_N^2 \sigma_N^2}.$$

To establish that the linear equilibrium is the unique, symmetric equilibrium, we express each investor's optimization problem as

$$U_t = \max_{X_t} E\left[e^{-\gamma\left(R\bar{W} + X_t\left(v_{t+1} + \sigma_D \varepsilon_{t+1}^D + P_{t+1} - RP_t\right)\right)}\right]$$

For an arbitary price function P_t, the FOC for the investor's holding of the risky asset X_t is

$$E\left[\left(v_{t+1} + \sigma_D \varepsilon_{t+1}^D + P_{t+1} - R^f P_t\right) e^{-\gamma X_t\left(v_{t+1} + \sigma_D \varepsilon_{t+1}^D + P_{t+1} - RP_t\right)} \mid \mathcal{F}_t\right] = 0.$$

Substituting this with the market-clearing condition

$$X_t = \left(1 - \vartheta_N\right) N_t + \vartheta_N \sigma_N G_t,$$

we arrive at

$$E\left[\left(v_{t+1} + \sigma_D \varepsilon_{t+1}^D + P_{t+1} - R^f P_t\right) e^{-\gamma X_t\left(v_{t+1} + \sigma_D \varepsilon_{t+1}^D + P_{t+1} - RP_t\right)} \mid \mathcal{F}_t\right] = 0.$$

Since P_{t+1} cannot be a function of ε_{t+1}^D, as P_{t+1} is forward-looking for the new generation of investors at time $t+1$, the above can be rewritten as

$$\begin{aligned} P_t &= \frac{1}{R^f} v_{t+1} - \frac{\gamma}{R^f} \sigma_D^2 \left(\left(1 - \vartheta_N\right) N_t + \vartheta_N \sigma_N G_t\right) \\ &\quad + \frac{1}{R^f} E\left[P_{t+1} \frac{e^{-\gamma\left(\left(1-\vartheta_N\right) N_t + \vartheta_N \sigma_N G_t\right) P_{t+1}}}{E\left[e^{-\gamma\left(\left(1-\vartheta_N\right) N_t + \vartheta_N \sigma_N G_t\right) P_{t+1}} \mid \mathcal{F}_t\right]} \mid \mathcal{F}_t\right]. \end{aligned} \tag{IA.2}$$

This defines a functional equation, whose fixed point is the price functional P_t. To see that the linear equilibrium we derived above solves this functional equation, we rewrite equation (IA.2) as

$$P_t = \frac{1}{R^f} v_{t+1} - \frac{\gamma}{R^f} \sigma_D^2 \left(\left(1 - \vartheta_N\right) N_t + \vartheta_N \sigma_N G_t\right) + \frac{1}{R^f} \left|\partial_u \log E\left[e^{uP_{t+1}} \mid \mathcal{F}_t\right]\right|_{u = -\gamma\left(\left(1-\vartheta_N\right) N_t + \vartheta_N \sigma_N G_t\right)},$$

and conjecture that $P_t = \frac{1}{R^f - \rho_v} v_{t+1} + p_N N_t + p_g G_t$, from which follows that p_N satisfies equation (IA.1). This verifies that the linear price equilibrium satisfies this more general equilibrium condition.

Now define the operator $T : \mathcal{B}(\mathbb{R}^2) \to \mathcal{B}(\mathbb{R}^2)$

$$Tf = \frac{1}{R^f}v - \frac{\gamma}{R^f}\sigma_D^2\left((1-\vartheta_N)N + \vartheta_N\sigma_N G\right) + \frac{1}{R^f}E\left[f'\frac{e^{-\gamma((1-\vartheta_N)N+\vartheta_N\sigma_N G)f'}}{E\left[e^{-\gamma((1-\vartheta_N)N+\vartheta_N\sigma_N G)f'} \mid v, N, G\right]} \mid v, N, G\right],$$

for $f \in \mathcal{B}(\mathbb{R}^2)$, where $\mathcal{B}(\mathbb{R}^2)$ is the space of continuous functions $\psi-$bounded in the $\psi-$norm $\|f\|_\psi = \sup_x \frac{|f(x)|}{\psi(x)}$, where $\psi(x) = 1 + \|x\|^b$ has polynomial growth for some $b \geq 1$. Since v_{t+1}, G_t, and N_t are Markov processes, $\{v_{t+1}, N_t, G_t\}$ are sufficient statistics for the conditioning in the above conditional expectations.

We now establish that T is a contraction map by verifying that T satisfies Blackwell's Sufficiency conditions. Since P_t defines an asset price, it must be the case that, if P_{t+1} (weakly) increases $\forall\{v_{t+2}, G_{t+1}, N_{t+1}\}$, which is a FOSD shift in the distribution of P_{t+1}, then it is (weakly) preferred by any averse agent whose utility is increasing in wealth. Consequently, investors would demand more to earn the higher return, which would bid up the price today. Thus, $Tf \geq Tg$ for $f \geq g$, and T satisfies monotonicity. Furthermore,

$$\begin{aligned}
T(f+c) &= \frac{1}{R^f}v - \frac{\gamma}{R^f}\sigma_D^2\left((1-\vartheta_N)N + \vartheta_N\sigma_N G\right) \\
&\quad + \frac{1}{R^f}E\left[(f'+c)\frac{e^{-\gamma((1-\vartheta_N)N+\vartheta_N\sigma_N G)(f'+c)}}{E\left[e^{-\gamma((1-\vartheta_N)N+\vartheta_N\sigma_N G)(f'+c)} \mid v, N, G\right]} \mid v, N, G\right] \\
&= \frac{1}{R^f}v - \frac{\gamma}{R^f}\sigma_D^2\left((1-\vartheta_N)N + \vartheta_N\sigma_N G\right) \\
&\quad + \frac{1}{R^f}E\left[f'\frac{e^{-\gamma((1-\vartheta_N)N+\vartheta_N\sigma_N G)f'}}{E\left[e^{-\gamma((1-\vartheta_N)N+\vartheta_N\sigma_N G)f'} \mid v, N, G\right]} \mid v, N, G\right] + \frac{1}{R^f}c \\
&= Tf + \frac{1}{R^f}c \\
&\leq Tf + \frac{1}{R^f}c\nu,
\end{aligned}$$

and T satisfies discounting since $R^f > 1$. Therefore, T is a strict contraction map by the Weighted Contraction Mapping Theorem of Boyd (1990). Since a contraction map has, at most, one fixed point, and an equilibrium with linear P_t exists, it must be the unique equilibrium in the economy, at least within the class of functions continuous and bounded in the $\psi-$norm.

Proof of Proposition 1

Note from the variance of the excess asset payoff that

$$Var\left[R_{t+1} \mid \mathcal{F}_t\right] = \sigma_D^2 + \left(\frac{1}{R^f - \rho_v}\right)^2\sigma_v^2 + p_N^2\sigma_N^2,$$

and thus the excess volatililty is driven by the $p_N^2\sigma_N^2$ term. Consider now the expression for the less negative root of p_N from Proposition 1 in the absence of government intervention:

$$p_N = \frac{\rho_N - R^f}{2\gamma\sigma_N^2} + \sqrt{\left(\frac{R^f - \rho_N}{2\gamma\sigma_N^2}\right)^2 - \left(\frac{\sigma_D^2}{\sigma_N^2} + \left(\frac{1}{R^f - \rho_v}\right)^2 \frac{\sigma_v^2}{\sigma_N^2}\right)}.$$

Given this expression, it follows that

$$\begin{aligned} 2p_N^2\sigma_N^2 &= \frac{1}{\sigma_N^2}\left(\frac{\rho_N - R^f}{\gamma}\right)^2 + \frac{\rho_N - R^f}{\gamma}\sqrt{\left(\frac{R^f - \rho_N}{\gamma\sigma_N^2}\right)^2 - 4\left(\frac{\sigma_D^2}{\sigma_N^2} + \left(\frac{1}{R^f - \rho_v}\right)^2 \frac{\sigma_v^2}{\sigma_N^2}\right)} \\ &\quad -2\left(\sigma_D^2 + \left(\frac{1}{R^f - \rho_v}\right)^2 \sigma_v^2\right). \end{aligned}$$

Differentiating with respect to σ_N^2, we find with some manipulation that

$$\begin{aligned} \frac{\partial p_N^2\sigma_N^2}{\partial\sigma_N^2} &= \frac{R^f - \rho_N}{2\gamma\sigma_N^2}\left[\sqrt{\left(\frac{R^f - \rho_N}{\gamma\sigma_N^2}\right)^2 - 4\left(\frac{\sigma_D^2}{\sigma_N^2} + \left(\frac{1}{R^f - \rho_v}\right)^2 \frac{\sigma_v^2}{\sigma_N^2}\right)}\right. \\ &\quad \left. + \frac{R^f - \rho_N}{\gamma\sigma_N^2}\left(\frac{1}{\sqrt{\left(\frac{R^f - \rho_N}{\gamma\sigma_N^2}\right)^2 - 4\left(\frac{\sigma_D^2}{\sigma_N^2} + \left(\frac{1}{R^f - \rho_v}\right)^2 \frac{\sigma_v^2}{\sigma_N^2}\right)}} - 1\right)\right]. \end{aligned}$$

Thus, from the second part of the above expression, it is sufficient for $\frac{\partial p_N^2\sigma_N^2}{\partial\sigma_N^2} > 0$ that

$$\sigma_N^4 + 4\left(\sigma_D^2 + \left(\frac{1}{R^f - \rho_v}\right)^2 \sigma_v^2\right)\sigma_N^2 - \left(\frac{R^f - \rho_N}{\gamma}\right)^2 > 0.$$

By Descartes' Rule of Signs, the above has only one positive root for σ_N such that $\frac{\partial p_N^2\sigma_N^2}{\partial\sigma_N^2} > 0$ if $\sigma_N > \sigma_N^* = \sqrt{4\left(\sigma_D^2 + \left(\frac{1}{R^f-\rho_v}\right)^2\sigma_v^2\right)^2 + \left(\frac{R^f-\rho_N}{\gamma}\right)^2} - 2\left(\sigma_D^2 + \left(\frac{1}{R^f-\rho_v}\right)^2\sigma_v^2\right)$. Provided that $\sigma_N > \sigma_N^*$, then $\frac{\partial p_N^2\sigma_N^2}{\partial\sigma_N^2} > 0$ and volatility is highest close to market breakdown, when $\left(\frac{R^f-\rho_N}{\gamma\sigma_N^2}\right)^2 - 4\left(\frac{\sigma_D^2}{\sigma_N^2} + \left(\frac{1}{R^f-\rho_v}\right)^2\frac{\sigma_v^2}{\sigma_N^2}\right) = \varepsilon$ for ε arbitrarily small. Market breakdown occurs when $\varepsilon = 0$, or

$$\sigma_N = \frac{Rv^f - \rho_N}{2\alpha\gamma\sqrt{\sigma_D^2 + \left(\frac{1}{R^f-\rho_v}\right)^2\sigma_v^2}}.$$

Furthermore, as $\varepsilon \to 0$, and $\sigma_N \to \frac{R^f - \rho_N}{2\alpha\gamma\sqrt{\sigma_D^2 + \left(\frac{1}{R^f - \rho_v}\right)^2 \sigma_v^2}}$, then

$$p_N^2 \sigma_N^2 \to \sigma_D^2 + \left(\frac{1}{R^f - \rho_v}\right)^2 \sigma_v^2.$$

Consequently, the maximum conditional excess payoff variance before breakdown occurs is $Var\left[R_{t+1} \mid \mathcal{F}_t\right] \to 2\left(\sigma_D^2 + \left(\frac{1}{R^f - \rho_v}\right)^2 \sigma_v^2\right)$.

Proof of Proposition A2

To arrive at the beliefs of investors, we first characterize the market beliefs based on only the public information set $\mathcal{F}_t^M$. To derive the market beliefs, we proceed in several steps. First, we assume the market posterior belief of (v_{t+1}, N_t) is jointly Gaussian, $(v_{t+1}, N_t) \sim \mathcal{N}\left(\left(\hat{v}_{t+1}^M, \hat{N}_t^M\right), \Sigma_t^M\right)$, where

$$\begin{aligned} \begin{bmatrix} \hat{v}_{t+1}^M \\ \hat{N}_t^M \end{bmatrix} &= E\left[\begin{bmatrix} v_{t+1} \\ N_t \end{bmatrix} \mid \mathcal{F}_t^M\right], \\ \Sigma_t^M &= \begin{bmatrix} \Sigma_t^{M,vv} & \Sigma_t^{M,vN} \\ \Sigma_t^{M,vN} & \Sigma_t^{M,NN} \end{bmatrix}. \end{aligned}$$

Standard results for the Kalman Filter establish that the law of motion of the conditional expectation of the market's posterior beliefs $\left(\hat{v}_{t+1}^M, \hat{N}_t^M\right)$ is

$$\begin{bmatrix} \hat{v}_{t+1}^M \\ \hat{N}_t^M \end{bmatrix} = \begin{bmatrix} \rho_v & 0 \\ 0 & \rho_N \end{bmatrix} \begin{bmatrix} \hat{v}_t^M \\ \hat{N}_{t-1}^M \end{bmatrix} + \mathbf{k}_t^M \begin{bmatrix} D_t - \hat{v}_t^M \\ \eta_t^H - p_v \rho_v \hat{v}_t^M - p_N \rho_N \hat{N}_{t-1}^M \end{bmatrix},$$

where

$$\begin{aligned} \mathbf{k}_t^M &= Cov\left[\begin{bmatrix} v_{t+1} \\ N_t \end{bmatrix}, \begin{bmatrix} D_t - \hat{v}_t^M \\ \eta_t^H - p_v \rho_v \hat{v}_t^M - p_N \rho_N \hat{N}_{t-1}^M \end{bmatrix} \mid \mathcal{F}_{t-1}^M\right] \\ &\quad \times Var\left[\begin{bmatrix} D_t - \hat{v}_t^M \\ \eta_t^H - p_v \rho_v \hat{v}_t^M - p_N \rho_N \hat{N}_{t-1}^M \end{bmatrix} \mid \mathcal{F}_{t-1}^M\right]^{-1}, \end{aligned}$$

is the Kalman Gain, and the conditional variance Σ_t^M evolves deterministically according to

$$\begin{aligned} \Sigma_t^M &= \begin{bmatrix} \rho_v & 0 \\ 0 & \rho_N \end{bmatrix} \Sigma_{t-1}^M \begin{bmatrix} \rho_v & 0 \\ 0 & \rho_N \end{bmatrix} + \begin{bmatrix} \sigma_v^2 & 0 \\ 0 & \sigma_N^2 \end{bmatrix} \\ &\quad - \mathbf{k}_t^M Cov\left[\begin{bmatrix} D_t - \hat{v}_t^M \\ \eta_t^H - p_v \rho_v \hat{v}_t^M - p_N \rho_N \hat{N}_{t-1}^M \end{bmatrix}, \begin{bmatrix} v_t \\ N_t \end{bmatrix} \mid \mathcal{F}_{t-1}^M\right]. \end{aligned}$$

It is straightforward to compute that

$$
\begin{aligned}
& Cov\left[\left[\begin{array}{c} v_{t+1} \\ N_t \end{array}\right], \left[\begin{array}{c} D_t - \hat{v}_t^M \\ \eta_t^H - p_v \rho_v \hat{v}_t^M - p_N \rho_N \hat{N}_{t-1}^M \end{array}\right] \mid \mathcal{F}_{t-1}^M\right] \\
= & \left[\begin{array}{cc} \rho_v \Sigma_{t-1}^{M,vv} & p_v\left(\rho_v \Sigma_{t-1}^{M,vv} + \sigma_v^2\right) + p_N \rho_v \rho_N \Sigma_{t-1}^{M,vN} \\ \rho_N \Sigma_{t-1}^{M,vN} & p_v \rho_v \rho_N \Sigma_{t-1}^{M,vN} + p_N\left(\rho_N^2 \Sigma_{t-1}^{M,NN} + \sigma_N^2\right) \end{array}\right],
\end{aligned}
$$

and that

$$
\begin{aligned}
\Omega_{t-1}^M &= Var\left[\left[\begin{array}{c} D_t - \hat{v}_t^M \\ \eta_t^M - p_v \rho_v \hat{v}_t^M - p_N \rho_N \hat{N}_{t-1}^M \end{array}\right] \mid \mathcal{F}_{t-1}^M\right] \\
&= \left[\begin{array}{cc} \Sigma_{t-1}^{M,vv} + \sigma_D^2 & p_v \rho_v \Sigma_{t-1}^{M,vv} + p_N \rho_N \Sigma_{t-1}^{M,vN} \\ p_v \rho_v \Sigma_{t-1}^{M,vv} + p_N \rho_N \Sigma_{t-1}^{M,vN} & \begin{array}{c} p_v^2\left(\rho_v \Sigma_{t-1}^{M,vv} + \sigma_v^2\right) + 2 p_v p_N \rho_v \rho_N \Sigma_{t-1}^{M,vN} \\ + p_N^2\left(\rho_N^2 \Sigma_{t-1}^{M,NN} + \sigma_N^2\right) \end{array} \end{array}\right],
\end{aligned}
$$

We consider the deterministic steady-state of the Kalman Filter and, consequently, drop all time t subscripts from conditional variances. We shall verify its existence at the end of the proof.

For $\eta_t^H \in \mathcal{F}_t^M \subseteq \mathcal{F}_t$, I can express η_t^H as

$$
\eta_t^H = p_v v_{t+1} + p_N N_t = p_v \hat{v}_{t+1}^M + p_N \hat{N}_t^M,
$$

from which follows that

$$
p_v\left(v_{t+1} - \hat{v}_{t+1}^M\right) + p_N\left(N_t - \hat{N}_t^M\right) = 0.
$$

As a consequence, it must be that the market beliefs about v_t and N_t are ex-post correlated after observing the stock price innovation process η_t^M, such that we have the three identities by taking its variance and its covariance with $v_{t+1} - \hat{v}_{t+1}^M$ and $N_t - \hat{N}_t^M$

$$
\begin{aligned}
\Sigma^{M,vN} &= -\frac{p_v}{p_N} \Sigma^{M,vv}, \\
\Sigma^{M,NN} &= -\frac{p_v}{p_N} \Sigma^{M,\theta N} = \left(\frac{p_v}{p_N}\right)^2 \Sigma^{M,\theta\theta}.
\end{aligned}
$$

Consequently, as in He and Wang (1995), we need to only compute $\Sigma^{M,vv}$.

Updating the market beliefs to the private beliefs of economic agents can be done in a manner similar to that in He and Wang (1995). Since the market belief acts as a normal prior for investor i who observes the normally distributed private signal s_t^i, they update their beliefs by Bayes' Law in accordance with a linear updating rule. The posterior of investor i

is $N\left(\hat{v}_{t+1}^{i}, \Sigma_{t}^{s}(i)\right)$, where $\hat{v}_{t+1}^{i} = E\left[v_{t+1} \mid \mathcal{F}_{t}^{i}\right]$ and $\Sigma^{s}(i) = E\left[\left(v_{t+1} - \hat{v}_{t+1}^{i}\right)^{2} \mid \mathcal{F}_{t}^{i}\right]$ are given by

$$\hat{v}_{t+1}^{i} = \hat{v}_{t+1}^{M} + \frac{\Sigma^{M,vv}}{\Sigma^{M,vv} + \tau_{s}^{-1}}\left(s_{t}^{i} - \hat{v}_{t+1}^{M}\right),$$

and

$$\Sigma^{s}(i)^{-1} = \left(\Sigma^{M,vv}\right)^{-1} + \tau_{s}.$$

This characterizes the beliefs of investors given the market beliefs.

Since the government does not trade in this benchmark, investors have no incentive to learn about the government's behavior, and therefore the information acquisition decision is trivial. Given that investors each acquire a private signal s_{t}^{i}, standard results for CARA utility with normally distributed prices and payoffs establish that the optimal trading policy of investor i, X_{t}^{i}, is given by

$$\begin{aligned} X_{t}^{i} &= \frac{E\left[D_{t+1} + P_{t+1} - R^{f}P_{t} \mid \mathcal{F}_{t}^{i}\right]}{\gamma Var\left[D_{t+1} + P_{t+1} \mid \mathcal{F}_{t}^{i}\right]} \\ &= \frac{\left(\begin{array}{c} \left(1 + p_{v}\left(\rho_{v} - R^{f}\right)\right)\left(\hat{v}_{t+1}^{i} - \hat{v}_{t+1}^{M}\right) + p_{N}\left(\rho_{N} - R^{f}\right)\hat{N}_{t}^{i} \\ + \left[\begin{array}{c} p_{\hat{v}} - p_{v} \\ 0 \end{array}\right]' \mathbf{k}^{M}\left[\begin{array}{c} \hat{v}_{t+1}^{i} - \hat{v}_{t+1}^{M} \\ p_{v}\rho_{v}\left(\hat{v}_{t+1}^{i} - \hat{v}_{t+1}^{M}\right) + p_{N}\rho_{N}\left(\hat{N}_{t}^{i} - \hat{N}_{t}^{M}\right) \end{array}\right] \end{array}\right)}{\gamma\varphi'\Omega(i)\varphi}, \end{aligned}$$

where

$$\varphi = \left[\begin{array}{c} 1 \\ 1 \end{array}\right] + \mathbf{k}^{M\prime}\left[\begin{array}{c} p_{\hat{v}} - p_{v} \\ 0 \end{array}\right],$$

and

$$\Omega(i) = \Omega^{M} - \left[\begin{array}{c} 1 \\ p_{v}\left(\rho_{v} - \rho_{N}\right) \end{array}\right]\frac{\left(\Sigma^{M,\theta\theta}\right)^{2}}{\Sigma^{M,\theta\theta} + \tau_{s}^{-1}}\left[\begin{array}{c} 1 \\ p_{v}\left(\rho_{v} - \rho_{N}\right) \end{array}\right]',$$

is the conditional variance of D_{t+1} and P_{t+1} with respect to $\mathcal{F}_{t}^{i}$. I can rewrite the above as

$$X_{t}^{i} = \frac{\left(1 + p_{v}\left(\rho_{v} - R^{f}\right) + \left[\begin{array}{c} p_{\hat{v}} - p_{v} \\ 0 \end{array}\right]' \mathbf{k}^{M}\left[\begin{array}{c} 1 \\ p_{v}\left(\rho_{v} - \rho_{N}\right) \end{array}\right]\right)\left(\hat{v}_{t+1}^{i} - \hat{v}_{t+1}^{M}\right) + p_{N}\left(\rho_{N} - R^{f}\right)\hat{N}_{t}^{i}}{\gamma\varphi'\Omega(i)\varphi},$$

by recognizing that $\Sigma^{M,vN} = -\frac{p_{v}}{p_{N}}\Sigma^{M,vv}$.

Substituting for $\hat{v}_{t+1}^{i}$, and recognizing from above that

$$\hat{N}_{t}^{M} = N_{t} + \frac{p_{v}}{p_{N}}\left(v_{t+1} - \hat{v}_{t+1}^{M}\right),$$

and therefore that

$$\hat{N}_{t}^{i} = \hat{N}_{t}^{M} - \frac{p_{v}}{p_{N}}\left(\hat{v}_{t+1}^{i} - \hat{v}_{t+1}^{M}\right) = N_{t} + \frac{p_{v}}{p_{N}}\left(v_{t+1} - \hat{v}_{t+1}^{M}\right) - \frac{p_{v}}{p_{N}}\left(\hat{v}_{t+1}^{i} - \hat{v}_{t+1}^{M}\right),$$

we arrive at

$$X_t^i = \frac{\left(\varphi' \begin{bmatrix} 1 \\ p_v\left(\rho_v - \rho_N\right) \end{bmatrix} \frac{\Sigma^{M,vv}}{\Sigma^{M,vv}+\tau_s^{-1}} \left(s_t^i - \hat{v}_{t+1}^M\right) + \left(\rho_N - R^f\right)\left(p_N N_t + p_v\left(v_{t+1} - \hat{v}_{t+1}^M\right)\right)\right)}{\gamma\varphi'\Omega\left(i\right)\varphi}.$$

Aggregating over the demand of investors and imposing market-clearing, we arrive at the two equations for p_v and p_N

$$\begin{aligned} \varphi' \begin{bmatrix} 1 \\ p_v\left(\rho_v - \rho_N\right) \end{bmatrix} \frac{\Sigma^{M,vv}}{\Sigma^{M,vv}+\tau_s^{-1}} + \left(\rho_N - R^f\right) p_v &= 0, \\ \frac{\left(\rho_N - R^f\right) p_N}{\gamma\varphi'\Omega\left(i\right)\varphi} &= 1. \end{aligned}$$

This completes our characterization of the linear equilibrium.

Proof of Proposition A3

To arrive at the beliefs of investors and the government, we first characterize the market beliefs based on the public information set $\mathcal{F}_t^M$. To derive the market beliefs, we proceed in several steps. First, we assume the market posterior belief of (v_{t+1}, N_t, G_{t+1}) is jointly Gaussian, $(v_{t+1}, N_t, G_{t+1}) \sim \mathcal{N}\left(\left(\hat{v}_{t+1}^M, \hat{N}_t^M, \hat{G}_{t+1}^M\right), \Sigma_t^M\right)$, where

$$\begin{aligned} \begin{bmatrix} \hat{v}_{t+1}^M \\ \hat{N}_t^M \\ \hat{G}_{t+1}^M \\ G_t \end{bmatrix} &= E\left[\begin{bmatrix} v_{t+1} \\ N_t \\ G_{t+1} \\ G_t \end{bmatrix} \mid \mathcal{F}_t^M\right], \\ \Sigma_t^M &= \begin{bmatrix} \Sigma_t^{M,vv} & \Sigma_t^{M,vN} & \Sigma_t^{M,vG_1} & 0 \\ \Sigma_t^{M,vN} & \Sigma_t^{M,NN} & \Sigma_t^{M,NG_1} & 0 \\ \Sigma_t^{M,vG_1} & \Sigma_t^{M,NG_1} & \Sigma_t^{M,G_1G_1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Standard results for the Kalman Filter then establish that the law of motion of the conditional expectation of the market's posterior beliefs $\left(\hat{v}_{t+1}^M, \hat{N}_t^M\right)$ is

$$\begin{bmatrix} \hat{v}_{t+1}^M \\ \hat{N}_t^M \\ \hat{G}_{t+1}^M \\ G_t \end{bmatrix} = \begin{bmatrix} \rho_v & 0 & 0 & 0 \\ 0 & \rho_N & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{v}_t^M \\ \hat{N}_{t-1}^M \\ \hat{G}_{t|t-1}^M \\ G_{t-1} \end{bmatrix} + \mathbf{K}_t^M \begin{bmatrix} D_t - \hat{v}_t^M \\ \eta_t^M - p_v\rho_v\hat{v}_t^M - p_N\rho_N\hat{N}_{t-1}^M \\ G_t - G_{t|t-1} \end{bmatrix},$$

where

$$\mathbf{K}_t^M = Cov\left[\begin{bmatrix} v_{t+1} \\ N_t \\ G_{t+1} \\ G_t \end{bmatrix}, \begin{bmatrix} D_t - \hat{v}_t^M \\ \eta_t^M - p_v\rho_v\hat{v}_{t-1}^M - p_N\rho_N\hat{N}_{t-1}^M \\ G_t - G_{t|t-1} \end{bmatrix} \mid \mathcal{F}_{t-1}^M\right]$$
$$\times Var\left[\begin{bmatrix} D_t - \hat{v}_t^M \\ \eta_t^M - p_v\rho_v\hat{v}_t^M - p_N\rho_N\hat{N}_{t-1}^M \\ G_t - G_{t|t-1} \end{bmatrix} \mid \mathcal{F}_{t-1}^M\right]^{-1},$$

is the Kalman Gain, and that the conditional variance Σ_t^M evolves deterministically according to

$$\Sigma_t^M = \begin{bmatrix} \rho_v & 0 & 0 & 0 \\ 0 & \rho_N & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Sigma_{t-1}^M \begin{bmatrix} \rho_v & 0 & 0 & 0 \\ 0 & \rho_N & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} \sigma_v^2 & 0 & 0 & 0 \\ 0 & \sigma_N^2 & 0 & 0 \\ 0 & 0 & \sigma_G^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$-\mathbf{K}_t^M Cov\left[\begin{bmatrix} D_t - \hat{v}_t^M \\ \eta_t^M - p_v\rho_v\hat{v}_t^M - p_N\rho_N\hat{N}_{t-1}^M \\ G_t - G_{t|t-1} \end{bmatrix}, \begin{bmatrix} v_{t+1} \\ N_t \\ G_{t+1} \end{bmatrix} \mid \mathcal{F}_{t-1}^M\right].$$

It is straightforward to compute that

$$Cov\left[\begin{bmatrix} v_{t+1} \\ N_t \\ G_{t+1} \\ G_t \end{bmatrix}, \begin{bmatrix} D_t - \hat{v}_t^M \\ \eta_t^M - p_v\rho_v\hat{v}_t^M - p_N\rho_N\hat{N}_{t-1}^M \\ G_t - G_{t|t-1} \end{bmatrix} \mid \mathcal{F}_{t-1}^M\right]$$
$$= \begin{bmatrix} \rho_v\Sigma_{t-1}^{M,vv} & p_v\left(\rho_v\Sigma_{t-1}^{M,vv} + \sigma_v^2\right) + p_N\rho_v\rho_N\Sigma_{t-1}^{M,vN} & \rho_v\Sigma_{t-1}^{M,vG_1} \\ \rho_N\Sigma_{t-1}^{M,vN} & p_v\rho_v\rho_N\Sigma_{t-1}^{M,vN} + p_N\left(\rho_N^2\Sigma_{t-1}^{M,NN} + \sigma_N^2\right) & \rho_N\Sigma_{t-1}^{M,NG_1} \\ 0 & p_G\sigma_G^2 & 0 \\ \Sigma_{t-1}^{M,vG_1} & p_v\rho_v\Sigma_{t-1}^{M,vG_1} + p_N\rho_N\Sigma_{t-1}^{M,NG_1} & \Sigma_{t-1}^{M,G_1G_1} \end{bmatrix},$$

and that

$$\Omega_{t-1}^M = Var\left[\begin{bmatrix} D_t - \hat{v}_t^M \\ \eta_t^M - p_v\rho_v\hat{v}_t^M - p_N\rho_N\hat{N}_{t-1}^M \\ G_t - G_{t|t-1} \end{bmatrix} \mid \mathcal{F}_{t-1}^M\right]$$
$$= \begin{bmatrix} \Sigma_{t-1}^{M,vv} + \sigma_D^2 & p_v\rho_v\Sigma_{t-1}^{M,vv} + p_N\rho_N\Sigma_{t-1}^{M,vN} & \Sigma_{t-1}^{M,vG_1} \\ p_v\rho_v\Sigma_{t-1}^{M,vv} + p_N\rho_N\Sigma_{t-1}^{M,vN} & \begin{pmatrix} p_v^2\left(\rho_v\Sigma_{t-1}^{M,vv} + \sigma_v^2\right) \\ +2p_vp_N\rho_v\rho_N\Sigma_{t-1}^{M,vN} \\ +p_N^2\left(\rho_N^2\Sigma_{t-1}^{M,NN} + \sigma_N^2\right) \\ +p_G^2\sigma_G^2 \end{pmatrix} & p_v\rho_v\Sigma_{t-1}^{M,vG_1} + p_N\rho_N\Sigma_{t-1}^{M,NG_1} \\ \Sigma_{t-1}^{M,vG_1} & p_v\rho_v\Sigma_{t-1}^{M,vG_1} + p_N\rho_N\Sigma_{t-1}^{M,NG_1} & \Sigma_{t-1}^{M,G_1G_1} \end{bmatrix}.$$

Since $\eta_t^M \in \mathcal{F}_t^M \subseteq \mathcal{F}_t$, I can express η_t^M as

$$\eta_t^M = p_v v_t + p_N N_t + p_G G_{t+1} = p_v \hat{v}_t^M + p_N \hat{N}_t^M + p_G \hat{G}_{t+1}^M,$$

from which follows that

$$p_v \left(v_t - \hat{v}_t^M\right) + p_N \left(N_t - \hat{N}_t^M\right) + p_G \left(G_{t+1} - \hat{G}_{t+1}^M\right) = 0.$$

As a consequence, it must be that the market beliefs about v_t and N_t are ex-post correlated after observing the stock price innovation process η_t^M, such that we have the three identities by taking its variance and its covariance with $v_{t+1} - \hat{v}_{t+1}^M$ and $N_t - \hat{N}_t^M$:

$$\begin{aligned}
\Sigma_t^{M,\theta N} &= -\frac{p_v}{p_N}\Sigma_t^{M,vv} - \frac{p_G}{p_N}\Sigma_t^{M,\theta G_1}, \\
\Sigma_t^{M,NN} &= -\frac{p_v}{p_N}\Sigma_t^{M,vN} - \frac{p_G}{p_N}\Sigma_t^{M,NG_1}, \\
\Sigma_t^{M,NG_1} &= -\frac{p_v}{p_N}\Sigma_t^{M,vG_1} - \frac{p_G}{p_N}\Sigma_t^{M,G_1G_1}.
\end{aligned}$$

This completes our characterization of the market's beliefs.

Proof of Proposition A4

Updating the market beliefs to each investor's private beliefs can be done in a manner similar to that in He and Wang (1995). Note that the market beliefs act as the prior for investor i who observes the normally distributed private signal s_t^i. The posterior of investor i is $N\left(\left(\hat{v}_{t+1}^i, \hat{N}_t^i, \hat{G}_{t+1|t}^i\right), \Sigma_t^i\right)$, where $\left(\hat{v}_{t+1}^i, \hat{N}_t^i, \hat{G}_{t+1|t}^i\right) = E\left[(v_t, N_t, G_{t+1}) \mid \mathcal{F}_t^i\right]$ and $\Sigma_t^s(i) = E\left[\begin{bmatrix} v_{t+1} - \hat{v}_{t+1}^i \\ N_t - \hat{N}_t^i \\ G_{t+1} - \hat{G}_{t+1|t}^i \end{bmatrix} \begin{bmatrix} v_{t+1} - \hat{v}_{t+1}^i \\ N_t - \hat{N}_t^i \\ G_{t+1} - \hat{G}_{t+1|t}^i \end{bmatrix}' \mid \mathcal{F}_t^i\right]$ are given by

$$\begin{bmatrix} \hat{v}_{t+1}^i \\ \hat{N}_t^i \\ \hat{G}_{t+1}^i \end{bmatrix} = \begin{bmatrix} \hat{v}_{t+1}^M \\ \hat{N}_t^M \\ \hat{G}_{t+1}^M \end{bmatrix} + \Gamma_t' \begin{bmatrix} s_t^i - \hat{v}_{t+1}^M \\ g_t^i - \hat{G}_{t+1}^M \end{bmatrix},$$

where

$$\begin{aligned}
\Gamma_t' &= Cov\left[\begin{bmatrix} v_{t+1} \\ N_t \\ G_{t+1} \end{bmatrix}, \begin{bmatrix} s_t - \hat{v}_{t+1}^M \\ g_t^i - \hat{G}_{t+1}^M \end{bmatrix} \mid \mathcal{F}_{t-1}^M\right] Var\left[\begin{bmatrix} s_t^i - \hat{v}_{t+1}^M \\ g_t^i - \hat{G}_{t+1}^M \end{bmatrix} \mid \mathcal{F}_t^M\right]^{-1} \\
&= \begin{bmatrix} \Sigma_t^{M,vv} & \Sigma_t^{M,vG_1} \\ \Sigma_t^{M,vN} & \Sigma_t^{M,NG_1} \\ \Sigma_t^{M,vG_1} & \Sigma_t^{M,G_1G_1} \end{bmatrix} \begin{bmatrix} \Sigma_t^{M,vv} + (a^i\tau_s)^{-1} & \Sigma_t^{M,vG_1} \\ \Sigma_t^{M,vG_1} & \Sigma_t^{M,G_1G_1} + \left[(1-a^i)\tau_g\right]^{-1} \end{bmatrix}^{-1},
\end{aligned}$$

and

$$\Sigma_t^s(i) = \Sigma_t^M - \Gamma_t^{i\prime} \begin{bmatrix} \Sigma_t^{M,vv} & \Sigma_t^{M,vG_1} \\ \Sigma_t^{M,vN} & \Sigma_t^{M,NG_1} \\ \Sigma_t^{M,vG_1} & \Sigma_t^{M,G_1G_1} \end{bmatrix}'.$$

Since G_t is publicly revealed, it is common knowledge and speculators need not update their beliefs about it with their private information. This characterizes the beliefs of investors given the market's beliefs.

Proof of Corollary 1

After the system has run for a sufficiently long time, initial conditions will diminish and the conditional variance of the Kalman Filter for the market beliefs Σ_t^M will settle down to its deterministic, covariance-stationary steady-state. To see this, let us conjecture that $\Sigma_t^M \to \Sigma^M$. In this proposed steady-state, $\Gamma_t \to \Gamma$, where Γ is given by

$$\Gamma = \begin{bmatrix} \Sigma^{M,vv} & \Sigma^{M,\theta G_1} \\ \Sigma^{M,vN} & \Sigma^{M,NG_1} \\ \Sigma^{M,vG_1} & \Sigma^{M,G_1G_1} \end{bmatrix} \begin{bmatrix} \Sigma^{M,vv} + (a^i\tau_s)^{-1} & \Sigma^{M,vG_1} \\ \Sigma^{M,vG_1} & \Sigma^{M,G_1G_1} + [(1-a^i)\tau_g]^{-1} \end{bmatrix}^{-1}.$$

Consequently, since Γ is indeed constant, so is $\Sigma^{M,vv}$. Furthermore, the steady-state Kalman Gain $\mathbf{K}^M$ is given by

$$\mathbf{K}^M = \begin{bmatrix} \rho_v\Sigma^{M,vv} & p_v\left(\rho_v\Sigma^{M,vv} + \sigma_v^2\right) + p_N\rho_v\rho_N\Sigma^{M,vN} & \rho_v\Sigma^{M,vG_1} \\ \rho_N\Sigma^{M,vN} & p_v\rho_v\rho_N\Sigma^{M,vN} + p_N\left(\rho_N^2\Sigma^{M,NN} + \sigma_N^2\right) & \rho_N\Sigma^{M,NG_1} \\ 0 & p_G\sigma_G^2 & 0 \\ \Sigma^{M,vG_1} & p_v\rho_v\Sigma^{M,\theta G_1} + p_N\rho_N\Sigma^{M,NG_1} & \Sigma^{M,G_1G_1} \end{bmatrix} \Omega^{M-1},$$

where

$$\Omega^M = \begin{bmatrix} \Sigma^{M,vv} + \sigma_D^2 & p_v\rho_v\Sigma^{M,vv} + p_N\rho_v\rho_N\Sigma^{M,vN} & \Sigma^{M,vG_1} \\ p_v\rho_v\Sigma^{M,vv} + p_N\rho_v\rho_N\Sigma^{M,vN} & \begin{pmatrix} p_v^2\left(\rho_v\Sigma^{M,vv} + \sigma_v^2\right) \\ +2p_vp_N\rho_v\rho_N\Sigma^{M,vN} \\ +p_N^2\left(\rho_N^2\Sigma^{M,NN} + \sigma_N^2\right) \\ +p_G^2\sigma_G^2 \end{pmatrix} & p_v\rho_v\Sigma^{M,vG_1} + p_N\rho_N\Sigma^{M,NG_1} \\ \Sigma^{M,vG_1} & p_v\rho_v\Sigma^{M,vG_1} + p_N\rho_N\Sigma^{M,NG_1} & \Sigma^{M,G_1G_1} \end{bmatrix}.$$

Consequently, since we have constructed a steady-state for the Kalman Filter for the market beliefs, such a steady-state exists.

Proof of Proposition A5

Similar to the problem for the government, it is convenient to define the state vector $\Psi_t = \left[\hat{v}_{t+1}^M, \hat{N}_t^M, \hat{G}_{t+1}^M, G_t\right]$ with law of motion

$$\Psi_{t+1} = \begin{bmatrix} \rho_v & 0 & 0 & 0 \\ 0 & \rho_N & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Psi_t + \mathbf{K}^M \varepsilon_{t+1}^M,$$

and $\varepsilon_{t+1}^M \mid \mathcal{F}_t^M \sim N\left(\mathbf{0}_{4\times 1}, \Omega^M\right)$ is given by

$$\varepsilon_{t+1}^M = \begin{bmatrix} D_{t+1} - \hat{v}_{t+1}^M \\ \eta_{t+1}^M - p_v \rho_v \hat{v}_{t+1}^M - p_N \rho_N \hat{N}_t^M \\ G_{t+1} - \hat{G}_{t+1}^M \end{bmatrix},$$

with Ω^M given in the proof of Corollary 1.

Given that excess payoffs are normally distributed, we can decompose R_{t+1} as

$$\begin{aligned} R_{t+1} &= E\left[R_{t+1} \mid \mathcal{F}_t^i\right] + \phi' \varepsilon_{t+1}^S \\ &= \varsigma \Psi_t + \phi' \omega \begin{bmatrix} \Sigma^{M,vv} + \left(a^i \tau_s\right)^{-1} & \Sigma^{M,vG_1} \\ \Sigma^{M,vG_1} & \Sigma^{M,G_1G_1} + \left[\left(1-a^i\right)\tau_g\right]^{-1} \end{bmatrix}^{-1} \begin{bmatrix} s_t^i - \hat{v}_{t+1}^M \\ g_t^i - \hat{G}_{t+1}^M \end{bmatrix} + \phi' \varepsilon_{t+1}^S \\ &= \varsigma \Psi_t + \frac{\phi' \omega \begin{bmatrix} \Sigma^{M,G_1G_1} + \left[\left(1-a^i\right)\tau_g\right]^{-1} & -\Sigma^{M,vG_1} \\ -\Sigma^{M,vG_1} & \Sigma^{M,vv} + \left(a^i \tau_s\right)^{-1} \end{bmatrix} \begin{bmatrix} s_t^i - \hat{v}_{t+1}^M \\ g_t^i - \hat{G}_{t+1}^M \end{bmatrix}}{\left(\Sigma^{M,vv} + \left(a^i \tau_s\right)^{-1}\right)\left(\Sigma^{M,G_1G_1} + \left[\left(1-a^i\right)\tau_g\right]^{-1}\right) - \left(\Sigma^{M,vG_1}\right)^2} + \phi' \varepsilon_{t+1}^S, \end{aligned}$$

with

$$\begin{aligned} \varsigma &= \begin{bmatrix} 1 + p_{\hat{v}}\left(\rho_v - R^f\right) & p_N\left(\rho_N - R^f\right) & p_g - R^f p_{\hat{G}} & -R^f p_g \end{bmatrix}, \\ \phi &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \mathbf{K}^{M\prime} \begin{bmatrix} p_{\hat{v}} - p_v \\ 0 \\ p_{\hat{G}} - p_G \\ p_g \end{bmatrix}. \end{aligned}$$

In this decomposition, we have updated the investor's beliefs sequentially from the market beliefs following Bayes' Rule as

$$\begin{aligned} & E\left[R_{t+1} \mid \mathcal{F}_t^i\right] \\ = \; & E\left[R_{t+1} \mid \mathcal{F}_t^M\right] + \phi' \omega \begin{bmatrix} \Sigma^{M,vv} + \left(a^i \tau_s\right)^{-1} & \Sigma^{M,\theta G_1} \\ \Sigma^{M,\theta G_1} & \Sigma^{M,G_1G_1} + \left[\left(1-a^i\right)\tau_g\right]^{-1} \end{bmatrix}^{-1} \begin{bmatrix} s_t^i - \hat{v}_{t+1}^M \\ g_t^i - \hat{G}_{t+1}^M \end{bmatrix} \\ = \; & \varsigma \Psi_t + \frac{\phi' \omega \begin{bmatrix} \Sigma^{M,G_1G_1} + \left[\left(1-a^i\right)\tau_g\right]^{-1} & -\Sigma^{M,vG_1} \\ -\Sigma^{M,vG_1} & \Sigma^{M,vv} + \left(a^i \tau_s\right)^{-1} \end{bmatrix} \begin{bmatrix} s_t^i - \hat{v}_{t+1}^M \\ g_t^i - \hat{G}_{t+1}^M \end{bmatrix}}{\left(\Sigma^{M,vv} + \left(a^i \tau_s\right)^{-1}\right)\left(\Sigma^{M,G_1G_1} + \left[\left(1-a^i\right)\tau_g\right]^{-1}\right) - \left(\Sigma^{M,vG_1}\right)^2}, \end{aligned}$$

where, as in Proposition 4,

$$\begin{aligned}\omega &= Cov\left[\varepsilon_{t+1}^{M}, \begin{bmatrix} v_{t+1} \\ G_{t+1} \end{bmatrix}' \mid \mathcal{F}_t^M\right] \\ &= \begin{bmatrix} \Sigma^{M,vv} & \Sigma^{M,vG_1} \\ p_v\rho_v\Sigma^{M,vv} + p_N\rho_N\Sigma^{M,\theta N} & p_v\rho_v\Sigma^{M,vG_1} + p_N\rho_N\Sigma^{M,NG_1} \\ \rho_v\Sigma^{M,vG_1} & \Sigma^{M,G_1G_1} \end{bmatrix}.\end{aligned}$$

Similarly, by Bayes' Rule, $\varepsilon_{t+1}^{S} \mid \mathcal{F}_t^i \sim \mathcal{N}\left(\mathbf{0}_{2\times 1}, \Omega^S\right)$, where

$$\Omega^S = \Omega^M - \frac{\omega \begin{bmatrix} \Sigma^{M,G_1G_1} + \left[(1-a^i)\tau_g\right]^{-1} & -\Sigma^{M,vG_1} \\ -\Sigma^{M,vG_1} & \Sigma^{M,vv} + \left(a^i\tau_s\right)^{-1} \end{bmatrix} \omega'}{\left(\Sigma^{M,vv} + \left(a^i\tau_s\right)^{-1}\right)\left(\Sigma^{M,G_1G_1} + \left[(1-a^i)\tau_g\right]^{-1}\right) - \left(\Sigma^{M,vG_1}\right)^2}.$$

Standard results establish that the investor's problem is equivalent to the mean-variance optimization program

$$\sup_{X_t(i)} \left\{R^f\bar{W} + X_t^i E\left[R_{t+1} \mid \mathcal{F}_t^i\right] - \frac{\gamma}{2}X_t^{i2}Var\left[R_{t+1} \mid \mathcal{F}_t^i\right]\right\}.$$

Importantly, since the investors have to form conditional expectations about excess payoffs at $t+1$, they must form conditional expectations about the government's future trading $E\left[G_{t+1} \mid \mathcal{F}_t^i\right]$. Given that the investors are price-takers, from the FOC we see that the optimal investment of investor i in the risky asset is given by

$$\begin{aligned}X_t^i &= \frac{E\left[R_{t+1} \mid \mathcal{F}_t^i\right]}{\gamma Var\left[R_{t+1} \mid \mathcal{F}_t^i\right]} \\ &= \frac{1}{\gamma}\frac{\varsigma\Psi_t + \frac{\phi'\omega \begin{bmatrix} \Sigma^{M,G_1G_1} + \left[(1-a^i)\tau_g\right]^{-1} & -\Sigma^{M,vG_1} \\ -\Sigma^{M,vG_1} & \Sigma^{M,vv} + \left(a^i\tau_s\right)^{-1} \end{bmatrix} \begin{bmatrix} s_t^i - \hat{v}_{t+1}^M \\ g_t^i - \hat{G}_{t+1}^M \end{bmatrix}}{\left(\Sigma^{M,vv}+(a^i\tau_s)^{-1}\right)\left(\Sigma^{M,G_1G_1}+\left[(1-a^i)\tau_g\right]^{-1}\right)-\left(\Sigma^{M,vG_1}\right)^2}}{\phi'\Omega^M\phi - \frac{\phi'\omega \begin{bmatrix} \Sigma^{M,G_1G_1} + \left[(1-a^i)\tau_g\right]^{-1} & -\Sigma^{M,vG_1} \\ -\Sigma^{M,\theta G_1} & \Sigma^{M,vv} + \left(a^i\tau_s\right)^{-1} \end{bmatrix}\omega'\phi}{\left(\Sigma^{M,vv}+(a^i\tau_s)^{-1}\right)\left(\Sigma^{M,G_1G_1}+\left[(1-a^i)\tau_g\right]^{-1}\right)-\left(\Sigma^{M,vG_1}\right)^2}}.\end{aligned}$$

This completes our characterization of the optimal trading policy of the investors.

Proof of Proposition A6

Each investor faces the optimization problem (A1) given in the main paper. It then follows that investor i will choose to learn about the payoff fundamental v_t (i.e, $a_t^i = 1$) with probability λ:

$$\lambda = \begin{cases} 1, & Q < 0 \\ (0,1), & Q = 0 \\ 0, & Q > 0, \end{cases},$$

where

$$Q = \phi' \left(M(0) - M(1) \right) \phi = \phi' \omega \begin{bmatrix} -\frac{1}{\Sigma^{M,vv} + \tau_s^{-1}} & 0 \\ 0 & \frac{1}{\Sigma^{M,G_1G_1} + \tau_g^{-1}} \end{bmatrix} \omega' \phi.$$

Given ω, we can expand out this condition to arrive at

$$\begin{aligned} Q &= \frac{\left(\begin{array}{c} \left(1 + (p_{\hat{v}} - p_v)\mathbf{K}^M_{1,1} + (p_{\hat{g}} - p_g)\mathbf{K}^M_{3,1} + (p_{\hat{G}} - p_G)\mathbf{K}^M_{4,1}\right)\Sigma^{M,\theta G_1} \\ + \left(1 + (p_{\hat{v}} - p_v)\mathbf{K}^M_{1,2} + (p_{\hat{g}} - p_g)\mathbf{K}^M_{3,2} + (p_{\hat{G}} - p_G)\mathbf{K}^M_{4,2}\right) \\ \times \left(p_v\rho_v\Sigma^{M,vG_1} + p_g\Sigma^{M,G_1G_1} + p_N\rho_N\Sigma^{M,NG_1}\right) \end{array} \right)^2}{\Sigma^{M,G_1G_1} + \tau_g^{-1}} \\ &\quad - \frac{\left(\begin{array}{c} \left(1 + (p_{\hat{v}} - p_v)\mathbf{K}^M_{1,1} + (p_{\hat{g}} - p_g)\mathbf{K}^M_{3,1} + (p_{\hat{G}} - p_G)\mathbf{K}^M_{4,1}\right)\Sigma^{M,vv} \\ + \left(1 + (p_{\hat{v}} - p_v)\mathbf{K}^M_{1,2} + (p_{\hat{g}} - p_g)\mathbf{K}^M_{3,2} + (p_{\hat{G}} - p_G)\mathbf{K}^M_{4,2}\right) \\ \times \left(p_v\rho_v\Sigma^{M,vv} + p_g\Sigma^{M,vG_1} + p_N\rho_N\Sigma^{M,vN}\right) \end{array} \right)^2}{\Sigma^{M,vv} + \tau_s^{-1}} \end{aligned}$$

Recognizing that $\phi'\omega = Cov\left[R_{t+1}, \begin{bmatrix} v_{t+1} \\ G_{t+1} \end{bmatrix} \mid \mathcal{F}_t^M\right]$, we can rewrite the above more generally as

$$Q = \frac{Cov\left[R_{t+1}, G_{t+1} \mid \mathcal{F}_t^M\right]^2}{\Sigma^{M,G_1G_1} + \tau_g^{-1}} - \frac{Cov\left[R_{t+1}, v_{t+1} \mid \mathcal{F}_t^M\right]^2}{\Sigma^{M,vv} + \tau_s^{-1}}.$$

Proof of Proposition A7

Since the government does not have any additional information to that of the market, it has the market beliefs. As described in the main paper, it is convenient to define the state vector Ψ_t, which follows a VAR(1) process in the covariance-stationary equilibrium of the economy given by Proposition A5.

Given the results in Proposition A5, the government's policy rule

$$X_t^G = \vartheta_{\hat{N}} \hat{N}_t^M + \sqrt{Var\left[\vartheta_{\hat{N}} \hat{N}_t^M \mid \mathcal{F}_{t-1}^M, \{a_t^i\}_i\right]} G_t,$$

and it follows that

$$\begin{aligned} Var\left[\Delta P_{t+1} \mid \mathcal{F}_t^M\right] &= \left(\phi - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)' \Omega^M \left(\phi - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right), \\ Var\left[X_{t+1}^G \mid \mathcal{F}_t^M\right] &= H, \end{aligned}$$

where $\vartheta = \begin{bmatrix} 0 & \vartheta_{\hat{N}} & 0 & 0 \end{bmatrix}'$ and

$$\begin{aligned} H &= \left(1 + \Sigma^{M,G_1G_1}\right) \vartheta' \mathbf{K}^M \Omega^M \mathbf{K}^M \vartheta \\ &\quad + 2\left(\sqrt{\vartheta' \mathbf{K}^M \Omega^M \mathbf{K}^M \vartheta}\right) \vartheta' \mathbf{K}^M \begin{bmatrix} \Sigma^{M,vG_1} \\ p_v\rho_v\Sigma^{M,vG_1} + p_N\rho_N\Sigma^{M,NG_1} \\ \Sigma^{M,G_1G_1} \end{bmatrix}. \end{aligned}$$

In addition, the price volatility can be expressed as

$$Var\left[P_{t+1} \mid \mathcal{F}_t^M\right] = \left(\phi - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)' \Omega^M \left(\phi - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$$

Finally, we can express the conditional uncertainty about the deviation in the asset price from its fundamentals as

$$\begin{aligned} F &= Var\left[P_{t+1} - p_{\hat{v}} v_{t+2} \mid \mathcal{F}_t^M\right] \\ &= Var\left[\left(\phi - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)' \varepsilon_{t+1}^M - p_{\hat{v}}\left(v_{t+2} - \rho_v \hat{v}_{t+1}^M\right) \mid \mathcal{F}_t^M\right] \\ &= \left(\phi - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)' \Omega^M \left(\phi - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + p_{\hat{v}}^2\left(\rho_v \Sigma^{M,vv} + \sigma_v^2\right) \\ &\quad -2p_{\hat{v}}\left(\phi - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)' \begin{bmatrix} \rho_v \Sigma^{M,vv} \\ p_v\left(\rho_v \Sigma^{M,vv} + \sigma_v^2\right) + p_N \rho_v \rho_N \Sigma^{M,vN} \\ \rho_v \Sigma^{M,vG_1} \end{bmatrix}. \end{aligned}$$

It follows in the covariance-stationary equilibrium that we can express the government's objective as

$$U^G = \sup_{\vartheta} -\gamma_\sigma \left(\phi - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)' \Omega^M \left(\phi - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) - \gamma_v F - \psi H.$$

References

Boyd, John H. (1990), Recursive Utility and the Ramsey Problem, *Journal of Economic Theory,* 50, 326-345.