A Variable Threat Game of *Ransom*

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I had the privilege and pleasure of John and Alicia Nash's friendship for almost a quarter century. When I think of John, the words that spring to my mind are modesty, courtesy, rationality, and precision; when I think of Alicia, the words are quiet courage and love.

For someone who made such pathbreaking and deep contributions to mathematics and game theory, John was incredibly modest. At conferences or seminars he never tried to attract attention to himself, and was surprised (but clearly happy) when attention found him, which it always did. He was unfailingly courteous and attentive to all – students and senior professors alike – and had an amazingly good memory for names and faces of people he had met some years ago. In matters of rationality and precision, John perhaps overcompensated for what he called his "years of irrational thinking." Indeed, he may have carried this logic too far in his advocacy of "ideal money," a monetary standard as precise as the standard meter and kilogram preserved in Paris. The very word "standard" connotes precision, and its implications had to be followed through logically to the end. Most of us would not want such a rigid monetary standard and would place more value on flexibility to respond to shocks to the economy, but John's logic gives us a clear and rigorous statement of the case for a rigid commitment.

Of course I knew of John's foundational research in game theory for almost half a century, and used the concepts of Nash equilibrium and the Nash bargaining solution more times than I can recall. In an earlier tribute I said that if John got a dollar every time someone wrote or said "Nash equilibrium," he would be a rich man.[1] The same goes for the Nash bargaining solution, which is extensively used in

[1] "John Nash – Founder of Modern Game Theory," in *Game Theory: A Festschrift in Honor of John Nash,* eds. Constantina Kottaridi and Gregorios Siouroun, Athens: Eurasia Publications, 2002, pp. 98-100. The book is in Greek; the English version is on my web site, http://www.princeton.edu/~dixitak/home/nashenco.pdf

labor economics, international trade, and many other areas of economics and political science.

No words can suffice to describe Alicia's courage and dedication to John and their son Johnny. I am sure John's recovery is mostly due to her care and love. She is the true heroine of the incredible story of their lives.

Abstraction versus Illustration

It has been said many times that the essence of mathematics is abstraction,[2] and many mathematicians revel in the most abstract possible definitions and propositions on any subject. But John rose to a higher level in recognizing that the best entrée into the world of mathematical concepts is through specific, simple, and memorable examples. His classic paper on bargaining illustrated the general solution concept with an exchange of goods like a ball, a bat, a pen, and a hat.[3] In a very helpful and perceptive quote he provided for my textbook, he stated this pedagogical philosophy succinctly: "The generous variety of illustrative cases has the effect that what is learned can be more easily retained than if there were only the assertions of theoretical concepts without enlightening examples." [4]

Therefore I have chosen to write about an example that vividly illustrates John's extension of his classic paper, namely variable threat bargaining.[5] Nash's original bargaining solution was formalized as a two-player cooperative game G, where the players communicate to agree upon their strategies, and these choices

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[2] For example, see *Mathematics: A Very Short Introduction* by Timothy Gowers, Oxford University Press, 2002.

[3] John F. Nash, Jr., "The bargaining problem," *Econometrica* 18:155-162, 1950.

[4] See the back cover of Avinash Dixit, Susan Skeath and David Reiley, *Games of Strategy,* New York: W. W. Norton, fourth edition 2015.

[5] John F. Nash, Jr., "Two-person cooperative games," *Econometrica* 21:128-140, 1953. My formal statement follows R. Duncan Luce and Howard Raiffa, *Games and Decisions,* New York: Wiley, 1957, pp. 140-141.

are externally enforceable. There is a specified payoff vector (v_1, v_2) which will result if the players fail to reach an agreement. This is often called the threat point, or BATNA (Best Alternative To Negotiated Agreement) in the jargon of the Harvard Business School's negotiation project.[6] The set of all feasible payoffs constitutes a compact set $P \subset \mathbb{R}^2$; the subset consisting of undominated payoffs is the bargaining frontier B. (Formally, $(b_1, b_2) \epsilon B$ if and only if there is no $(p_1, p_2) \epsilon P$ such that $p_1 \geq b_1$and $p_2 \geq b_2$ with at least one of the inequalities strict; informally, B is the north-east frontier of P.) The Nash solution is the (x_1, x_2) in P that maximizes the product $(x_1 - v_1)(x_2 - v_2)$. A generalized version has the solution maximizing $(x_1 - v_1)^\theta (x_2 - v_2)^{1-\theta}$, where $0 < \theta < 1$, and $\theta, 1 - \theta$ can represent the relative bargaining strengths of the two parties, or their relative merits in the eyes of an arbitrator.[7] (The original Nash solution is equivalent to setting $\theta = 1/2$.) Is trivial to prove that $(x_1, x_2) \epsilon B$.

In variable threat bargaining, a non-cooperative game G^* precedes G. Its Nash equilibrium payoffs (v_1, v_2) constitute the BATNA of G. When choosing his/her strategy in G^*, each player will seek to achieve the outcome that will yield the best payoff for him/her in the ensuing Nash cooperative solution of G.[8]

What is all this telling us about threats in bargaining? Most people, even mathematicians, on a first reading will be somewhat baffled by the abstract formulation. A vivid and memorable example will clarify it.

[6] Roger Fisher and William Ury, *Getting to Yes,* Boston: Houghton Mifflin, 1981.

[7] See Roger Myerson, *Game Theory,* Cambridge, MA: Harvard University Press, 1991, pp. 379, 390.

[8] This process of looking ahead to the outcome of a later game to choose strategies in a prior game is an early instance of the concept of subgame perfectness, later made rigorous and famous by Reinhard Selten in "Reexamination of the perfectness concept in extensive games," *International Journal of Game Theory* 4(1):25-55, 1975.

Ransom

In the movie *Ransom,* the son of multimillionaire Tom Mullen (played by Mel Gibson) has been kidnapped. The man holding him is demanding a ransom of two million dollars. Mullen goes on live TV with the money spread out on a table before him, and makes the following announcement: "The whole world now knows ... my son, Sean Mullen, was kidnapped, for ransom, three days ago. This is a recent photograph of him. Sean, if you're watching, we love you. And this ... well, this is what waits for the man that took him. This is your ransom. Two million dollars in unmarked bills, just like you wanted. But this is as close as you'll ever get to it. You'll never see one dollar of this money, because no ransom will ever be paid for my son. Not one dime, not one penny. Instead, I'm offering this money as a reward on your head. Dead or alive, it doesn't matter. So congratulations, you've just become a two million dollar lottery ticket ... except the odds are much, much better. Do you know anyone that wouldn't turn you in for two million dollars? I don't think you do. I doubt it. So wherever you go and whatever you do, this money will be tracking you down for all time. And to ensure that it does, to keep interest alive, I'm running a full-page ad in every major newspaper every Sunday ... for as long as it takes. But ... and this is your last chance ... you return my son, alive, uninjured, I'll withdraw the bounty. With any luck you can simply disappear. Understand ... you will never see this money. Not one dollar. So you still have a chance to do the right thing. If you don't, well, then, God be with you, because nobody else on this Earth will be." [9]

Let us represent this in game-theoretic language. Call Tom Mullen player 1; the kidnapper Jimmy Shaker (player by Gary Sinise) is player 2. Figure 1 shows their payoffs. The origin is at the point of their initial wealths. Before the kidnapping, Mullen also has his son; denote his value or utility from that in money-equivalent terms by a. So the payoff point in the status quo ex ante is $(a, 0)$, or the point Q in

[9] The text comes from http://www.imdb.com/title/tt0117438/?ref_=nv_sr_1, accessed February 16, 2016.

the figure. The line through Q with slope -1 shows all attainable payoff points starting at Q and transferring money between the parties, and is therefore the bargaining frontier.

After the kidnapping, if negotiation fails, Mullen will lose his son but Shaker won't get any money, so Shaker's threat point T is the origin. He asks for $2 million. If this goes through and Mullen gets his son back, the payoffs will be $(a-2{,}2)$, shown as the point P.

Mullen's strategy changes the threat point. Now if the negotiation fails, Mullen will lose his son and end up paying the $2 million bounty to the person who kills Shaker (probably one of Shaker's confederates), while Shaker will lose his life. Let Shaker's valuation of his own life be denoted by b. Then the payoffs at the new BATNA are $(-2,-b)$. The figure shows this as the point T*.

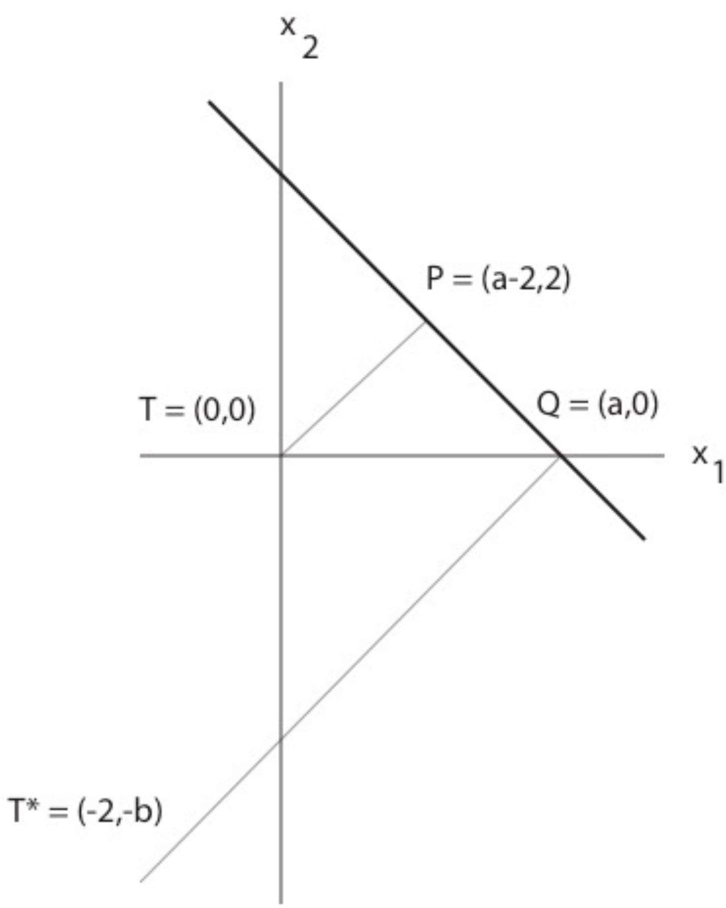

Figure 1: Various payoff points in the *Ransom* game

With this new threat point, Mullen offers Shaker his solution to the bargaining problem, namely going back to the status quo point Q: bring my son back unharmed and I will withdraw the bounty on your head.[10]

The figure is drawn as if these solutions conform to the Nash bargaining solution. It is easy to verify that this implicitly sets $a = 4$ and $b = 6$. For other values of a and b, the Nash solution for Mullen's counterproposal need not be exactly at Q. And the game G may be played in some way other than Nash bargaining, for example the threatener may be able to make a take-it-or-leave-it offer to the other player. But the following reasoning yields some general conclusions applicable to all such variants.

When will Mullen's strategy give him an outcome better than the one he would get by acceding to Shaker's original demand? If the Nash solution for the threat point T* is to the south-east of that for T along the bargaining frontier. This happens if the line T*Q lies below the line TP, that is, if $b > 2$, that is, if Shaker values his own life more than the ransom money. Since T* is to the south-west of T, in changing the threat point from T to T* Mullen worsens both BATNAs. His strategy aims to achieve Shaker's acceptance of the alternative proposal, because it carries the threat of an even bigger loss for Shaker than for himself if the negotiation fails. In other words, Mullen is implicitly saying to Shaker: "This will hurt you more than it will hurt me." We often hear such statements made in arguments and disputes; now we see the strategic role they play in negotiations.

If $b > 6$, the line from T* will meet the bargaining frontier at a point south-east of $(a, 0)$. It will therefore correspond to a negative payoff for Shaker, that is, he will end up with wealth below his original level. It is as if he is paying Mullen to take his son back! This may be impossible, and if Mullen's threat T* is that severe, the

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10 Movies have their own requirements of dramatic tension and denouement that override game-theoretic logic. To conform with those demands, *Ransom* does not have any efficient resolution on the bargaining frontier where one of the players accedes to the other's demand, but twists that end in chases and gunfights. But that is not material to the basic bargaining game I want to illustrate.

outcome may be a corner solution at Q. However, it is not outside the realm of possibilities that kidnappers agree to pay to be rid of their hostage, at least in some other branches of fiction.[11]

The variable threat strategy of "This will hurt you more than it will hurt me" has been used in real life. For example, smart labor unions threaten or launch strikes at times when that will deliver the biggest hit to the firms' profits. British coal miners' union did this consistently in the 1970s. Conversely, Mrs. Thatcher's strategy of provoking the union's leader Arthur Scargill into striking in the spring and summer of 1984 was instrumental in the collapse of the strike, and led to a collapse of the union itself.

The same strategy was used in the baseball strike of 1980.[12] The strike started during the exhibition games of preseason. The players returned to work (actually, to play) at the start of the regular season, but resumed the strike after Memorial Day. This curious discontinuous strike can be understood when we examine the time-varying costs of the strike to the two sides. During the exhibition games period, the players are not paid salaries but the owners earn substantial revenues from fans who combine a vacation in a warmer clime with following their favorite team's stars and prospects. Once the regular season starts, the players get salaries, but attendances at games, and therefore the owners' revenues, grow substantially only after Memorial Day. Therefore the discontinuous strike was the players' correct strategy to maximize the owners' loss relative to their own.

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[11] In O. Henry's short story "The Ransom of Red Chief," two small-time crooks kidnap a banker's 10-year-old son. He turns out to be a brat who makes their lives so impossible that they pay the father to take him back. The text is available from http://fiction.eserver.org/short/ransom_of_red_chief.html , accessed February 17, 2016.

[12] Lawrence M. DeBrock and Alvin E. Roth, "Strike Two: Labor-management negotiations in major league baseball," *Bell Journal of Economics* 12(2):413-425, 1981.

Concluding Comments

You may have already forgotten the formal definitions of variable threat bargaining, but I guarantee that you will not forget the examples. And with the examples in mind, any mathematician will easily be able to reconstruct the formalism. Therefore I hope I have convinced readers of the merits of the pedagogical philosophy I am happy to have shared with John Nash: vivid examples can convey concepts and even formal methods of mathematical theories better and more memorably than purely algebraic or symbolic statements.