Whence the complex numbers?

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After all these years, we still do not fully understand why the complex numbers $\mathbb{C}$ play such a central role in our best theories of physical reality, namely quantum mechanics and quantum field theory. Hopefully something in future physics — e.g. strings? mirror symmetry? quantum foam? — will clarify the issue. For the time being, we must rest content with cataloging reasons why the real numbers $\mathbb{R}$ will not suffice.

One of the reasons that $\mathbb{R}$ will not suffice is because we cannot represent the uncertainty relations. To be more precise, a real vector space V does not admit quantities q and p that are canonically conjugate. This fact can easily be seen by noting that each quantity corresponds to an orthonormal basis, and canonical conjugacy of quantities corresponds to the bases being "perfectly skewed" relative to each other.

Proposition. *Let V be a 3-dimensional vector space over $\mathbb{R}$. There are no two orthonormal bases $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ such that $\langle x_i, y_j \rangle^2 = \frac{1}{3}$ for all i, j.*

Proof. If there were two such bases then we would have

$$y_1 = \langle x_1, y_1 \rangle x_1 + \langle x_2, y_1 \rangle x_2 + \langle x_3, y_1 \rangle x_3,$$
$$y_2 = \langle x_1, y_2 \rangle x_1 + \langle x_2, y_2 \rangle x_2 + \langle x_3, y_2 \rangle x_3,$$

but since these two vectors are orthogonal, it follows that

$$0 = \langle x_1, y_1 \rangle \langle x_1, y_2 \rangle + \langle x_2, y_1 \rangle \langle x_2, y_2 \rangle + \langle x_3, y_1 \rangle \langle x_3, y_2 \rangle.$$

Now each term on the right hand side is either $\frac{1}{3}$ or $-\frac{1}{3}$, and obviously these can sum only to $-1, -\frac{1}{3}, \frac{1}{3}$ or 1, a contradiction. □