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Journal Title: C*-algebras and their applications to statistical mechanics and quantum field theory
Article Author: Roberts, J. E.
Article Title: Statistics and the intertwiner calculus

Journal Vol: Journal Issue:
Journal Month: Journal Year: 1976
Article Pages: 203-225

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Statistics and the Intertwiner Calculus.

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1. – The origin of statistics.

One of the insights provided by the study of superselection sectors concerns the origin of what is termed the « statistics » of a particle. You must all have met this term and know that the particles occurring in nature are divided into bosons and fermions according as their n-particle wave functions are totally symmetric or totally antisymmetric. Now just as a theory should determine its particle states so should it determine the statistics of these particles. Ordinary quantum mechanics ignores this challenge saying in effect that the statistics of particles is one of the parameters determining the theory, the one telling you what symmetry the n-particle states have. Quantum field theory does a little better: it says that the statistics of the particles in the theory is determined by the commutation relations of the fields at spacelike separations. The commutation relations are part of the input but one might have one fundamental field determining the statistics of many different particles. Furthermore for fields with a finite number of components one derives a connexion between spin and statistics.

In adopting the philosophy that the local observables determine the theory, we are forced to meet this challenge in full. In the case of observables, commutativity at spacelike separations has a physical interpretation in terms of relativistic causality and so is not a free parameter.

If we want to pass from commutation relations to particle statistics then, just as in quantum field theory, we have to construct the n-particle scattering states by a limiting process. This is a rather lengthy procedure but fortunately the concept of statistics does not have to be restricted to particle states. Instead we shall introduce statistics in terms of localized states where the role of particle states of identical particles is played by localized states carrying identical charge quantum numbers, *i.e.* belonging to the same sector. In this way we shall arrive at the statistics of a sector and can postpone the problem of showing that the statistics of a particle is the same as the statistics of the of the sector to which it belongs (*).

Underlying any notion of statistics is a notion of the product of states. In our case we need to define the product of localized states of the form $\omega_0 \circ \varrho$, where ϱ is a localized morphism (**). Notice that the state $\omega_0 \circ \varrho$ determines ϱ uniquely because the vacuum is cyclic for the algebra $\mathfrak{A}(\mathcal{O}')$ for any double cone $\mathcal{O}$. Given any two states $\omega_i = \omega_0 \circ \varrho_i$ localized in double cones $\mathcal{O}_i$, $i = 1, 2$ with $\mathcal{O}_1 \subset \mathcal{O}_2'$, we have a well-defined product state $\omega = \omega_0 \circ \varrho_1 \varrho_2$. Now $\varrho_1 \varrho_2 = \varrho_2 \varrho_1$ so you might think we are predicting Bose statistics. However the statistics is latent and to see it one must consider the product of state vectors rather than states.

A *state vector* is simply a nonzero vector in a Hilbert space carrying a representation of the observable algebra. As all the representations we consider are defined by localized morphisms and act in the same space, we must take particular care to indicate the representation when talking about a state vector. Thus for our purposes, I shall define a state vector $\boldsymbol{\Psi}$ to be a pair $\boldsymbol{\Psi} = \{\varrho; \Psi\}$ with $0 \neq \Psi \in \mathscr{H}_0$ and ϱ a localized morphism. The corresponding state of $\mathfrak{A}$ is then just $\|\Psi\|^{-2}(\Psi, \varrho(A)\Psi)$. Hence we look upon state vectors as elements of a vector bundle whose fibers $\mathscr{H}_\varrho$ are copies of $\mathscr{H}_0$ labelled by the localized morphismsm.

We consider the product of states $\omega_0 \circ \varrho_2' \times \omega_0 \circ \varrho_1' = \omega_0 \circ \varrho_1' \varrho_2'$ where $\varrho_i' \simeq \varrho_i$ is localized in $\mathcal{O}_i$ and $\mathcal{O}_1 \subset \mathcal{O}_2'$. We then have unitaries U_i such that $\varrho_i = \sigma_{U_i} \varrho_i'$ and hence $\{\varrho_i; U_i \Omega\}$ is a state vector inducing $\omega_0 \circ \varrho_i'$. We define

$$\{\varrho_2;\, U_2\Omega\} \times \{\varrho_1;\, U_1\Omega\} = \{\varrho_1\varrho_2;\, \varrho_1(U_2)\, U_1\Omega\}\,, \tag{1.1}$$

and note that $\{\varrho_1\varrho_2, \varrho_1(U_2)\, U_1\Omega\}$ induces $\omega_0 \circ \varrho_1' \varrho_2'$. It turns out that this product is not commutative and this is the origin of the different statistics. The definition (1.1) is simple but the reason why it is essentially forced on you is that it extends to a densely defined *bilinear* mapping $\mathscr{H}_{\varrho_2} \times \mathscr{H}_{\varrho_1} \to \mathscr{H}_{\varrho_1\varrho_2}$,

$$\{\varrho_2;\, U_2A_2\Omega\} \times \{\varrho_1;\, U_1A_1\Omega\} = \{\varrho_1\varrho_2;\, \varrho_1(U_2A_2)\, U_1A_1\Omega\}\,, \qquad A_i \in \mathfrak{A}(\mathcal{O}_i)\,. \tag{1.2}$$

Note that the case where A_1 and A_2 are unitary is covered by (1.1).

If you are unhappy that something physical like statistics should manifest itself in the passage from states to state vectors let me remind you that the

(*) This is one of the topics dealt with by HAAG in his lectures.

(**) A general introduction to superselection sectors explaining the connexion with localized morphisms may be found in Doplicher's lectures. I have made free use of the terminology and results contained there. Apart from one or two remarks in this introductory Section, I shall restrict myself to assumptions 1 to 4 stated by DOPLICHER. These correspond to the assumptions used in [1].

difference between integral and half-integral spin appears in the same way. It does not show up directly in the states which are always invariant under a rotation of 2π but shows up because one cannot always find a representation of the Poincaré group by unitary (hence *linear*) operators on state vectors and must pass to the covering group.

We now examine the lack of commutativity in the product (1.2): $\varrho_1'(A_2) = A_2$ by choice of supports, so $\varrho_1(A_2)\,U_1 = U_1 A_2$, hence

$$\{\varrho_2;\; U_2 A_2 \Omega\}\times\{\varrho_1;\; U_1 A_1\, \Omega\} = \{\varrho_1\varrho_2;\; \varrho_1(U_2)\,U_1 A_2 A_1 \Omega\}\,,$$
$$\{\varrho_1;\; U_1 A_1 \Omega\}\times\{\varrho_2;\; U_2 A_2 \Omega\} = \{\varrho_2\varrho_1;\; \varrho_2(U_1)\,U_2 A_1 A_2 \Omega\}\,.$$

Noting that $A_1 A_2 = A_2 A_1$, we see that the unitary operator $\varepsilon(\varrho_1, \varrho_2) = \varrho_2(U_1)\,U_2\,U_1^{-1}\varrho_1(U_2^{-1})$ maps $\varrho_1(U_2)\,U_1 A_2 A_1 \Omega$ into $\varrho_2(U_1)\,U_2 A_1 A_2 \Omega$. It also intertwines $\varrho_1\varrho_2$ and $\varrho_2\varrho_1$:

$$\varepsilon(\varrho_1, \varrho_2)\,\varrho_1\varrho_2(A) = \varrho_2\varrho_1(A)\,\varepsilon(\varrho_1, \varrho_2)\,, \qquad A \in \mathfrak{A}\,, \tag{1.3}$$

and we shall see that it is even independent of the choice of spacelike double cones $\mathcal{O}_1$ and $\mathcal{O}_2$.

To analyse statistics rather than the product of state vectors we need to take $\varrho_1 = \varrho_2$ (identical charges) and we must also generalize to arbitrary permutations of n variables. At this point the time has come to be a little more systematic and begin studying intertwining operators.

2. – Intertwiners and permutation symmetry.

Let ϱ and ϱ' be localized morphisms. An operator $S \in \mathscr{B}(\mathscr{H}_0)$ is said to intertwine from ϱ to ϱ' if

$$\varrho'(A)\,S = S\varrho(A)\,, \qquad A \in \mathfrak{A}\,. \tag{2.1}$$

A triple $\boldsymbol{S} = (\varrho'|S|\varrho)$ satisfying (2.1) will be called an *intertwiner* and the linear space of intertwiners from ϱ to ϱ' will be denoted by $\mathscr{I}(\varrho, \varrho')$. There is obviously an adjoint of intertwiners defined by $\boldsymbol{S}^* = (\varrho|S^*|\varrho')$ and a composition of intertwiners defined if the adjoining morphisms coincide

$$(\varrho''|S'|\varrho')\circ(\varrho'|S|\varrho) = (\varrho''|S'S|\varrho)\,. \tag{2.2}$$

Those who are familiar with category theory may profitably think in terms of the set of intertwiners as a category whose « objects » are localized morphisms and whose « morphisms » are the intertwiners but I do not intend to stress this aspect.

What I do wish to stress is that the set of intertwiners will turn out to have a rich algebraic structure, quite as rich say as that of a Tomita algebra. Furthermore, as far as I know, all manifestations of superselection structure can be expressed purely in terms of the algebraic structure of the set of intertwiners.

Consider first the localization properties of intertwiners. If ϱ and ϱ' are localized in $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively and $\boldsymbol{S} = (\varrho'|S|\varrho)$, then S is a bilocal operator in the sense that $S \in \{\mathfrak{A}(\mathcal{O}_1') \cap \mathfrak{A}(\mathcal{O}_2')\}'$. $\mathcal{O}_1$ will be called a right support for $\boldsymbol{S}$ and $\mathcal{O}_2$ a left support. If $\mathcal{O} \supset \mathcal{O}_1 \cap \mathcal{O}_2$ then $S \in \mathfrak{A}(\mathcal{O}')' = \mathfrak{A}(\mathcal{O})$ by duality. It follows, in particular, that one may apply localized morphisms to S and this allows us to define a product of intertwiners.

Given $\boldsymbol{S}_i = (\varrho_i'|S_i|\varrho_i)$, $i = 1, 2$, we set

$$\boldsymbol{S}_1 \times \boldsymbol{S}_2 = (\varrho_1'\varrho_2'|S_1\varrho_1(S_2)|\varrho_1\varrho_2) , \tag{2.3}$$

and can verify that this cross-product is associative and that

$$(\boldsymbol{S}_1 \times \boldsymbol{S}_2)^* = \boldsymbol{S}_1^* \times \boldsymbol{S}_2^* , \tag{2.4}$$

$$(\boldsymbol{S}_1' \circ \boldsymbol{S}_1) \times (\boldsymbol{S}_2' \circ \boldsymbol{S}_2) = (\boldsymbol{S}_1' \times \boldsymbol{S}_2') \circ (\boldsymbol{S}_1 \times \boldsymbol{S}_2) , \tag{2.5}$$

the latter relation being valid whenever the left-hand side is defined.

We say that $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are *causally disjoint* if $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ have mutually spacelike right supports and mutually spacelike left supports.

Lemma 2.1. – If $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are causally disjoint, then $\boldsymbol{S}_1 \times \boldsymbol{S}_2 = \boldsymbol{S}_2 \times \boldsymbol{S}_1$.

Proof. We must show that when $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are causally disjoint:

$$S_1\varrho_1(S_2) = S_2\varrho_2(S_1) . \tag{2.6}$$

This relation is trivial if one can find spacelike double cones $\hat{\mathcal{O}}_1$ and $\hat{\mathcal{O}}_2$, where $\hat{\mathcal{O}}_i$ contains both a right and a left support of $\boldsymbol{S}_i$. The idea of the proof is to reduce everything to this trivial case. Replace $\boldsymbol{S}_2$ by $\boldsymbol{S}_3 = \boldsymbol{S}_2 \circ \boldsymbol{U}$ where $\boldsymbol{U} = (\varrho_2|U|\varrho_3)$ is a unitary intertwiner. Let $\mathcal{O}_i$ denote the right support of $\boldsymbol{S}_i$ and suppose that the smallest double cone containing $\mathcal{O}_2$ and $\mathcal{O}_3$ is still spacelike to $\mathcal{O}_1$. Then U is localized spacelike to $\mathcal{O}_1$. Setting

$$\boldsymbol{I}_\varrho = (\varrho|I|\varrho) , \tag{2.7}$$

we get from (2.5)

$$\boldsymbol{S}_1 \times \boldsymbol{S}_3 = \boldsymbol{S}_1 \times \boldsymbol{S}_2 \circ \boldsymbol{I}_{\varrho_1} \times \boldsymbol{U} , \qquad \boldsymbol{S}_3 \times \boldsymbol{S}_1 = \boldsymbol{S}_2 \times \boldsymbol{S}_1 \circ \boldsymbol{U} \times \boldsymbol{I}_{\varrho_1} .$$

But $\boldsymbol{I}_{\varrho_1}\times\boldsymbol{U}=\boldsymbol{U}\times\boldsymbol{I}_{\varrho_1}$, this is the trivial case. Hence the question of commutativity of the cross-product is unchanged if we pass from $\boldsymbol{S}_2$ to $\boldsymbol{S}_3$ shifting the right support in the manner described. By a succession of shifts we may suppose that a right and a left support of $\boldsymbol{S}_2$ are both contained in a double cone $\hat{\mathcal{O}}_2$ spacelike to a left support of $\boldsymbol{S}_1$ (*). Applying the same procedure to a right support of $\boldsymbol{S}_1$ we may reduce the problem to the trivial case and thus prove the Lemma.

We can now describe the dependence of the cross-product on the order of the factors in the general case. Given n intertwiners $\boldsymbol{S}_k$ and a permutation

$$p=\begin{pmatrix}12\ldots n\\ p(1)\ldots p(n)\end{pmatrix}$$

let me write as a shorthand

$$\boldsymbol{S}(p)=\boldsymbol{S}_{p^{-1}(1)}\times\ldots\times\boldsymbol{S}_{p^{-1}(n)}\,. \tag{2.8}$$

The identity permutation will be denoted by $e\in\boldsymbol{P}_n$.

Theorem 2.2. – Given $p\in\boldsymbol{P}_n$ there is a unique intertwiner-valued function $(\varrho_1,\ldots,\varrho_n)\to\boldsymbol{\varepsilon}_p(\varrho_1,\ldots,\varrho_n)$ such that

(2.9) *a*) $$\boldsymbol{S}(p)\circ\boldsymbol{\varepsilon}_p(\varrho_1,\ldots,\varrho_n)=\boldsymbol{\varepsilon}_p(\varrho_1',\ldots,\varrho_n')\circ\boldsymbol{S}(e)$$

for all $\boldsymbol{S}_k=(\varrho_k'|S_k|\varrho_k)$, $k=1,2,\ldots,n$.

b) Given ϱ_k we can find $\varrho_k'\simeq\varrho_k$, $k=1,2,\ldots,n$ such that the ϱ_k' have mutually spacelike supports and

$$\boldsymbol{\varepsilon}_p(\varrho_1',\ldots,\varrho_n')=\boldsymbol{I}_{\varrho_1'\ldots\varrho_n'}\,.$$

Proof. The uniqueness claim tells us how to go about defining $\boldsymbol{\varepsilon}_p$: given ϱ_k, $k=1,2,\ldots,n$ pick any unitary intertwiners $\boldsymbol{U}_k=(\tau_k|U_k|\varrho_k)$ such that the τ_k have mutually spacelike supports and set

$$\boldsymbol{\varepsilon}_p(\varrho_1,\ldots,\varrho_n)=\boldsymbol{U}(p)^{-1}\circ\boldsymbol{U}(e)\,. \tag{2.10}$$

Hence $\boldsymbol{\varepsilon}_p(\varrho_1',\ldots,\varrho_n')=\boldsymbol{U}'(p)^{-1}\circ\boldsymbol{U}'(e)$ where $\boldsymbol{U}_k'=(\tau_k'|U_k'|\varrho_k')$ is unitary and the τ_k' have mutually spacelike supports. Set $\boldsymbol{T}_k=\boldsymbol{U}_k'\circ\boldsymbol{S}_k\circ\boldsymbol{U}_k^{-1}$ then by Lemma 2.1, $\boldsymbol{T}(p)=\boldsymbol{T}(e)$ and re-arranging using (2.5) gives (2.9). Lemma 2.1 also shows that $\boldsymbol{\varepsilon}_p(\varrho_1,\ldots,\varrho_n)=\boldsymbol{I}_{\varrho_1\ldots\varrho_n}$ if the ϱ_k have mutually spacelike supports. To prove

(*) This argument would not apply in a 2-dimensional space-time world.

uniqueness suppose $\boldsymbol{\varepsilon}'_p$ satisfies *a*) and *b*), pick ϱ'_k as in *b*) and take $S_k = (\varrho'_k|S_k|\varrho_k)$ to be unitary, then (2.9) gives

$$\boldsymbol{\varepsilon}'_p(\varrho_1, \ldots, \varrho_n) = S(p)^{-1} \circ S(e) = \boldsymbol{\varepsilon}_p(\varrho_1, \ldots, \varrho_n) ,$$

as required.

The uniqueness result immediately gives for $p \in \mathbf{P}_n$,

Corollary 2.3.

$$\boldsymbol{\varepsilon}_p(\varrho_1, \ldots, \varrho_n)^* = \boldsymbol{\varepsilon}_p(\varrho_1, \ldots, \varrho_n)^{-1} ,$$

$$\boldsymbol{\varepsilon}_q(\varrho_{p^{-1}(1)}, \ldots, \varrho_{p^{-1}(n)}) \circ \boldsymbol{\varepsilon}_p(\varrho_1, \ldots, \varrho_n) = \boldsymbol{\varepsilon}_{qp}(\varrho_1, \ldots, \varrho_n) , \qquad q \in \mathbf{P}_n ,$$

$$\boldsymbol{\varepsilon}_q(\varrho_1, \ldots, \varrho_m) \times \boldsymbol{\varepsilon}_p(\varrho_{m+1}, \ldots, \varrho_{m+n}) = \boldsymbol{\varepsilon}_{q \times p}(\varrho_1, \ldots, \varrho_{m+n}) , \qquad q \in \mathbf{P}_m ,$$

$$\boldsymbol{\varepsilon}_p(\varrho_1, \ldots, \varrho_i \varrho', \ldots, \varrho_n) = \boldsymbol{\varepsilon}_r(\varrho_1, \ldots, \varrho_i, \varrho', \ldots, \varrho_n) .$$

$$\boldsymbol{\varepsilon}_p(\varrho_1, \ldots, \varrho_{i-1}, \iota, \varrho_{i+1}, \ldots, \varrho_n) = \boldsymbol{\varepsilon}_s(\varrho_1, \ldots, \varrho_{i-1}, \varrho_{i+1}, \ldots, \varrho_n) ,$$

where $q \times p \in \mathbf{P}_{m+n}$, $r \in \mathbf{P}_{n+1}$, $s \in \mathbf{P}_{n-1}$ are the obvious permutations.

We see from this result that computing an expression involving just compositions of cross-products of values of $\boldsymbol{\varepsilon}$'s (and note in this context that $I_{\varrho_1 \ldots \varrho_n}$ is just a value of $\boldsymbol{\varepsilon}_e$, $e \in \mathbf{P}_n$) is a triviality: one need only look at what variables are involved and what the overall permutation is. I refer to this result as $\boldsymbol{\varepsilon}$-coherence.

By way of explanation of the name, the cross-product makes the category of localized morphisms and intertwiners into a (strict) monoidal category (the terminology is that of EILENBERG and KELLY [2, 3]). $\boldsymbol{\varepsilon}$ gives a natural transformation making the monoidal structure symmetric. In this context there is a coherence theorem first proved by MACLANE with a similar content [4]. However in our case it is easier to derive the coherence result directly rather than appeal to the theorem.

In the special case $n = 2$ and p the transposition I write

$$\boldsymbol{\varepsilon}_p(\varrho_1, \varrho_2) = \boldsymbol{\varepsilon}(\varrho_1, \varrho_2) = (\varrho_2 \varrho_1|\varepsilon(\varrho_1, \varrho_2)|\varrho_1 \varrho_2) .$$

If $\varrho_1 = \varrho_2$, I also write $\boldsymbol{\varepsilon}_\varrho$ in place of $\boldsymbol{\varepsilon}(\varrho, \varrho)$ or more generally for n variables

$$\boldsymbol{\varepsilon}_p(\varrho, \ldots, \varrho) = (\varrho^n|\varepsilon^{(n)}_\varrho(p)|\varrho^n) = \boldsymbol{\varepsilon}^{(n)}_\varrho(p) .$$

In this case we have from Theorem 2.2 and Corollary 2.3,

Theorem 2.4.

a) $p \to \varepsilon^{(n)}_\varrho(p)$ is a unitary representation of $\mathbf{P}_n$,

b) $\varepsilon^{(n)}_\varrho(p) \in \varrho^n(\mathfrak{A})'$, *c*) if $\boldsymbol{W} = (\varrho'|W|\varrho)$ is unitary and $\boldsymbol{W}^{\times n}$ denotes the n-fold cross-product of $\boldsymbol{W}$ with itself

$$\boldsymbol{\varepsilon}_{\varrho'}^{(n)}(p) = \boldsymbol{W}^{\times n} \circ \boldsymbol{\varepsilon}_{\varrho}^{(n)}(p) \circ \boldsymbol{W}^{\times n *} .$$

Thus attached to each ϱ we have a string of unitary representations $\varepsilon_{\varrho}^{(n)}$ whose unitary equivalence class depends only on $\hat{\varrho}$, the equivalence class of ϱ.

3. – Classification of statistics.

Bearing in mind our opening discussion, if we want to consider the statistics associated with the sector given by ϱ, we should look at products of states $\omega_0 \circ \varrho_i$ where $\varrho_i \simeq \varrho$ and the ϱ_i are localized in mutually spacelike double cones $\mathcal{O}_i$, $i = 1, 2, ..., n$. If $\boldsymbol{U}_i = (\varrho_i|U_i|\varrho)$ is a unitary intertwiner then $\boldsymbol{\Psi}_i = \boldsymbol{U}_i^* \{\varrho_i; \Omega\}$ represents the state $\omega_0 \circ \varrho_i$ as a vector state of the representation ϱ. By (1.1), the product of these state vectors in the order determined by $p \in \boldsymbol{P}_n$ is

$$|p\rangle = \boldsymbol{\Psi}_{p^{-1}(n)} \times ... \times \boldsymbol{\Psi}_{p^{-1}(1)} = \boldsymbol{U}^*(p) \{\varrho_1 ... \varrho_n; \Omega\} . \tag{3.1}$$

Theorem 2.2 tells us that the vectors $|p\rangle$ are transformed into each other by the permutation operators $\boldsymbol{\varepsilon}_{\varrho}^{(n)}(q)$

$$\boldsymbol{\varepsilon}_{\varrho}^{(n)}(q)|p\rangle = |qp\rangle , \tag{3.2}$$

and that the $\varepsilon_{\varrho}^{(n)}(q)$ commute with all the observables in the representation ϱ^n. Thus these permutations are the analogues of the place permutations of the wave functions of n identical particles in wave mechanics.

The unitary equivalence class of $\varepsilon_{\varrho}^{(n)}$ depends only on the sector $\hat{\varrho}$ and we shall determine this class for each n. If $E \neq 0$ is the central projection corresponding to an irreducible component of $\varepsilon_{\varrho}^{(n)}$, then $E \in \varrho^n(\mathfrak{A})'$ and there is an isometric intertwiner $\boldsymbol{W} = (\varrho^n|W|\varrho')$ with $WW^* = E$ so E projects onto a subspace with the same dimension as $\mathscr{H}_0$. Hence it suffices to determine the quasi-equivalence class of $\varepsilon_{\varrho}^{(n)}$. Irreducible representations of $\varepsilon_{\varrho}^{(n)}$ correspond to Young tableaux with n squares so we consider our goal achieved when we describe the Young tableaux associated with $\varepsilon_{\varrho}^{(n)}$ as n varies. To do this we recall the properties of a left inverse φ of ϱ. If $A \in \varrho^n(\mathfrak{A})'$ then $\varphi(A) \in \varrho^{n-1}(\mathfrak{A})'$ so φ gives positive mappings

$$... \varrho^n(\mathfrak{A})' \xrightarrow{\varphi} \varrho^{n-1}(\mathfrak{A})' \xrightarrow{\varphi} ... \xrightarrow{\varphi} \varrho(\mathfrak{A})' \xrightarrow{\varphi} \mathfrak{A}' = \boldsymbol{C}I .$$

Now $\varepsilon_{\varrho}^{(n)}(p) \in \varrho^n(\mathfrak{A})'$ so if ϱ is irreducible, $\varphi^{n-1}(\varepsilon_{\varrho}^{(n)}(p))$ is a scalar and defines a function of positive type over $\boldsymbol{P}_n$. The problem is solved by computing this function explicitly.

Lemma 3.1. Let φ be a left inverse of an irreducible ϱ and $p \in \mathbf{P}_n$, then

$$\varphi(\varepsilon_\varrho^{(n)}(p)) = \varepsilon_\varrho^{(n-1)}(p') , \qquad p(1) = 1 ,$$

$$\varphi(\varepsilon_\varrho^{(n)}(p)) = \lambda\varepsilon_\varrho^{(n-1)}(p') , \qquad p(1) \neq 1 ,$$

where $\lambda I = \varphi(\varepsilon_\varrho)$ and $p' \in \mathbf{P}_{n-1}$ is the permutation obtained from p by deleting 1 from its cycle in the decomposition of p into disjoint cycles and then writing t for $t+1$, $1 \leqslant t \leqslant n-1$.

Proof. If $p(1) = 1$, $\varepsilon_\varrho^{(n)}(p) = \varrho(\varepsilon_\varrho^{(n-1)}(p'))$ by $\boldsymbol{\varepsilon}$-coherence. If $p(1) \neq 1$, write $p = p_1 p_2 p_3$ whre $p_2 = (12)$, $p_1 = (2p(1))$ and the symbol (st) denotes the transposition of s and t. This defines p_3 and one checks that $p_3(1) = 1$. $\boldsymbol{\varepsilon}$-coherence now gives $\varepsilon_\varrho^{(n)}(p) = \varrho(\varepsilon_\varrho^{(n-1)}(p_1'))\varepsilon_\varrho\varrho(\varepsilon_\varrho^{(n-1)}(p_3'))$ so using the definition of a left inverse we have $\varphi(\varepsilon_\varrho^{(n)}(p)) = \lambda\varepsilon_\varrho^{(n-1)}(p_1'p_3')$. The proof is completed by checking that $p_1'p_3' = p'$.

Theorem 3.2. If φ is a left inverse of an irreducible ϱ, then $\varphi(\varepsilon_\varrho) = \lambda_\varrho I$ where $\lambda_\varrho \in \{0\} \cup \{\pm d^{-1}\colon d \text{ integer}\}$. λ_ϱ depends only on the equivalence class of ϱ and determines the statistics of the corresponding sector as follows: the Young tableaux associated with the representation $\varepsilon_\varrho^{(n)}$ of $\mathbf{P}_n$, $n \geqslant 1$ are all Young tableaux:

a) whose columns have length $\leqslant d$, if $\lambda_\varrho = d^{-1}$ (para-Bose statistics of order d);

b) whose rows have length $\leqslant d$, if $\lambda_\varrho = -d^{-1}$ (para-Fermi statistics of order d);

c) without restriction, if $\lambda_\varrho = 0$ (infinite statistics).

Proof. If $\varphi(\varepsilon_\varrho) = \lambda I$, then ω_λ^n, the function on $\mathbf{P}_n$ defined by

$$\varphi^n(\varepsilon_\varrho^{(n)}(p)) = \omega_\lambda^n(p) I , \tag{3.3}$$

can be computed using Lemma 3.1 iteratively, it is a class function multiplicative on disjoint cycles and taking the value λ^{k-1} on a k-cycle. However since φ is a positive linear mapping ω_λ^n must be a function of positive type on $\mathbf{P}_n$ for all values of n. This implies that $\lambda \in \{0\} \cup \{\pm d^{-1}\colon d \text{ integer}\}$; the explicit computation needed is given in an Appendix where it is also shown that the quasi-equivalence class of the representation of $\mathbf{P}_n$ generated by ω_λ^n corresponds to the Young tableaux of order n determined by λ according to *a*), *b*) and *c*). The quasi-equivalence class of $\varepsilon_\varrho^{(n)}$ could *a priori* contain more Young tableaux although for $\lambda = 0$ this is trivially impossible. But if $\lambda \neq 0$, φ is faithful ([1]; Lemma 3.8) so if E is a central projection of the group algebra of $\mathbf{P}_n$, $\varepsilon_\varrho^{(n)}(E) = 0$ if and only if $\varphi^n(\varepsilon_\varrho^{(n)}(E)) = 0$. Hence λ determines the quasi-equivalence class of $\varepsilon_\varrho^{(n)}$ for all n. Looked at the other way, λ must be independent of the choice of φ and by Theorem 2.4 even of ϱ within its equivalence class.

If $\xi = \hat{\varrho}$, $\lambda_\xi = \lambda_\varrho$ is called the *statistics parameter* of the sector ξ. Note that Theorem 3.2 remains valid if ϱ is reducible provided there is a left inverse φ of ϱ such that $\varphi(\varepsilon_\varrho)$ is a scalar, and that we restrict attention to such left inverses. We make use of this remark to make a definition: ϱ is said to have *infinite statistics* if there is a left inverse φ of ϱ with $\varphi(\varepsilon_\varrho) = 0$. If there is no such left inverse ϱ is said to have *finite statistics*. If every left inverse φ of ϱ has $\varphi(\varepsilon_\varrho) = 0$, ϱ is said to have *purely infinite statistics*.

We note for future use that if ϱ has a left inverse with $\varphi(\varepsilon_\varrho) = \pm I$, then ϱ must be an automorphism. In fact $\varphi(\varepsilon_\varrho \mp I) = 0$ implies $\varepsilon_\varrho = \pm I$ since φ is faithful. The result now follows from ([1]; Prop. 2.7).

4. – Morphisms with finite statistics and their left inverses.

The question of the existence of localized morphisms with purely infinite statistics is still open. What little is known of their properties makes it clear that they do not behave in the same way as morphisms with finite statistics.

We shall concentrate our attention on morphisms with finite statistics. The first few results are designed to clarify what is involved in the distinction between finite and infinite statistics.

Lemma 4.1. Let φ be a left inverse of ϱ and $\boldsymbol{S} = (\varrho|S|\varrho')$, $\boldsymbol{T} = (\varrho|T|\varrho')$ then

a) $\varphi(SAS^*) = \varphi'(A)\varphi(SS^*)$, where φ' is a left inverse of ϱ';

b) $\varphi(S\varepsilon_{\varrho'}T^*) = T^*\varphi(\varepsilon_\varrho)S$;

c) if ϱ' is irreducible, S an isometry and $E = SS^*$, then $\varphi(E\varepsilon_\varrho E) = E\varphi(\varepsilon_\varrho)E = \lambda_{\varrho'}\varphi(E)E$.

Proof.

a) Since $A \to \varphi(SAS^*)$ is positive, $\varphi(SS^*) = 0$ implies $\varphi(SAS^*) = 0$. Hence we may suppose that the scalar $\varphi(SS^*) \neq 0$.

b) By Theorem 2.2,

$$\boldsymbol{S} \times \boldsymbol{I}_{\varrho'} \circ \boldsymbol{\varepsilon}_{\varrho'} \circ \boldsymbol{T}^* \times \boldsymbol{I}_{\varrho'} = \boldsymbol{\varepsilon}(\varrho', \varrho) \circ \boldsymbol{I}_{\varrho'} \times \boldsymbol{S} \circ \boldsymbol{T}^* \times \boldsymbol{I}_{\varrho'} = \boldsymbol{\varepsilon}(\varrho', \varrho) \circ \boldsymbol{T}^* \times \boldsymbol{I}_\varrho \circ \boldsymbol{I}_\varrho \times \boldsymbol{S} = \boldsymbol{I}_\varrho \times \boldsymbol{T}^* \circ \boldsymbol{\varepsilon}_\varrho \circ \boldsymbol{I}_\varrho \times \boldsymbol{S} .$$

Applying φ to this equation we get the desired result.

c) $\lambda_{\varrho'}\varphi(E) = \varphi'(\varepsilon_{\varrho'})\varphi(SS^*) = \varphi(S\varepsilon_{\varrho'}S^*)$ using a). Hence by b), $\lambda_{\varrho'}\varphi(E) = \varphi(S\varepsilon_{\varrho'}S^*) = S^*\varphi(\varepsilon_\varrho)S$ and thus

$$\lambda_{\varrho'}\varphi(E)E = S\lambda_{\varrho'}\varphi(E)S^* = E\varphi(\varepsilon_\varrho)E = \varphi(E\varepsilon_\varrho E)\,.$$

Lemma 4.2.

a) If ϱ has a component with infinite statistics, then ϱ has infinite statistics.

b) If ϱ has purely infinite statistics, then ϱ has no component with finite statistics.

c) If $\varrho(\mathfrak{A})'$ contains a countable set of mutually orthogonal projections then ϱ has infinite statistics.

Proof. Let $\boldsymbol{S} = (\varrho|S|\varrho')$ be isometric and φ' a left inverse of ϱ'. Use Lemma 4.1*a* with S^* in place of S to define a left inverse φ of ϱ. Then $\varphi(\varepsilon_\varrho) = \varphi'(S^*\varepsilon_\varrho S) = S\varphi'(\varepsilon_{\varrho'})S^*$, by Lemma 4.1*b*. Since S is isometric $\varphi(\varepsilon_\varrho) = 0$ if and only if $\varphi'(\varepsilon_{\varrho'}) = 0$. This proves a) and b). To prove c) let $\{E_i\}$ be a sequence of mutually orthgonal projections, $\boldsymbol{W}_i = (\varrho|W_i|\varrho_i)$ isometries with $W_iW_i^* = E_i$ and φ_i a left inverse of ϱ_i. Set $\varphi^{(n)}(A) = \sum_{i=1}^{n} n^{-1}\varphi_i(W_i^*AW_i)$, then $\varphi^{(n)}$ is a left inverse of ϱ and by Lemma 4.1 b), $\varphi^{(n)}(\varepsilon_\varrho) = \sum_{i=1}^{n} n^{-1}W_i\varphi_i(\varepsilon_{\varrho_i})W_i^*$. Since $\|\varphi_i(\varepsilon_{\varrho_i})\| \leqslant 1$ and the E_i are mutually orthogonal, $\|\varphi^{(n)}(\varepsilon_\varrho)\| \leqslant n^{-1}$. Thus a limit point φ of $\varphi^{(n)}$ in $\mathscr{M}$ must be a left inverse with $\varphi(\varepsilon_\varrho) = 0$. Hence ϱ has infinite statistics.

Corollary 4.3. Let ϱ have finite statistics then ϱ is a finite direct sum of irreducible morphisms with finite statistics.

From now on except where the contrary is explicitly stated all localized morphisms will be supposed to have finite statistics. We define $\Lambda(\varrho)$ to be the central element of $\varrho(\mathfrak{A})'$ with

$$\Lambda(\varrho)E = \lambda_E E\,, \tag{4.1}$$

where E is a minimal projection and λ_E is the statistics parameter of a component equivalent to $\varrho|E\mathscr{H}_0$. If $\boldsymbol{\Lambda}(\varrho) = (\varrho|\Lambda(\varrho)|\varrho)$, then we have for $\boldsymbol{S} = (\varrho|S|\varrho')$

$$\boldsymbol{\Lambda}(\varrho)\circ\boldsymbol{S} = \boldsymbol{S}\circ\boldsymbol{\Lambda}(\varrho')\,, \tag{4.2}$$

an equation which could equally well have been used to define $\boldsymbol{\Lambda}(\varrho)$ for reducible ϱ. Note that as we have restricted ourselves to finite statistics $\boldsymbol{\Lambda}(\varrho)$ is invertible. For a reducible morphism ϱ, $\Lambda(\varrho)$ takes the place of a statistics parameter and determines the statistics of any component of ϱ. An effective way of computing $\Lambda(\varrho)$ is to use a suitable left inverse. We begin by classifying left inverses. If φ is a left invers of ϱ let ω_φ be the state on $\varrho(\mathfrak{A})'$ defined by

$$\omega_\varphi(S)I = \varphi(S) . \tag{4.3}$$

Let Tr denote the trace on $\varrho(\mathfrak{A})'$ normalized to be 1 on minimal projections.

Theorem 4.4. If φ is a left inverse of ϱ, then

$$\varphi(\varepsilon_\varrho) = \Lambda(\varrho) X_\varphi , \tag{4.4}$$

where X_φ is a positive operator with $\mathrm{Tr}\,(X_\varphi) = 1$ (a density matrix) and

$$\omega_\varphi(S) = \mathrm{Tr}\,(SX_\varphi) . \tag{4.5}$$

The affine mappings $\varphi \to X_\varphi$ and $\varphi \to \omega_\varphi$ are 1-1 and onto the set of density matrices of $\varrho(\mathfrak{A})'$ and the set of states of $\varrho(\mathfrak{A})'$ respectively.

Proof. Let $\{E_i\}$ be a maximal set of mutually orthogonal projections in $\varrho(\mathfrak{A})'$ picked to be eigenprojections of the self-adjoint operator $\varphi(\varepsilon_\varrho)$:

$$\varphi(\varepsilon_\varrho) = \sum_i E_i \varphi(\varepsilon_\varrho) E_i = \sum_i \lambda_{E_i} \varphi(E_i) E_i = \Lambda(\varrho) \sum_i \varphi E_i) E_i ,$$

where we have used Lemma 4.1 *c*). If we set $X_\varphi = \sum_i \varphi(E_i) E_i$, then $\mathrm{Tr}\,(X_\varphi) I = \sum_i \varphi(E_i) = \varphi(I) = I$ so X_φ is a density matrix. Now given a density matrix $X = \sum_i a_i E_i$ where $\{E_i\}$ is a maximal set of mutually orthogonal projections of $\varrho(\mathfrak{A})'$ write $E_i = W_{i,n} W_{i,n}^*$ where $\boldsymbol{W}_{i,n} = (\varrho | W_{i,n} | \varrho_{i,n})$ are isometric intertwiners. Consider any limit point φ_0 of $A \to \sum_i a_i W_{i,n}^* A W_{i,n}$ in $\mathscr{M}$ as the right supports of $\boldsymbol{W}_{i,n}$ tend spacelike to infinity. Then φ_0 is a left inverse of ϱ. Now $\varphi_0(A) = \sum_{j,k} \varphi_0(E_j A E_k) = \sum_i \varphi_0(E_i A E_i)$. Applying this to $A = \varepsilon_\varrho$ and using Lemma 4.1 *c*) we get

$$\varphi_0(\varepsilon_\varrho) = \boldsymbol{\Lambda}(\varrho) \sum_i \varphi_0(E_i) E_i = \boldsymbol{\Lambda}(\varrho) \sum_i a_i E_i = \boldsymbol{\Lambda}(\varrho) X .$$

On the other hand applying this to $A = S \in \varrho(\mathfrak{A})'$ we get

$$\varphi_0(S) = \sum_i \varphi_0(E_i S E_i) = \sum_i a_i s_i I ,$$

where s_i is defined by $E_i S E_i = s_i E_i$, so $\omega_{\varphi_0}(S) = \mathrm{Tr}\,(SX)$. The set of left inverses satisfying (4.4) with $X_\varphi = X$ is a nonvoid convex compact subset of

$\mathscr{M}$ and so has an extremal point φ say. Now

$$\varphi(A^*A) = \sum_i a_i \varphi(A^* W^*_{i,n} W_{i,n} A) \geqslant \sum_i a_i \varphi(W_{i,n} A)^* \varphi(W_{i,n} A) .$$

But if $A \in \mathfrak{A}(\mathscr{O})$ and n is sufficiently large

$$W_{i,n} A = \varrho(A) W_{i,n} = \varrho(A) \varepsilon_\varrho \varrho(W_{i,n})$$

so

$$\varphi(W_{i,n} A) = A\varphi(\varepsilon_\varrho) W_{i,n} = a_i \lambda_{E_i} A W_{i,n} .$$

Hence

$$\varphi(A^*A) \geqslant \sum_i a_i^2 \lambda_{E_i}^2 W^*_{i,n} A^* A W_{i,n} \geqslant \mu \sum_i a_i W A^* A W_{i,n} ,$$

where $\mu = \min\{a_i^2 \lambda_{E_i}^2 : a_i \neq 0\} > 0$ so

$$\varphi(A^*A) \geqslant \mu \varphi_0(A^*A) .$$

This holds for all $A \in \mathfrak{A}$ by taking norm limits, so $\varphi = \varphi_0$. Hence there is a unique left inverse φ satisfying $\varphi(\varepsilon_\varrho) = \Lambda(\varrho) X$ completing the proof.

Note that our first inequality involving μ also shows that φ is faithful if and only if no a_i is zero, *i.e.* if and only if $\varphi(\varepsilon_\varrho)$ is invertible.

In [1] a left inverse φ of ϱ is called *standard* if $\varphi(\varepsilon_\varrho)^2$ is a multiple of the identity. These are now easily characterized: define $\varkappa(\varrho)$ to be the central element of $\varrho(\mathfrak{A})'$ defined by

$$\varkappa(\varrho) E = \operatorname{sign} \lambda_E E ,$$

where E is a minimal projection. If $\boldsymbol{\varkappa}(\varrho) = (\varrho|\varkappa(\varrho)|\varrho)$ and $\boldsymbol{S} = (\varrho|S|\varrho')$

$$\boldsymbol{\varkappa}(\varrho) \circ \boldsymbol{S} = \boldsymbol{S} \circ \boldsymbol{\varkappa}(\varrho') . \tag{4.6}$$

Theorem 4.5. A morphism ϱ with finite statistics has a unique standard left inverse φ, φ is faithful and

$$d(\varrho)\, \varphi(\varepsilon_\varrho) = \varkappa(\varrho) , \tag{4.7}$$

where $d(\varrho) = \operatorname{Tr} |\Lambda(\varrho)|^{-1}$. Also

$$\varphi(\varepsilon_\varrho \varkappa(\varrho)) = \varphi(\varkappa(\varrho) \varepsilon_\varrho) = d(\varrho)^{-1} I . \tag{4.8}$$

If $S = (\varrho|S|\varrho')$ is isometric, then

$$d(\varrho)\,\omega_\varphi(SS^*) = d(\varrho')\,, \tag{4.9}$$

$$d(\varrho)\varphi(SAS^*) = d(\varrho')\varphi'(A)\,, \qquad A \in \mathfrak{A}\,, \tag{4.10}$$

where φ' is the standard left inverse of ϱ'.

Proof. If φ is standard, $\Lambda(\varrho)^2 X_\varphi^2$ is a multiple of the identity so $X_\varphi = |\Lambda(\varrho)|^{-1} d(\varrho)^{-1}$. This shows that φ is unique and faithful and proves (4.7). Now by (2.9), $\varepsilon_\varrho \varkappa(\varrho) = \varrho(\varkappa(\varrho))\varepsilon_\varrho$, applying φ to this equation gives (4.8). Let φ' be as in Lemma 4.1 *a*), then by Lemma 4.1 *b*), $d(\varrho)\varphi(SS^*)\varphi'(\varepsilon_\varrho) = d(\varrho)\varphi(S\varepsilon_\varrho S^*) = S^*\varkappa(\varrho)S = \varkappa(\varrho')$. Hence φ' is the standard left inverse of ϱ' and $d(\varrho)\varphi(SS^*) = d(\varrho')I$. This proves (4.9) and (4.10) and completes the proof.

The integer $d(\varrho)$ is called the *statistical dimension* of ϱ; it is just the sum of the statistical dimensions of the constituents of ϱ. If

$$\varrho \simeq \bigoplus_{i=1}^{n} \varrho_i, \quad \text{then} \quad d(\varrho) = \sum_{i=1}^{n} d(\varrho_i).$$

The relation between cross-products and left inverses is governed by the next Lemma where we set $\varphi(\boldsymbol{\varepsilon}_\varrho) = (\varrho|\varphi(\varepsilon_\varrho)|\varrho)$.

Lemma 4.6. Let φ_1 and φ_2 be left inverses of ϱ_1 and ϱ_2 respectively, then $\varphi_2\varphi_1$ is a left inverse for $\varrho_1\varrho_2$ and

$$\varphi_2\varphi_1(\boldsymbol{\varepsilon}_{\varrho_1\varrho_2}) = \varphi_1(\boldsymbol{\varepsilon}_{\varrho_1}) \times \varphi_2(\boldsymbol{\varepsilon}_{\varrho_2})\,.$$

Proof. We have by (2.9),

$$\varrho_1\varphi_2(\varepsilon_{\varrho_2}) = \varepsilon(\varrho_2, \varrho_1)\varphi_2(\varepsilon_{\varrho_2})\varepsilon(\varrho_1, \varrho_2)\,,$$

so

$$\varphi_1(\varepsilon_{\varrho_1})\varrho_1\varphi_2(\varepsilon_{\varrho_2}) = \varphi_1(\varepsilon_{\varrho_1})\varepsilon(\varrho_2, \varrho_1)\varphi_2(\varepsilon_{\varrho_2})\varepsilon(\varrho_1, \varrho_2) =$$
$$= \varphi_2\big(\varepsilon(\varrho_1, \varrho_2)\varphi_1(\varepsilon_{\varrho_1})\varepsilon(\varrho_2, \varrho_1)\varrho_2(\varepsilon(\varrho_2, \varrho_1))\varepsilon_{\varrho_2}\varrho_2(\varepsilon(\varrho_1, \varrho_2))\big)\,.$$

Pulling φ_1 out and using $\boldsymbol{\varepsilon}$-coherence this expression reduces to $\varphi_2\varphi_1(\varepsilon_{\varrho_1\varrho_2})$ as required.

Corollary 4.7.

a) If φ_1 and φ_2 are standard, $\varphi_2\varphi_1$ is standard.

b) The product of sectors with finite statistics has finite statistics.

c) If $\varrho = \varrho_1\varrho_2$, $d(\varrho) = d(\varrho_1)d(\varrho_2)$ and $\boldsymbol{\varkappa}(\varrho_1\varrho_2) = \boldsymbol{\varkappa}(\varrho_1) \times \boldsymbol{\varkappa}(\varrho_2)$.

We now know that the product of sectors with finite statistics is just a finite direct sum of sectors with finite statistics.

When we restrict our attention to intertwiners between morphisms with finite statistics we do not lose our most important piece of algebraic structure, the cross-product. In fact we gain some structure; here is the first example: we put a «bitrace» on the set of intertwiners by considering $\mathscr{I}(\varrho, \varrho')$ as a Hilbert space with the scalar product

$$(\boldsymbol{S}, \boldsymbol{T}) = d(\varrho)\,\omega_\varphi(S^* T)\,, \tag{4.11}$$

where φ is the standard left inverse of ϱ. The properties of this bitrace are summed up in

Proposition 4.8.

$$(\boldsymbol{S}, \boldsymbol{X}\circ\boldsymbol{T}) = (\boldsymbol{X}^*\circ\boldsymbol{S}, \boldsymbol{T})\,, \tag{4.12}$$

$$(\boldsymbol{S}, \boldsymbol{T}) \quad = (\boldsymbol{T}^*, \boldsymbol{S}^*)\,. \tag{4.13}$$

If $\boldsymbol{S}_i, \boldsymbol{T}_i \in \mathscr{I}(\varrho_i, \varrho_i')$, $i = 1, 2$, then

$$(\boldsymbol{S}_1 \times \boldsymbol{S}_2, \boldsymbol{T}_1 \times \boldsymbol{T}_2) = (\boldsymbol{S}_1, \boldsymbol{T}_1)(\boldsymbol{S}_2, \boldsymbol{T}_2)\,. \tag{4.14}$$

Proof. Equation (4.12) follows at once from the definition. By (4.8)

$$d(\varrho)\varphi(S^* T) = d(\varrho)\,d(\varrho')\,\varphi\varphi'\big(\varrho'(S)^* \varkappa(\varrho')\,\varepsilon_{\varrho'}\,\varrho'(T)\big)\,. \tag{4.15}$$

Now

$$\varrho'(S)^*\varkappa(\varrho')\,\varepsilon_{\varrho'}\,\varrho'(T) = \varkappa(\varrho')\,\varrho'(S)^*\,T\varepsilon(\varrho', \varrho) = \varkappa(\varrho')\,T\varrho(S)^*\varepsilon(\varrho', \varrho) = T\varkappa(\varrho)\,\varrho(S)^*\varepsilon(\varrho', \varrho)$$

by (4.6). But by Corollary 4.7 *a*) and (4.10)

$$\varphi'\varphi\big(\varepsilon(\varrho', \varrho)A\varepsilon(\varrho, \varrho')\big) = \varphi\varphi'(A)\,, \qquad A \in \mathfrak{A}\,.$$

Hence

$$\varphi\varphi'\big(\varrho'(S)^*\varkappa(\varrho')\,\varepsilon_{\varrho'}\,\varrho'(T)\big) = \varphi'\varphi\big(\varepsilon(\varrho', \varrho)\,T\varkappa(\varrho)\,\varrho(S)^*\big) =$$
$$= \varphi'\varphi\big(\varrho(T)\,\varepsilon_\varrho\,\varkappa(\varrho)\,\varrho(S)^*\big) = d(\varrho)^{-1}\varphi'(TS^*)$$

by (4.8). Comparing with (4.15) we have $d(\varrho)\varphi(S^* T) = d(\varrho')\,\varphi'(TS^*)$ which is (4.13). The remaining eq. (4.14) is a trivial consequence of Corollary 4.7.

5. – Charge conjugation.

There is an important analogy between the type of structure we have been deriving here and the representation theory of a compact group $\mathscr{G}$. In this analogy, localized morphisms with finite statistics correspond to finite-dimensional, continuous unitary representations of $\mathcal{G}$. The product of morphisms corresponds to the tensor product of representations and intertwiners correspond to intertwining operators ($\mathcal{G}$-module homorphisms). In the case of models where the observables are constructed from fields by the principle of gauge invariance, $\mathcal{G}$ is the gauge group and this analogy can be given a precise mathematical form [5].

Corresponding to our basic operators $\varepsilon(\varrho_1, \varrho_2)$ we have for any two representations ϱ_1, ϱ_2, say of $\mathcal{G}$ in Hilbert spaces $H(\varrho_1)$ and $H(\varrho_2)$ a unitary operator $\theta(\varrho_1, \varrho_2)$ defined by

$$\theta(\varrho_1, \varrho_2) x_1 \otimes x_2 = x_2 \otimes x_1 , \tag{5.1}$$

or more generally for n representations and $p \in \boldsymbol{P}_n$

$$\theta_p(\varrho_1, \varrho_2, \dots, \varrho_n) x_1 \otimes \dots \otimes x_n = x_{p^{-1}(1)} \otimes \dots \otimes x_{p^{-1}(n)} . \tag{5.2}$$

At this point the analogy is no longer perfect because the corresponding representations of the permutation group are all of para-Bose type $\lambda > 0$ and never of the para-Fermi type $\lambda < 0$. For this reason it is sometimes convenient to modify $\boldsymbol{\varepsilon}$ to make the analogy with group theory more transparent and at the same time to simplify certain formulae. We define

$$2\delta \boldsymbol{n}(\varrho_1, \varrho_2) = (\boldsymbol{I}_{\varrho_1} \times \boldsymbol{I}_{\varrho_2}) + (\boldsymbol{I}_{\varrho_1} \times \boldsymbol{\varkappa}(\varrho_2)) + (\boldsymbol{\varkappa}(\varrho_1) \times \boldsymbol{I}_{\varrho_2}) - (\boldsymbol{\varkappa}(\varrho_1) \times \boldsymbol{\varkappa}(\varrho_2)) \tag{5.3}$$

and set

$$\hat{\boldsymbol{\varepsilon}}(\varrho_1, \varrho_2) = \boldsymbol{\varepsilon}(\varrho_1, \varrho_2) \circ \delta \boldsymbol{n}(\varrho_1, \varrho_2) = \delta \boldsymbol{n}(\varrho_2, \varrho_1) \circ \boldsymbol{\varepsilon}(\varrho_1, \varrho_2) . \tag{5.4}$$

We have for $\boldsymbol{S}_i = (\varrho_i'|S_i|\varrho_i)$, $i = 1, 2$,

$$\hat{\boldsymbol{\varepsilon}}(\varrho_1', \varrho_2') \circ (\boldsymbol{S}_1 \times \boldsymbol{S}_2) = (\boldsymbol{S}_2 \times \boldsymbol{S}_1) \circ \hat{\boldsymbol{\varepsilon}}(\varrho_1, \varrho_2) . \tag{5.5}$$

Further

$$\hat{\boldsymbol{\varepsilon}}(\varrho_1, \varrho_2) \circ \hat{\boldsymbol{\varepsilon}}(\varrho_2, \varrho_1) = \boldsymbol{I}_{\varrho_1\varrho_2} , \tag{5.6}$$

$$\hat{\boldsymbol{\varepsilon}}(\varrho_1\varrho_2, \varrho_3) = \hat{\boldsymbol{\varepsilon}}(\varrho_1, \varrho_3) \times \boldsymbol{I}_{\varrho_2} \circ \boldsymbol{I}_{\varrho_1} \times \hat{\boldsymbol{\varepsilon}}(\varrho_2, \varrho_3) , \tag{5.7}$$

where we use Corollary 4.7 c) to derive (5.7). Equations (5.5) to (5.7) suffice to show that we can define an intertwiner-valued function $(\varrho_1, \dots, \varrho_n) \to \hat{\boldsymbol{\varepsilon}}_p(\varrho_1, \dots, \varrho_n)$ satisfying the analogues of (2.9) and Corollary 2.3. This is $\hat{\boldsymbol{\varepsilon}}$-coherence. Of course we could also write down an explicit formula for $\hat{\boldsymbol{\varepsilon}}_p$.

We recall that nowhere in the proof of Theorem 3.2 did we need to use the fact that $\boldsymbol{\varepsilon}$ was defined to have $\boldsymbol{\varepsilon}(\varrho_1, \varrho_2) = \boldsymbol{I}_{\varrho_1\varrho_2}$ if ϱ_1 and ϱ_2 are spacelike separated. Thus we can easily classify the representations $p \to \hat{\varepsilon}_\varrho^{(n)}(p)$ of $\boldsymbol{P}_n$ by computing $\varphi(\hat{\varepsilon}_\varrho)$. Since

$$2\hat{\varepsilon}_\varrho = \varepsilon_\varrho\big(I + \varrho(\varkappa(\varrho)) + \varkappa(\varrho) - \varkappa(\varrho)\,\varrho(\varkappa(\varrho))\big)\,,$$

(4.7) and (4.8) give

$$\varphi(\hat{\varepsilon}_\varrho) = d(\varrho)^{-1} I\,. \tag{5.8}$$

Thus we find just the result for para-Bose statistics of order $d(\varphi)$.

To return to our analogy, now that we have improved it by introducing $\hat{\boldsymbol{\varepsilon}}$ in place of $\boldsymbol{\varepsilon}$, there is one feature of the finite-dimensional, continuous unitary representations of $\mathscr{G}$ which we have so far not seen in our abstract structure. Each representation of $\mathscr{G}$ has a corresponding complex conjugate representation. Let me recall one method of constructing this complex conjugate representation (up to equivalence) for, say, a unitary matrix group of dimension d. Let ϱ' be the representation on the d-dimensional space of totally antisymmetric tensors of rank $d-1$, then $g \to \det(g)^{-1}\varrho'(g)$ is the conjugate representation. However $g \to \det(g)$ is itself nothing more than the representation γ, say, on the 1-dimensional space of totally antisymmetric tensors of rank d. Since both ϱ' and γ are constructed using antisymmetrized tensor powers of the original representation, it is clear how the analogous localized morphisms ϱ' and γ may be defined.

Let E_a^n denote the totally antisymmetric projection in the group algebra of $\boldsymbol{P}_n$. Given a localized morphism ϱ with $d(\varrho) = d < \infty$ set

$$E = \hat{\varepsilon}_\varrho^{(d)}(E_a^d)\,, \qquad E' = \hat{\varepsilon}_\varrho^{(d-1)}(E_a^{d-1})\,. \tag{5.9}$$

Let $\boldsymbol{W} = (\varrho^d|W|\gamma)$ and $\boldsymbol{W}' = (\varrho^{d-1}|W'|\varrho')$ be isometric intertwiners with $WW^* = E$ and $W'W'^* = E'$. Using (4.9), (3.3) and Corollary 4.7 we have $d(\gamma) = d^d\omega_a^d(E_a^d)$ and $d(\varrho') = d^{d-1}\omega_a^{d-1}(E_a^{d-1})$. These integers are computed in the Appendix and we find $d(\gamma) = 1$ and $d(\varrho') = d$. The crucial point is that by the remark at the end of Sect. **3**, $d(\gamma) = 1$ implies that γ is an automorphism. So we set

$$\bar{\varrho} = \varrho'\gamma^{-1}\,, \tag{5.10}$$

and by Corollary 4.7, $d(\bar{\varrho}) = d = d(\varrho)$. Now

$$\varrho(W')^* WA = \varrho(W')^* W\gamma\gamma^{-1}(A) = \varrho(W')^* \varrho^d\gamma^{-1}(A)\,W = \varrho\bar{\varrho}(A)\,\varrho(W')^* W\,. \tag{5.11}$$

Actually $\varrho(W')^* W$ is an isometry; one can compute this directly at this stage but it will follow from the main theorem where I characterize $\bar{\varrho}$ abstractly in terms of ϱ up to an equivalence. We first need a simple lemma.

Lemma 5.1. Let ϱ have finite statistics and let $\boldsymbol{S} = (\bar{\varrho}\varrho|S|\iota)$ and define

$$\bar{\boldsymbol{S}} = \hat{\boldsymbol{\varepsilon}}(\bar{\varrho}, \varrho)\circ\boldsymbol{S}\,. \tag{5.12}$$

Then

$$S = d(\varrho)\varphi(\bar{S})\,, \tag{5.13}$$

where φ is the standard left inverse of ϱ and

$$S^*\bar{\varrho}(\bar{S}) = d(\varrho)\varphi(\bar{S}\bar{S}^*) = \bar{S}^*\varrho(\hat{\varepsilon}_{\bar{\varrho}})\bar{S}\,. \tag{5.14}$$

Further the expressions in (5.14) equal the identity operator if and only if

$$\bar{S}^*\varrho(A)\bar{S} = d(\bar{\varrho})\bar{\varphi}(A)\,, \qquad A \in \mathfrak{A}\,, \tag{5.15}$$

where $\bar{\varphi}$ is the standard left inverse of $\bar{\varrho}$.

Proof. Using (5.5) and (5.6) we have

$$\bar{S} = \hat{\varepsilon}(\varrho, \varrho\bar{\varrho})\varrho(\bar{S}) = \varrho(\hat{\varepsilon}(\varrho, \bar{\varrho}))\hat{\varepsilon}_{\varrho}\varrho(\bar{S})\,,$$
$$\bar{S} = \hat{\varepsilon}(\bar{\varrho}, \varrho\bar{\varrho})\bar{\varrho}(\bar{S}) = \varrho(\hat{\varepsilon}_{\bar{\varrho}})\hat{\varepsilon}(\bar{\varrho}, \varrho)\bar{\varrho}(S)\,.$$

Applying φ to the first equation and using (5.8) gives (5.13), whilst the second equation shows that $S^*\bar{\varrho}(\bar{S}) = \bar{S}^*\varrho(\hat{\varepsilon}_{\bar{\varrho}})\bar{S}$. However $\varphi(\bar{S}\bar{S}^*) = \varphi(\bar{S}^*\varrho\bar{\varrho}(\bar{S})) = \varphi(\bar{S})^*\bar{\varrho}(\bar{S}) = d(\varrho)^{-1}S^*\bar{\varrho}(\bar{S})$ and we now have (5.14). Since $A \to \bar{S}^*\varrho(A)\bar{S}$ is anyway a multiple of a left inverse of $\bar{\varrho}$, $\bar{S}^*\varrho(\hat{\varepsilon}_{\bar{\varrho}})\bar{S} = I$ and (5.8) imply (5.15).

Theorem 5.2. Let ϱ have finite statistics then there exists a $\bar{\varrho}$ and intertwiners $\boldsymbol{R} = (\bar{\varrho}\varrho|R|\iota)$, $\bar{\boldsymbol{R}} = (\varrho\bar{\varrho}|\bar{R}|\iota)$ satisfying

$$\bar{\boldsymbol{R}} = \hat{\boldsymbol{\varepsilon}}(\bar{\varrho}, \varrho)\circ\boldsymbol{R}\,, \tag{5.16}$$

$$\bar{R}^*\varrho(R) = I\,, \qquad R^*\bar{\varrho}(\bar{R}) = I\,, \tag{5.17}$$

$$R^*\bar{\varrho}(A)R = d(\varrho)\varphi(A)\,, \qquad \bar{R}^*\varrho(A)\bar{R} = d(\bar{\varrho})\bar{\varphi}(A)\,, \tag{5.18}$$

where φ and $\bar{\varphi}$ are the standard left inverses of ϱ and $\bar{\varrho}$ respectively.

Proof. We define $\bar{\varrho}$ by (5.10) and set $\bar{R} = d(\varrho)^{\frac{1}{2}}\varrho(W')^*W$ and then use (5.16) to define R. We know from (5.11) that $\bar{R}$ intertwines from ι to $\varrho\bar{\varrho}$ and hence R intertwines from ι to $\bar{\varrho}\varrho$. Now $d(\varrho)\varphi(\bar{R}\bar{R}^*) = d(\varrho)^2 W'^*\varphi(E)W'$ and by Lemma 3.1 for $\hat{\varepsilon}_{\varrho}^{(d)}$ we have $d(\varrho)^2\varphi(E) = E'$, so $d(\varrho)\varphi(\bar{R}\bar{R}^*) = I$. By Lemma 5.1 this proves the right-hand equation of (5.17) and (5.18). Now apply Lemma 5.1 with ϱ and $\bar{\varrho}$ interchanged; $\bar{R}^*\varrho(R) = d(\bar{\varrho})\bar{\varphi}(RR^*) = \bar{R}^*\varrho(RR^*)\bar{R}$

so $\bar{R}^* \varrho(R)$ is a projection. Now $\varphi(\bar{R}^* \varrho(R)) = \varphi(\bar{R}^*) R = d(\varrho)^{-1} R^* R = I$ since $d(\varrho) = d(\bar{\varrho})$. But φ is faithful so $\bar{R}^* \varrho(R) = I$. By Lemma 5.1 this proves the left-hand equations of (5.17) and (5.18) and with them the Theorem.

We say that $\boldsymbol{R} = (\bar{\varrho}\varrho|R|\iota)$ satisfies the *conjugate equations* if (5.17) holds with $\bar{\boldsymbol{R}}$ defined by (5.16). We know from Lemma 5.1 that (5.18) then holds as well. I shall call $\bar{\varrho}$ a *conjugate* for ϱ if there is an intertwiner $\boldsymbol{R} = (\bar{\varrho}\varrho|R|\iota)$ satisfying the conjugate equations. We see from (5.17) that $R\Omega$ separates $\bar{\varrho}(\mathfrak{A})'$ so that $\bar{\varrho}$ is equivalent to the representation associated with the state $\omega_0 \circ \varphi$.

The structure of the set of intertwiners is much enriched now that we know that there are conjugates.

Proposition 5.3. Let $\boldsymbol{R} = (\bar{\varrho}\varrho|R|\iota)$ satisfy the conjugate equations and set

$$\nu(\boldsymbol{S}) = \boldsymbol{I}_{\bar{\varrho}} \times \boldsymbol{S} \circ \boldsymbol{R} \times \boldsymbol{I}_{\varrho_1}, \qquad \boldsymbol{S} \in \mathscr{I}(\varrho\varrho_1, \varrho_2). \tag{5.19}$$

Then ν is a unitary mapping from $\mathscr{I}(\varrho\varrho_1, \varrho_2)$ to $\mathscr{I}(\varrho_1, \bar{\varrho}\varrho_2)$ whose inverse is given by

$$\nu^{-1}(\boldsymbol{S}') = \bar{\boldsymbol{R}}^* \times \boldsymbol{I}_{\varrho_2} \circ \boldsymbol{I}_{\varrho} \times \boldsymbol{S}', \qquad \boldsymbol{S}' \in \mathscr{I}(\varrho_1, \bar{\varrho}\varrho_2). \tag{5.20}$$

Proof. We see from (5.19) and (5.20) that $\nu(\boldsymbol{S}) \in \mathscr{I}(\varrho_1, \bar{\varrho}\varrho_2)$ and $\nu^{-1}(\boldsymbol{S}') \in \in \mathscr{I}(\varrho\varrho_1, \varrho_2)$. Now

$$\nu\nu^{-1}(\boldsymbol{S}') = \boldsymbol{I}_{\bar{\varrho}} \times \bar{\boldsymbol{R}}^* \times \boldsymbol{I}_{\varrho_2} \circ \boldsymbol{I}_{\bar{\varrho}\varrho} \times \boldsymbol{S}' \circ \boldsymbol{R} \times \boldsymbol{I}_{\varrho_1} = \boldsymbol{I}_{\bar{\varrho}} \times \bar{\boldsymbol{R}}^* \times \boldsymbol{I}_{\varrho_2} \circ \boldsymbol{R} \times \boldsymbol{I}_{\bar{\varrho}\varrho_2} \circ \boldsymbol{S}',$$

so by (5.17), $\nu\nu^{-1}(\boldsymbol{S}') = \boldsymbol{S}'$ and ν^{-1} is a right inverse for ν. However by definition of the scalar product on $\mathscr{I}(\varrho_1, \bar{\varrho}\varrho_2)$

$$(\nu(\boldsymbol{S}), \nu(\boldsymbol{S})) = d(\varrho_1)\varphi_1(R^* \bar{\varrho}(S^* S) R) = d(\varrho_1) d(\varrho) \varphi_1 \varphi(S^* S) = (\boldsymbol{S}, \boldsymbol{S}),$$

since by Corollary 4.7, $\varphi_1 \varphi$ is the standard left inverse of $\varrho\varrho_1$ and $d(\varrho\varrho_1) = = d(\varrho) d(\varrho_1)$. Hence ν is isometric completing the proof.

Actually this result shows in particular that the two functors $\boldsymbol{T} \to \boldsymbol{I}_{\varrho} \times \boldsymbol{T}$ and $\boldsymbol{T} \to \boldsymbol{I}_{\bar{\varrho}} \times \boldsymbol{T}$ on the category of localized morphisms are right and left adjoints of one another. This implies that the symmetric monoidal category of localized morphisms with finite statistics is even a closed category [2].

If ϱ_1 and ϱ_2 in Proposition 5.3 do not necessarily have finite statistics ν and ν^{-1} are still inverses of one another. This allows us to draw a few conclusions about purely infinite statistics. If ϱ_1 has purely infinite statistics and ϱ has finite statistics $\varrho\varrho_1$ has purely infinite statistics. This follows from Lemma 4.2 *b*) and its converse which I have not proved here but which is a consequence of ([1]; Lemma 6.1). Using the argument in the Appendix of [6] as well it follows that the product of a covariant morphism with positive energy and purely infinite statistics with any other localized morphism has purely infinite statistics.

A simple corollary of Proposition 5.3 is that if ϱ and $\bar{\varrho}$ are conjugates then ι is contained precisely once in $\bar{\varrho}\varrho$ if and only if ϱ is irreducible. Hence if ϱ is irreducible so is $\bar{\varrho}$. Proposition 5.3 may also be used to prove the uniqueness of conjugates up to unitary equivalence:

Proposition 5.4. Let $\boldsymbol{R} = (\bar{\varrho}\varrho|R|\iota)$ and $\boldsymbol{R}_1 = (\varrho_1\varrho|R_1|\iota)$ satisfy the conjugate equations then there is a unique unitary $\boldsymbol{U} = (\varrho_1|U|\bar{\varrho})$ such that

$$\boldsymbol{R}_1 = \boldsymbol{U} \times \boldsymbol{I}_\varrho \circ \boldsymbol{R}\,, \tag{5.21}$$

$$\overline{\boldsymbol{R}}_1 = \boldsymbol{I}_\varrho \times \boldsymbol{U} \circ \overline{\boldsymbol{R}}\,. \tag{5.22}$$

Proof. Since $\overline{\boldsymbol{R}} = \hat{\boldsymbol{\varepsilon}}(\bar{\varrho}, \varrho) \circ \boldsymbol{R}$ and $\overline{\boldsymbol{R}}_1 = \hat{\boldsymbol{\varepsilon}}(\varrho_1, \varrho) \circ \boldsymbol{R}_1$ (5.5) shows that (5.21) and (5.22) are equivalent. On the other hand setting $\varrho_2 = \iota$ in Proposition 5.3 we see from (5.19) and (5.20) that $\boldsymbol{U} = \nu(\overline{\boldsymbol{R}}_1^*)^*$ is the unique intertwiner from $\bar{\varrho}$ to ϱ_1 satisfying (5.22). Hence

$$\boldsymbol{U} \circ \boldsymbol{U}^* = \nu(\overline{\boldsymbol{R}}_1^*)^* \circ \nu(\overline{\boldsymbol{R}}_1^*) = \boldsymbol{R}^* \times \boldsymbol{I}_{\varrho_1} \circ \boldsymbol{I}_\varrho \times (\overline{\boldsymbol{R}}_1 \overline{\boldsymbol{R}}_1^*) \circ \boldsymbol{R} \times \boldsymbol{I}_{\varrho_1} = d(\varrho)\varphi(\overline{\boldsymbol{R}}_1 \overline{\boldsymbol{R}}_1^*)\,.$$

Since $\boldsymbol{R}_1$ satisfies the conjugate equations $\boldsymbol{U} \circ \boldsymbol{U}^* = \boldsymbol{I}_{\varrho_1}$ by (5.14). Thus ϱ_1 is a component of $\bar{\varrho}$ but the symmetry between $\boldsymbol{R}$ and $\boldsymbol{R}_1$ shows that ϱ_1 and $\bar{\varrho}$ are actually equivalent, hence $\boldsymbol{U}$ is unitary.

For each ϱ with finite statistics we now pick a solution $\boldsymbol{R}_\varrho$ of the conjugate equations. This then allows us to define a conjugation on intertwiners. If $\boldsymbol{T} = (\varrho_2|T|\varrho_1)$ we define

$$\boldsymbol{T}^\dagger = \boldsymbol{I}_{\bar{\varrho}_2} \times \overline{\boldsymbol{R}}^*_{\varrho_1} \circ \boldsymbol{I}_{\bar{\varrho}_2} \times \boldsymbol{T}^* \times \boldsymbol{I}_{\bar{\varrho}_1} \circ \boldsymbol{R}_{\varrho_2} \times \boldsymbol{I}_{\bar{\varrho}_1}\,. \tag{5.23}$$

Actually we could have defined † as the composition of the following mappings:

$$\mathscr{I}(\varrho_1, \varrho_2) \xrightarrow{\nu} \mathscr{I}(\iota, \bar{\varrho}_1, \bar{\varrho}_2) \xrightarrow{\hat{\boldsymbol{\varepsilon}}(\bar{\varrho}_1, \varrho_2)\circ} \mathscr{I}(\iota, \varrho_2\bar{\varrho}_1) \xrightarrow{*} \mathscr{I}(\varrho_2\bar{\varrho}_1, \iota) \xrightarrow{\nu} \mathscr{I}(\bar{\varrho}_1, \bar{\varrho}_2)\,,$$

so Propositions 4.8 and 5.3 imply that † is antiunitary.

Lemma 5.5 Given $\boldsymbol{T} \in \mathscr{I}(\varrho_1, \varrho_2)$, $\boldsymbol{T}^\dagger$ defined by (5.23) is the unique element of $\mathscr{I}(\bar{\varrho}_1, \bar{\varrho}_2)$ satisfying either

$$\boldsymbol{T}^\dagger \times \boldsymbol{I}_{\varrho_1} \circ \boldsymbol{R}_{\varrho_1} = \boldsymbol{I}_{\bar{\varrho}_2} \times \boldsymbol{T}^* \circ \boldsymbol{R}_{\varrho_2}\,, \tag{5.24}$$

or

$$\boldsymbol{I}_{\varrho_2} \times \boldsymbol{T}^{\dagger *} \circ \overline{\boldsymbol{R}}_{\varrho_2} = \boldsymbol{T} \times \boldsymbol{I}_{\bar{\varrho}_1} \circ \overline{\boldsymbol{R}}_{\varrho_1}\,. \tag{5.25}$$

Proof. Substitute (5.23) into the left-hand side of (5.24), move $\boldsymbol{R}_{\varrho_1}$ to the left and use (5.17) to prove (5.24). Conversely if $\boldsymbol{S} = (\bar{\varrho}_2|S|\bar{\varrho}_1)$ and $\boldsymbol{S}\times\boldsymbol{I}_{\varrho_1}\circ\boldsymbol{R}_{\varrho_1} = \boldsymbol{I}_{\bar{\varrho}_2}\times\boldsymbol{T}^*\circ\boldsymbol{R}_{\varrho_2}$ compute the right-hand side of (5.23) in terms of $\boldsymbol{S}$, move $\boldsymbol{S}$ to the left and use (5.17) to deduce $\boldsymbol{S} = \boldsymbol{T}^\dagger$. Similarly we may prove that (5.23) and (5.25) are equivalent.

Proposition 5.6. The antiunitary map † satisfies

$$\boldsymbol{S}^\dagger\circ\boldsymbol{T}^\dagger = (\boldsymbol{S}\circ\boldsymbol{T})^\dagger\,, \tag{5.26}$$

where $\boldsymbol{S} = (\varrho_3|S|\varrho_2)$, $\boldsymbol{T} = (\varrho_2|T|\varrho_1)$. Further

$$\boldsymbol{T}^{\dagger *} = \boldsymbol{T}^{*\dagger}\,. \tag{5.27}$$

Proof. Using $(\boldsymbol{S}\circ\boldsymbol{T})^* = \boldsymbol{T}^*\circ\boldsymbol{S}^*$ and (5.24) to pass backwards and forwards between † and * leads to (5.26). Now

$$\boldsymbol{T}^{*\dagger} = \boldsymbol{I}_{\bar{\varrho}_1}\times\overline{\boldsymbol{R}}^*_{\varrho_2}\circ\boldsymbol{I}_{\bar{\varrho}_1}\times\boldsymbol{T}\times\boldsymbol{I}_{\bar{\varrho}_2}\circ\boldsymbol{R}_{\varrho_1}\times\boldsymbol{I}_{\bar{\varrho}_2}\,.$$

If we compare this with the expression for $\boldsymbol{T}^{\dagger *}$ given by (5.23) it is clear how to proceed: we must use (5.16) to pass from $\overline{\boldsymbol{R}}_{\varrho_2}$ to $\boldsymbol{R}_{\varrho_2}$ and from $\boldsymbol{R}_{\varrho_1}$ to $\overline{\boldsymbol{R}}_{\varrho_1}$ and then use (5.5) and (5.7) to change the order of the cross-products.

It is possible to make † into an involution by making a convention. The conjugation depends on the choice of solutions of the conjugate equations although not in any very essential way as is clear from Proposition 5.4. If we make the convention that $\boldsymbol{R}_{\bar{\varrho}} = \overline{\boldsymbol{R}}_\varrho$ then $\bar{\bar{\varrho}} = \varrho$ and from (5.24) and (5.25)

$$\boldsymbol{T}^{\dagger\dagger}\times\boldsymbol{I}_{\bar{\varrho}_1}\circ\overline{\boldsymbol{R}}_{\varrho_1} = \boldsymbol{T}^{\dagger\dagger}\times\boldsymbol{I}_{\bar{\varrho}_1}\circ\boldsymbol{R}_{\bar{\varrho}_1} = \boldsymbol{I}_{\varrho_2}\times\boldsymbol{T}^{\dagger *}\circ\boldsymbol{R}_{\bar{\varrho}_2} = \boldsymbol{I}_{\varrho_2}\times\boldsymbol{T}^{\dagger *}\circ\overline{\boldsymbol{R}}_{\varrho_2} = \boldsymbol{T}\times\boldsymbol{I}_{\bar{\varrho}_1}\circ\overline{\boldsymbol{R}}_{\varrho_1}$$

hence

$$\boldsymbol{T}^{\dagger\dagger} = \boldsymbol{T}\,, \tag{5.28}$$

and † is an involution.

Notice however that if ϱ and $\bar{\varrho}$ are equivalent we are still not necessarily at liberty to take $\bar{\varrho} = \varrho$ if we stick to this convention because this would mean that $\overline{\boldsymbol{R}}_\varrho = \boldsymbol{R}_\varrho$. This is precisely the starting remark for proving a variant of Carruthers' theorem [**6**; Theorem **6.5**].

Proposition 5.7. Suppose that ϱ leads to a self-conjugate sector, *i.e.* that $\bar{\varrho}$ and ϱ are equivalent and irreducible. Then

a) if $\boldsymbol{V} = (\bar{\varrho}|V|\varrho)$, $\boldsymbol{V}^\dagger = \pm\boldsymbol{V}^*$;

b) if $\boldsymbol{S} = (\varrho^2|S|\iota)$ satisfies the conjugate equations, $\overline{\boldsymbol{S}} = \pm\boldsymbol{S}$.

Furthermore the sign in *a*) or *b*) is the same and depends only on the sector in question.

Proof. Since ϱ is irreducible, $V^\dagger = \mu V^*$ where $\mu \in \mathbf{C}$. But

$$V = V^{\dagger\dagger} = \bar{\mu} V^{*\dagger} = \bar{\mu} V^{\dagger *} = \bar{\mu}^2 V ,$$

so $\mu = \pm 1$, proving *a*). However if S satisfies the conjugate equations, then by Proposition 5.4 there is a unitary $V^\dagger = (\varrho | V^\dagger | \bar{\varrho})$ with $S = V^\dagger \times I_\varrho \circ R_\varrho$. Hence by (5.24) and (5.21)

$$S = I_\varrho \times V^* \circ R_{\bar{\varrho}} = \mu (I_\varrho \times V^\dagger) \circ \bar{R}_\varrho = \mu \bar{S} .$$

This proves *a*) and *b*) and the equality of the signs in *a*) and *b*). However from *b*) the sign can at most depend on ϱ and using (5.26) and *a*) it is clear that it is independent of the choice of ϱ within its equivalence class.

A self-conjugate sector is said to be *real* or *pseudoreal* according as the sign in the above Proposition is positive or negative. Of course in the group-theoretical analogy the sign is positive for a self-conjugate irreducible representation of a group if and only if one can choose a basis in which the group is represented by real matrices.

This concludes my description of the algebraic structure of superselection sectors expressed in terms of the intertwiner calculus. In the lectures of HAAG devoted to certain standard problems of quantum field theory withint his framework (see also [6]) you will see how various invariants of the superselection structure, the statistics parameter, the statistical dimension or the « bitrace » on the interwiners often appear naturally when treating these problems.

It would be nice to be able to give a complete list of invariants of the superselection structure perhaps in terms of additional structure on the set $\Delta_t/\mathscr{I}$. To illustrate how we have been enriching this structure, the product of morphisms makes $\Delta_t/\mathscr{I}$ into an Abelian semi-group, adding the direct sum makes it into a semi-ring and if we include the operations of symmetrization or antisymmetrization associated with $\boldsymbol{\varepsilon}$ or $\hat{\boldsymbol{\varepsilon}}$ we would make $\Delta_t/\mathscr{I}$ into a λ–semi-ring. Let me remark that there is a natural conjecture namely that (restricting to finite statistics) the category of localized morphisms and intertwiners with its structure is equivalent to the category $\mathfrak{U}(\mathscr{G})$ of finite-dimensional, continuous unitary representations of a compact group $\mathscr{G}$, the gauge group of the theory. If this is the case then we can also construct a field algebra $\mathfrak{F}$ and a representation of $\mathscr{G}$ by automorphisms of $\mathfrak{F}$ and the observable algebra $\mathfrak{A}$ can be identified with the fixed-point algebra of $\mathfrak{F}$ under the action of $\mathscr{G}$. In essence the construction can be regarded as the crossed product of $\mathfrak{A}$ by an action of $\mathfrak{U}(\mathscr{G})$.

Appendix

This Appendix is devoted to explicit calculations on class functions ω_λ^n over $\mathbf{P}_n$ which arise in the classification of statistics. ω_λ^n is defined to be the class function multiplicative on disjoint cycles and taking the value λ^{k-1} on a k-cycle. The same symbol is used to denote the extension of ω_λ^n to a linear functional over the complex group algebra of $\mathbf{P}_n$.

Consider the natural inclusion homomorphism $p \to p^+$ from $\mathbf{P}_{n-1}$ to $\mathbf{P}_n$. The mapping $q \to q^-$ from $\mathbf{P}_n$ to $\mathbf{P}_{n-1}$ got by writing q as a product of disjoint cycles and then removing n from its cycle is not a homomorphism, however

$$(qp^+)^- = q^-p\,, \quad (p^+q)^- = pq^-\,, \quad p \in \mathbf{P}_{n-1}\,, \quad q \in \mathbf{P}_n\,. \tag{A.1}$$

Extend these mappings by linearity to the complex group algebra of $\mathbf{P}_n$ and $\mathbf{P}_{n-1}$.

A Young tableau T will be identified with the corresponding symmetrizer in the group algebra

$$T = \sum_{p,q} \operatorname{sign}(q) qp\,,$$

where p and q run respectively over all permutations permuting the rows or columns of the tableau respectively. Suppose that removing the square labelled n from T gives another Young tableau T'. One would expect T' to be much the same as T^-. In fact if p permutes the rows of T', then p^+ permutes the rows of T so that by [7, Lemma 4.3A], $Tp^+ = T$ and by (A.1), $T^-p = T^-$. Similarly if q permutes the columns of T', then $q^+T = \operatorname{sign}(q^+)T$, so by (A.1) $qT^- = \operatorname{sign}(q)T^-$. Hence by [7; Lemma 4.3$A$], T^- is just a multiple of T' and looking at the coefficient of the identity permutation in T^- we see that

$$T^- = \bigl(1 + (c - r)\bigr)T'\,, \tag{A.2}$$

where c and r are respectively the lengths of the column and row containing the square labelled n.

It is now easy to compute $\omega_\lambda^n(T)$ iteratively. In the first place $\omega_\lambda^n(T)$ does not depend on the way T is labelled because ω_λ^n is a class function. Now

$$\omega_\lambda^{n-1}(p^-) = \omega_\lambda^n(p) \qquad \text{if } p(n) = n\,,$$

$$\omega_\lambda^{n-1}(p^-) = \lambda\omega_\lambda^n(p) \qquad \text{if } p(n) \neq n\,.$$

However $T - T'^+$ is a linear combination of permutations p with $p(n) \neq n$, hence $\omega_\lambda^n(T - T'^+) = \lambda\omega_\lambda^{n-1}(T^- - T') = \lambda(c - r)\omega_\lambda^{n-1}(T')$ by (A.2). So we may calculate $\omega_\lambda^n(T)$ by induction using

$$\omega_\lambda^n(T) = \bigl(1 + \lambda(c - r)\bigr)\omega_\lambda^{n-1}(T')\,. \tag{A.3}$$

We deduce

a) if $\lambda = d^{-1}$, d an integer, $\omega_\lambda^n(T) \geqslant 0$ for all n and T and $\omega_\lambda^n(T) = 0$ if and only if T has a column of length $> d$;

b) if $\lambda = -d^{-1}$, d an integer, $\omega_\lambda^n(T) \geqslant 0$ for all n and T and $\omega_\lambda^n(T) = 0$ if and only if T has a row of length $> d$;

c) if $\lambda \notin \{\pm d^{-1}: d \text{ integer}\}$ then $\omega_\lambda^n(T) \neq 0$ for all n and T and $\omega_\lambda^n(T) > 0$ for all n and T if and only if $\lambda = 0$.

A function on $\boldsymbol{P}_n$ is of positive type if and only if its extension to the group algebra is positive. For a class function it is enough to test positivity on the minimal central projections or equivalently on the Young tableaux (see the explicit formulae in [7; Ch. IV 3]). Thus ω_λ^n is a function of positive type over $\boldsymbol{P}_n$ if and only if $\lambda \in \{0\} \cup \{\pm d^{-1}: d \text{ integer}\}$.

We conclude with a small computation used in Sect. **5**. Let E_a^n denote the totally antisymmetric projection in the group algebra of $\boldsymbol{P}_n$; the corresponding Young tableau differs by a normalization factor $n!$ and we deduce from (A.3) that

$$n!\,\omega_\lambda^n(E_a^n) = (1-\lambda)(1-2\lambda)\dots\bigl(1-(n-1)\lambda\bigr)\,. \tag{A.4}$$

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