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System Reduction and Solution Algorithms for Singular Linear Difference Systems under Rational Expectations

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Abstract. A first-order linear difference system under rational expectations is,

$$AEy_{t+1}|I_t = By_t + C(\mathbf{F})Ex_t|I_t\,,$$

where y_t is a vector of endogenous variables; x_t is a vector of exogenous variables; $Ey_{t+1}|I_t$ is the expectation of y_{t+1} given date t information; and $C(\mathbf{F})Ex_t|I_t = C_0x_t + C_1Ex_{t+1}|I_t + \cdots + C_nEx_{t+n}|I_t$. If the model is solvable, then y_t can be decomposed into two sets of variables: dynamic variables d_t that evolve according to $Ed_{t+1}|I_t = Wd_t + \Psi_d(\mathbf{F})Ex_t|I_t$ and other variables that obey the dynamic identities $f_t = -Kd_t - \Psi_f(\mathbf{F})Ex_t|I_t$. We develop an algorithm for carrying out this decomposition and for constructing the implied dynamic system. We also provide algorithms for (i) computing perfect foresight solutions and Markov decision rules; and (ii) solutions to related models that involve informational subperiods.

Key words: system reduction, algorithm, models, solutions, in practice

1. Introduction

Many dynamic linear rational expectations models can be cast in the form:

$$AEy_{t+1}|I_t = By_t + C_0x_t + C_1Ex_{t+1}|I_t + \cdots + C_nEx_{t+n}|I_t, \quad t = 0, 1, 2, \ldots \quad (1)$$

where A, B and $\{C_i\}$ are matrices of constants; y_t is a column vector of endogenous variables; x_t is a column vector of exogenous variables; and $E \cdot |I_t$ denotes the

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(rational) expectation conditional on I_t. While seemingly special, this framework is general enough to accommodate models with (i) expectations of variables more than one period into the future, (ii) lags of endogenous variables, (iii) lags of expectations of endogenous variables, and many other complications. Accordingly, (1) has proven to be a convenient representation for analyzing properties of rational expectations models, such as the existence and uniqueness of solutions. It further provides a useful starting point for the design of methods for computing such rational expectations solutions.

In general, economic models give rise to matrices A and B which are singular, so that (1) is called a singular linear difference system under rational expectations. The analysis of Blanchard and Kahn (1980) studied the existence and uniqueness of stable rational expectations solutions to (1) under the assumption that A was nonsingular and their analysis also suggested how to compute such solutions in this case. This article describes a general system reduction algorithm that simplifies the solution and analysis of (1), by isolating a reduced-dimension, nonsingular dynamic system in a subset of variables, d_t, of the full vector of endogenous variables y_t. Once the rational expectations solution to this smaller system is obtained, it is easy to also calculated the solution for the remaining variables, f_t, as these are governed by dynamic identities. This article also discusses two algorithms for computing the rational expectations solution to the reduced system and shows as well how to solve related 'timing' models, i.e., variants of (1) in which there are distinct informational subperiods.

Before turning to the details, it is important to stress that system reduction has long been a standard component of the analysis of linear rational expectations models (and dynamic systems, more generally). For example, in their study of the business cycle implications of the neoclassical growth model, King, Plosser and Rebelo (1988a, b) studied a model that included output, consumption, investment, labor supply and the capital stock. Transformations of these endogenous variables, together with a Lagrange multiplier (the shadow value of a unit of capital) made up the vector y_t, and the exogenous variables x_t where real shocks including the level of total factor productivity and government purchases. One of the first steps in their analysis was to show that output, consumption, investment and the labor supply could be solved out of the model. This allowed them to analyze a smaller dynamical system that included only two endogenous variables: the capital stock and its shadow value. In the notation of the last paragraph, their system reduction used a decomposition where f_t included output, consumption, investment and the labor supply, and d_t included the capital stock and its shadow value.

System reduction in this King–Plosser–Rebelo (KPR) model was relatively easy – a few lines of algebra – both because the number of variables in the model was small and because the workings of the model were relatively simple. However, modern quantitative macroeconomic models are much larger (sometimes involving more than a hundred equations) and much more complicated. In these models, it can become very difficult to know which of the endogenous variables belong in f_t

and which belong in d_t, how to eliminate f_t from the model, and the form of the resulting dynamic system for d_t.

The plan of the article is as follows. Section 2 uses two basic real business cycle models to provide motivate the representation (1) and to illustrate the problems which can arise in application of the conventional approach to system reduction employed in KPR and in many other studies. We prove that if (1) has a unique solution, then there exists a reduced-dimension dynamical system for d_t satisfies the Blanchard–Kahn (1980) conditions for solvability. Section 3 presents the computational algorithm for carrying out system reduction that we have found to be both fast and numerically stable, even in very large models. The algorithm is iterative and we show that it converges in a finite number of iterations if there exists any solution to (1). Section 4 details how to solve the reduced and complete model under rational expectations and Section 5 extends the method to 'timing' models. Finally, Section 6 provides a brief summary and conclusion.

2. Background

Let $\mathbf{F}$ denote the expectations lead operator (Sargent, 1979), defined as $\mathbf{F}^h E w_{t+n}|I_t = E w_{t+n+h}|I_t$ for any variable w_t. (Notice that the operator shifts the dating of the variable but does not change the dating of the conditioning information.) Using this operator (1) can be written more compactly as

$$AEy_{t+1}|I_t = By_t + C(\mathbf{F})Ex_t|I_t, \quad t = 0, 1, 2, \ldots \tag{2}$$

Throughout, we will assume that (y_t, x_t, I_{t-1}) are contained in I_t as well as other information to be specified in more detail below (we also sometimes write expectations as $E_t y_{t+1}$). Some of the endogenous variables in the model are predetermined, i.e., do not respond at t to shifts in x_t: we call these variables k_t and assume that they are ordered last in the vector y_t.

2.1. APPROXIMATING RECURSIVE MODELS

The behavioral equations of many other macroeconomic models – notably recursive optimizing models – can be expressed as a nonlinear system of expectational equations,

$$E_t F(Y_{t+1}, Y_t, X_{t+1}, X_t) = 0. \tag{3}$$

In this expression Y_t is a column vector of endogenous variables and X_t is a column vector of exogenous variables. Models of the form (3) are typically solved by linear or loglinear approximation around a stationary point at which $F(Y, Y, X, X) = 0$. A linear approximation of $F = 0$ about (Y, X) is

$$\begin{aligned} 0 \;=\; & F_1(Y_{t+1} - Y) + F_2(Y_t - Y) \\ & + F_3(X_{t+1} - X) + F_4(X_t - X)\,, \end{aligned}$$

where F_1 denotes the matrix of partial derivatives of the model equations with respect to the elements of Y_{t+1} and so forth. Accordingly, any model of the generic form (3) has a related approximate linear rational expectations system of the form $AE_t y_{t+1} = By_t + C_0 x_t + C_1 E_t x_{t+1}$, with suitable redefinitions of variables (as deviations from stationary levels $y_t = Y_t - Y$ and $x_t = X_t - X$) and suitable redefinitions of matrices (so that $A = F_1$, $B = -F_2$, $C_1 = -F_3$, and $C_0 = -F_4$).

A basic example of this class of models is the basic neoclassical model used in real business cycles. Its general structure is discussed in King, Plosser and Rebelo (1988a, b). The approximation and the solution of the basic RBC model is discussed in the KPR technical appendix published elsewhere in this volume, which we henceforth call KPR-TA. In the next subsection, we review the model analyzed by KPR-TA, focusing in particular on their system reduction. We then present a generalization of this model to two locations of production and display the complications that this introduces in the system reduction step. Complications like this motivate the general algorithm presented in Section 3.

2.2. THE STOCHASTIC GROWTH MODEL

It is convenient to use dynamic programming to describe the KPR-TA version of the neoclassical model because this leads naturally to recursive behavioral equations of the form (3).[1]

The Bellman equation is

$$v(k_t, \xi_t) = \max_{c_t, i_t} \{u(c_t) + \beta E_t v(k_{t+1}, \xi_{t+1})\},$$

where c_t is consumption, i_t is investment, and k_t is the predetermined capital stock. The value function v depends on capital and on information that is useful for forecasting the exogenous variables of the model, which we call ξ_t, and is taken to evolve according to a Markov process. The expectation $E_t v(k_{t+1}, \xi_{t+1})$ is short-hand for $Ev(k_{t+1}, \xi_{t+1})|(k_t, \xi_t)$.

This maximization is subject to two constraints. The resources available for consumption and investment are affected by productivity shocks (a_t), which is assumed for simplicity to be the sole source of uncertainty:

$$c_t + i_t = a_t f(k_t).$$

The capital stock evolves as the result of investment less depreciation, which occurs at proportional rate δ:

$$k_{t+1} - k_t = i_t - \delta k_t.$$

2.2.1. *Restrictions on Macroeconomic Variables*

To find restrictions on the paths of efficient consumption, investment and capital, it is convenient to form the Lagrangian

$$\begin{aligned} L \;=\; & u(c_t) + \beta E_t v(k_{t+1}, \xi_{t+1}) \\ & + p_t[a_t f(k_t) - c_t - i_t] \\ & + \lambda_t[(1-\delta)k_t + i_t - k_{t+1}] \end{aligned}$$

and the resulting first order conditions are

$$\begin{aligned} c_t \quad &: \; 0 = Du(c_t) - p_t \\ i_t \quad &: \; 0 = -p_t + \lambda_t \\ p_t \quad &: \; 0 = a_t f(k_t) - c_t - i_t \\ k_{t+1} \quad &: \; 0 = -\lambda_t + \beta E_t \tfrac{\partial v(k_{t+1}, \xi_{t+1})}{\partial k_{t+1}} \\ \lambda_t \quad &: \; 0 = (1-\delta)k_t + i_t - k_{t+1} \end{aligned}$$

The capital efficiency condition can be rewritten as $\lambda_t = \beta E_t\{\lambda_{t+1}(1-\delta) + p_{t+1}a_{t+1}Df(k_{t+1})]\}$ using the envelope theorem.

Consequently, one sector stochastic growth model fits neatly into the general form (3), which is $E_t F(Y_{t+1}, Y_t, X_{t+1}, X_t) = 0$, and there are five conditions that restrict the evolution of the vector endogenous variables $Y_t = [c_t, i_t, p_t, \lambda_t, k_t]'$ given the vector of exogenous variables $X_t = [a_t]$.

2.2.2. *A Loglinear Approximation*

Using the circumflex notation of KPR-TA to denote proportionate deviations from stationary values, the first three equations are approximated as

$$\widehat{c_t} = \left(-\frac{1}{\sigma}\right)\widehat{p_t} \tag{4}$$

$$\widehat{p_t} = \widehat{\lambda_t} \tag{5}$$

$$\widehat{a_t} + s_k\widehat{k_t} = s_i\widehat{i_t} + s_c\widehat{c_t}\,. \tag{6}$$

The parameters s_i and s_c are the output shares of investment and consumption; s_k is the capital income share; and $(1/\sigma)$ is the intertemporal substitution elasticity in preferences.

Similarly, the last two equations are approximated as

$$E_t\widehat{k}_{t+1} = (1-\delta)\widehat{k_t} + \delta\widehat{i_t} \tag{7}$$

$$\gamma\eta E_t\widehat{k}_{t+1} + \gamma E_t\widehat{a}_{t+1} + \gamma E_t\widehat{p}_{t+1} + (1-\gamma)E_t\widehat{\lambda}_{t+1} = \widehat{\lambda_t}\,, \tag{8}$$

where $\gamma = 1 - \beta(1-\delta)$ and $\eta = kD^2f(k)/Df(k)$ is the elasticity of the marginal product of capital with respect to the capital stock.

2.2.3. *Writing the System in Standard Form*

Taken together, (4)–(8) can be written in the first-order form $AE_t y_{t+1} = By_t + C_0 x_t + C_1 E_t x_{t+1}$ as:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \gamma & 1-\gamma & \delta\gamma \end{bmatrix} E_t \begin{bmatrix} \widehat{c}_{t+1} \\ \widehat{i}_{t+1} \\ \widehat{p}_{t+1} \\ \widehat{\lambda}_{t+1} \\ \widehat{k}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/\sigma & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ s_c & s_i & 0 & 0 & -s_k \\ 0 & 1 & 0 & 0 & 1-\delta \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \widehat{c}_t \\ \widehat{i}_t \\ \widehat{p}_t \\ \widehat{\lambda}_t \\ \widehat{k}_t \end{bmatrix}$$
$$+ \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} E_t \widehat{a}_t + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\gamma \end{bmatrix} E_t \widehat{a}_{t+1}$$

Inspecting this expression, note that the A matrix for the system is singular, as it contains rows of zeros. Thus, the model is 'singular'. But, a system reduction approach can easily isolate a reduced-dimension nonsingular system in this model.

The classical system reduction method, used in KPR-TA, is to look for a vector of controls, flows, and point-in-time shadow prices. In the current context, this vector is $f_t = (c_t \; i_t \; p_t)'$. The first three loglinear equations of the model ((4) through (6)) have rows of zeros in the A matrix: thus, one can use these equations to 'solve out' for the three f_t variables as functions of x_t and the remaining variables $d_t = (\lambda_t \quad k_t)'$. When these solutions are substituted into the remaining equations, the results is a two variable nonsingular system in d_t: this is the system reduction step in KPR-TA.

Applied to the one sector growth model, the system reduction method described in the next section will implement this approach automatically, enabling a researcher to simply specify behavioral equations in an arbitrary order and not to distinguish between f_t and d_t. However, it is useful to note that there is not a unique method for system reduction or a unique 'reduced' representation of (2). For example, in this one sector growth model, one can select any two of c_t, p_t and λ_t to be f_t elements and the third to be the element of d_t. This indeterminacy is a familiar one, as it is reflected in the differing, but formally equivalent analytical presentations of the global dynamics of the growth model which alternatively use the phase plane in (c, k) space or in (λ, k) space. While the reduced systems are different, the RE solutions based on such alternative reductions will be identical.

2.3. CONVENTIONAL SYSTEM REDUCTION

As we have emphasized, system reduction involves writing the model in a way that decomposes the vector of endogenous variables y_t into a vector f_t that is easily solved out of the model and another vector d_t that summarizes the model's

intrinsic dynamics. In the one sector growth model analysis in KPR-TA, as well as in many other contexts in economics and engineering, this decomposition follows from detailed knowledge about how the dynamic system works for all parameter values. For example, the analysis in the KPR-TA example used two related ideas. First, some behavioral equations contain no elements of $E_t y_{t+1}$, i.e., involved rows of zeros in A. Second, from aspects of the economic problem, some elements of y_t are controls and point-in-time shadow prices and, therefore, it is natural to think about solving out for these variables using the 'non dynamic' subset of model equations. Using this decomposition of y_t (2) has the form:

$$\begin{bmatrix} 0 & 0 \\ A_{df} & A_{dd} \end{bmatrix} E_t \begin{bmatrix} f_{t+1} \\ d_{t+1} \end{bmatrix} = \begin{bmatrix} B_{ff} & B_{fd} \\ B_{df} & B_{dd} \end{bmatrix} \begin{bmatrix} f_t \\ d_t \end{bmatrix} + \begin{bmatrix} C_f(\mathbf{F}) \\ C_d(\mathbf{F}) \end{bmatrix} E_t x_t \,. \tag{9}$$

Notice that this general system, like the example, has a singular A matrix due to rows of zeros: conventional system reduction assumes that there are as many such equations, $n(f)$, as there are elements of f_t.[2] Notice also that there is a subset of variables, f_t, which are ordered first.

The conventional system reduction procedure solves the equations for f_t

$$f_t = -B_{ff}^{-1} B_{fd} d_t - B_{ff}^{-1} C_f(\mathbf{F}) E_t x_t$$

and uses this solution to obtain the reduced dynamical system:

$$a E_t d_{t+1} = b d_t + \Psi_d(\mathbf{F}) E_t x_t \,, \tag{10}$$

where $a = [A_{dd} - A_{df} B_{ff}^{-1} B_{fd}]$; $b = [B_{dd} - B_{df} B_{ff}^{-1} B_{fd}]$; and $\Psi_d(\mathbf{F}) = [C_d(\mathbf{F}) - B_{df} B_{ff}^{-1} C_f(\mathbf{F}) + A_{df} B_{ff}^{-1} C_f(\mathbf{F})\mathbf{F}]$.

This process requires two important constraints on the model. First, B_{ff} must be nonsingular. Second, reduction of the dynamic system is typically completed by inverting a to get a dynamic system in the form $E d_{t+1} | I_t = a^{-1} b d_t + a^{-1} \Psi_d(\mathbf{F}) E x_t | I_t$; this requires that a is nonsingular. Finally, note that the reduction process typically adds a lead, i.e., there is an additional power of $\mathbf{F}$ in $\Psi_d(\mathbf{F})$ in (10).

This discussion of system reduction has stressed that some of the variables are solved out of the model. However, conventional system reduction may alternatively be viewed as the application of $T(\mathbf{F}) = G\mathbf{F} + H$ to the dynamic system with

$$T(\mathbf{F}) = G\mathbf{F} + H = \begin{bmatrix} 0 & 0 \\ A_{df} B_{ff}^{-1} & 0 \end{bmatrix} \mathbf{F} + \begin{bmatrix} B_{ff}^{-1} & 0 \\ -B_{df} B_{ff}^{-1} & I \end{bmatrix} .$$

That is, the dynamic system $(A\mathbf{F} - B) E_t y_t = C(\mathbf{F}) E_t x_t$ is transformed into a new system by multiplying both sides by $T(\mathbf{F})$. Notice that since $|T(z)| = |B_{ff}^{-1}| = \tau$, the characteristic polynomial of the transformed system, $|T(z)(Az - B)| = \tau |Az - B|$, has the same roots as the characteristic polynomial of the original system. It is also notable that the application of $T(\mathbf{F})$ to the dynamic system $[A\mathbf{F} - B] E y_t | I_t = C(\mathbf{F}) E x_t | I_t$ has different effects on the order of the system's

internal and exogenous dynamics. Since there is a special structure to $T(\mathbf{F})$, it follows that $T(\mathbf{F})[A\mathbf{F} - B] = [A^*\mathbf{F} - B^*]$, i.e., premultiplication does not change the first-order nature of the difference equation because of the nature G and H. However, $T(\mathbf{F})C(\mathbf{F})$ is typically higher order than $C(\mathbf{F})$, which is another way of stating that a lead is added by conventional system reduction.

The discussion below switches between these two notions of system reduction: 'solving equations' is generally more intuitive, but 'multiplication by matrix polynomials in $\mathbf{F}$' can be more revealing about the consequences of these operations.

2.4. TWO LOCATIONS IN THE GROWTH MODEL

To motivate the need for a general system reduction algorithm, it is useful to discuss an extension of the growth model to two locations of economic activity. This provides a concrete example of the problems that can arise using the convention system reduction outlined in the last subsection.[3] Specifically, consider a model with two locations of production of the same final consumption/investment good with relative sizes π_1 and π_2, with $\pi_1 + \pi_2 = 1$. There are then two capital stocks k_{1t} and k_{2t}, two investment flows i_{1t} and i_{2t} which augment these capital stocks, two shadow values of capital λ_{1t} and λ_{2t}, two productivity levels a_{1t} and a_{2t}. Assume that the final consumption/investment good can be costlessly moved across locations of economic activity.

The Bellman equation for optimal consumption, investment and capital accumulation is

$$v(k_{1t,}k_{2t}, \xi_t) = \max_{c_i, i_{1t}, i_{2t}} \{u(c_t) + \beta E_t v(k_{1,t+1,}k_{2,t+1}, \xi_{t+1})\}$$

subject to production and capital accumulation constraints. The Lagrangian form of the dynamic program is

$$\begin{aligned} L \;=\; & u(c_t) + \beta E_t v(k_{1,t+1,}k_{2,t+1}, \xi_{t+1}) \\ & + p_t[\pi_1 a_{1t} f(k_{1t}) + \pi_2 a_{2t} f(k_{2t}) - c_t - \pi_1 i_{1t} - \pi_2 i_{2t}] \\ & + \lambda_{1t}\pi_1[(1-\delta)k_{1t} + i_{1t} - k_{1,t+1}] \\ & + \lambda_{2t}\pi_2[(1-\delta)k_{2t} + i_{2t} - k_{2,t+1}]\,. \end{aligned}$$

Since this a direct generalization of the previous model, the full set of approximate equations that describe the model's dynamics will not be presented. The key point is that, given this is a two location generalization of the standard growth model, then a natural way of writing the vector f_t is $f_t = (c_t i_{1t} i_{2t} p_t)'$, i.e., using the fact that these are the controls and point-in-time shadow prices. Correspondingly, a natural way of writing d_t is $d_t = (\lambda_{1t}\lambda_{2t}k_{1t}k_{2t})'$, since these are the states and costates.

The four efficiency conditions corresponding to elements of f_t are

$$
\begin{aligned}
&c_t : Du(c_t) - p_t = 0\\
&i_{1t} : \pi_1[-p_t + \lambda_{1t}] = 0\\
&i_{2t} : \pi_2[-p_t + \lambda_{2t}] = 0\\
&p_t : [\pi_1 a_{1t} f(k_{1t}) + \pi_2 a_{2t} f(k_{2t}) - c_t - \pi_1 i_{1t} - \pi_2 i_{2t}] = 0
\end{aligned}
$$

which take the log-linear approximate forms,

$$
\begin{aligned}
&c_t : \widehat{c_t} = \left(-\tfrac{1}{\sigma}\right)\widehat{p_t}\\
&i_{1t} : \widehat{p_t} - \widehat{\lambda}_{1t} = 0\\
&i_{2t} : \widehat{p_t} - \widehat{\lambda}_{2t} = 0\\
&p_t : [\pi_1(\widehat{a}_{1t} + s_k\widehat{k}_{1t}) + \pi_2(\widehat{a}_{1t} + s_k\widehat{k}_{1t}) - s_c\widehat{c_t} - \pi_1 s_i\widehat{i}_{1t} - \pi_2 s_i\widehat{i}_{2t}] = 0\,.
\end{aligned}
$$

Each of these corresponds to a row of zeros in the A matrix of the 8 by 8 approximate linear system, which is consistent with the idea that $f_t = (c_t i_{1t} i_{2t} p_t)'$.

Conventional system reduction fails using these definitions of f and d: when the model is written in the form (9), the equations $0 = B_{ff} f_t + B_{fd} d_t + C_f(F) E_t x_t$ include the pair of equations $p_t = \lambda_{1t}$ and $p_t = \lambda_{2t}$. Thus, B_{ff} is singular, violating one of the necessary constraints for conventional system reduction. (The economics behind this singularity is clear: the two investment goods are perfect substitutes from the standpoint of resource utilization so that their shadow prices must be equated if there is a positive amount of each investment, i.e., $\lambda_{1t} = \lambda_{2t}$.).[4]

2.5. RE SOLVABILITY AND SYSTEM REDUCTION

These examples have shown that, in general, macroeconomic models may imply that both the matrices A and B are singular. The next section describes a generalized system reduction algorithm that finds a reduced dynamic system,

$$
E d_{t+1} | I_t = W d_t + \Psi_d(\mathbf{F}) E x_t | I_t \tag{11}
$$

with other variables that evolve according to

$$
f_t = -K d_t - \Psi_f(\mathbf{F}) E x_t | I_t. \tag{12}
$$

This representation is constructed by re-ordering the non-predetermined elements of y_t.[5]

Before presenting that algorithm, we present a result that provides a useful link between the properties of the reduced system and the original system. In particular it answers the following question. Suppose that there is a unique solution to (2). Then, will it always be possible to find a representation like (11)–(12) and, if so, what characteristics will it share with the original model?[6]

THEOREM 1. *If there exists a unique rational expectations solution to the singular linear difference equation system*

$$AEy_{t+1}|I_t = By_t + C(\mathbf{F})Ex_t|I_t$$

then there is an equivalent dynamic system

$$\begin{aligned} Ed_{t+1}|I_t &= Wd_t + \Psi_d(\mathbf{F})Ex_t|I_t \\ f_t &= -Kd_t - \Psi_f(\mathbf{F})Ex_t|I_t \end{aligned}$$

that can be derived from the original model by transforming its equations. This reduced system has eigenvalues, solutions to $|Iz - W| = 0$*, which are the same as the finite eigenvalues of the original system, which are solutions to* $|Az - B| = 0$*. Further, the solvability of the original system implies that the reduced system also satisfies the Blanchard–Khan conditions for its solvability.*

This theorem is proved in Appendix A, using results from a previous analysis of the theoretical conditions under which a singular linear difference system under rational expectations can be solved uniquely (King and Watson, 1998). It establishes that the system reduction algorithm developed in the next section is always aiming for a well-defined target, if the model is uniquely solvable. After developing the iterative steps of the algorithm, we will see that it will always find a representation (11)–(12) if any RE solution exists from general initial conditions, not just if there is a unique one.

3. The Reduction Algorithm

The reduction algorithm begins with the dynamic system (2) and produces a sequence of equivalent dynamic systems ending in the transformed system (11, 12). Before getting to the details, it is worth providing an outline of the algorithm. First, it finds rows of zeros in the matrix A or introduces rows of zeros into it: this step essentially defines identities that link together the variables of the model at a point in time. Second, it 'solves out' for as many nonpredetermined variables as possible from these identities, potentially reordering the vector of variables y_t for this purpose. This process continues until (11) is reached.

This section discusses (i) the concept of equivalent dynamic systems; (ii) the steps in the reduction algorithm; and (iii) a proof that any solvable model will result in a reduced dynamic system of the form (11, 12). The reduction algorithm that we construct in this section is closely related to the 'shuffle algorithm' developed by Luenberger (1978); Luenberger works directly with the matrices A, B, and C, applying transformations that 'shuffle' the rows of the matrices so as to extract an identity linking elements of y, solves the identity and then searches for additional ones.[7] By contrast, we use the Singular Value Decomposition (SVD) to introduce multiple rows of zeros into A and use the QR factorization to solve the related identities for elements of f_t. Further, in our algorithm, we impose constraints on the

admissible transformations that rule out moving any of the predetermined variables (k) into f.

3.1. EQUIVALENT DYNAMIC SYSTEMS

The various systems produced by the system reduction algorithm are equivalent in the sense that there is one-to-one transformation connecting them. Specifically, two types of transformations are used in the algorithm. The first type of transformations involves non-singular matrices T and V designed to introduce rows of zeros into A. That is: it transforms the equations (T) and the variables (V) of the model:

$$T[A\mathbf{F} - B]V^{-1}VEy_t|I_t = TC(\mathbf{F})Ex_t|I_t\,,$$

to yield a new dynamic system of the form,

$$[A^*\mathbf{F} - B^*]Ey_t^*|I_t = C^*(\mathbf{F})Ex_t|I_t\,,$$

with $[A^*\mathbf{F} - B^*] = T[A\mathbf{F} - B]V^{-1}$, $C^*(\mathbf{F}) = TC(\mathbf{F})$, and new variables, $Ey_t^*|I_t = V\ Ey_t|I_t$. Since T and V are non-singular, the transformation produces an equivalent system. Further, both the new system and the original system have the same roots to their determinantal polynomials, since $|A^*z - B^*| = |T||Az - B||V^{-1}|$. The algorithm restricts V to simply reorder the variables so that the reduced dimension system is in terms of the model's original variables. The second set of transformations are those of conventional system reduction. Section (2.3) showed how this transformation could be summarized by multiplying the model's equations by a specific polynomial $T(\mathbf{F}) = G\mathbf{F} + H$, with two key properties. First, $|T(z)| = \tau \neq 0$, so that again, this transformation does not alter the roots of the model's determinantal equation. Second, the matrices G and H are chosen so that the system remains first order.

Each iteration of the algorithm begins with a dynamic system with the following characteristics: some elements of f have already been identified, and comprise the vector $\overline{f}$, and the remaining elements of y are $\overline{d}$. The transformations that have been previously undertaken have not altered the fact that the predetermined variable k are the last elements of the transformed y vector, which we call $\overline{y}$; this also implies that the initial elements of $\overline{d}$ are nonpredetermined variables, which we call $\overline{\lambda}$. Thus, the system is of the form:

$$\overline{A}E\overline{y}_{t+1}|I_t = \overline{B}\overline{y}_t + \overline{C}(\mathbf{F})Ex_t|I_t\,, \tag{13}$$

where

$$\overline{A} = \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}, \overline{y} = \begin{bmatrix} \overline{f} \\ \overline{d} \end{bmatrix}, \overline{B} = \begin{bmatrix} I & \kappa \\ 0 & b \end{bmatrix}, \overline{C}(\mathbf{F}) = \begin{bmatrix} C_{\overline{f}}(\mathbf{F}) \\ C_{\overline{d}}(\mathbf{F}) \end{bmatrix}.$$

Technically, the vector $\overline{y} = \overline{V}y$, where $\overline{V}$ is a permutation matrix determined in previous iterations, and $\overline{f}$ denotes the flow variables determined in previous

iterations. The vector $\overline{d}$ is partitioned as $\overline{d} = (\overline{\lambda}' \quad k')'$, so that the predetermined variables k always appear as the final $n(k)$ elements. The iteration then proceeds by moving some of the elements of $\overline{\lambda}$ into $\overline{f}$. The algorithm terminates when this is no longer possible. As we show below, the resulting value of a will be non-singular if the original system satisfies the restriction $|Az - B| \neq 0$ and a solution to the model exists.

3.2. THE ALGORITHM STEPS

The algorithm includes five steps:

Step 1: Initialization

Begin by setting $\overline{y} = \overline{d} = y$, $\overline{A} = a = A$, $\overline{B} = b = B$, $\overline{C}(\mathbf{F}) = C_{\overline{d}}(\mathbf{F}) = C(\mathbf{F})$ and $C_{\overline{f}}(\mathbf{F}) = 0$. Since $\overline{f}$ is initially absent from the model, $n(\overline{f}) = 0$ and $n(\overline{d}) = n(y)$. If a is non-singular, proceed to Step 5, otherwise proceed to Step 2.

Step 2: Uncovering New Candidate Flows

Use the singular value decomposition to transform a so that there are some rows of zeros, i.e., that there are some nondynamic equations in the transformed system. Let the singular value decomposition of a be $a = U * S * V'$, with $U * U' = I$, $V * V' = I$, and S is a diagonal matrix with the (nonnegative) singular values on the diagonal: these singular values are ordered in increasing size, so that $s_j \geq s_i$ for $j \geq i$.[8] Supposing that r of these are nonzero (which is equivalently the number of positive singular values or the rank of a) then it is possible to induce $n(\overline{d}) - r$ rows of zeros into a. More specifically, multiplying the full system (13), which includes the elements of $\overline{f}$, by

$$T_1 = \begin{bmatrix} I_{n(f)} & 0 \\ 0 & U' \end{bmatrix},$$

produces a transformed version of (13), $T_1\overline{A}E\overline{y}_{t+1}|I_t = T_1\overline{B}\overline{y}_t + T_1\overline{C}(\mathbf{F})Ex_t|I_t$. By construction, the dynamic component of this new system has the form:

$$\begin{bmatrix} 0 & 0 \\ a_{2\lambda} & a_{2k} \end{bmatrix} E_t \begin{bmatrix} \overline{\lambda}_{t+1} \\ k_{t+1} \end{bmatrix} = \begin{bmatrix} b_{1\lambda} & b_{1k} \\ b_{2\lambda} & b_{2k} \end{bmatrix} \begin{bmatrix} \overline{\lambda}_t \\ k_t \end{bmatrix} + \begin{bmatrix} \Psi_1(\mathbf{F}) \\ \Psi_2(\mathbf{F}) \end{bmatrix} E_t x_t \,. \tag{14}$$

Let $n(f_c) = n(\overline{d}) - r$ denote the number of equations in the blocks with a leading subscript 1 (the subscript 'c' denotes the idea that these are candidate flows); these are those model equations with rows of zeros in (14). Now, we want to solve for some elements of $\overline{\lambda}_t$ as flows, which is done in next two steps.

Step 3: Isolating New Flows

Focusing on the first block of equations of (14): $b_{1\lambda}\overline{\lambda}_t + b_{1k}\ k_t = -\Psi_1(\mathbf{F})E_t x_t$, first determine the implied number of linearly independent restrictions on $\overline{\lambda}_t$ (call this $n(f_n)$ for the number of new flows). Transform the equations and variables so as to facilitate the solution of the equations using the QR factorization of $b_{1\lambda}$: $QR = b_{1\lambda}P$.[9] Since P is an $n(\overline{\lambda}) \times n(\overline{\lambda})$ permutation matrix, this factorization implies that the elements of $\overline{\lambda}_t$ can be reordered by multiplying by P'. Thus, for the full vector $\overline{y}$, the reordering is produced by:

$$V_1 = \begin{bmatrix} I & 0 & 0 \\ 0 & P' & 0 \\ 0 & 0 & I \end{bmatrix}.$$

That is, rewrite the dynamic system $T_1\overline{A}E\overline{y}_{t+1}|I_t = T_1\overline{B}y_t + T_1\overline{C}(\mathbf{F})Ex_t|I_t$ as $T_1\overline{A}EV_1^{-1}V_1\overline{y}_{t+1}|I_t = T_1\overline{B}V_1^{-1}V_1\overline{y}_t + T_1\overline{C}(\mathbf{F})Ex_t|I_t$. While reordering $\overline{\lambda}$, V_1 preserves the ordering of $\overline{f}$ and k. Also reorganize the system's equations using the QR factorization of $b_{1\lambda}$

$$T_2 = \begin{bmatrix} I & 0 & 0 \\ 0 & Q' & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Hence, the transformations of the system through Step 2 are: $T_2T_1\overline{A}EV_1^{-1}V_1\overline{y}_{t+1}|I_t = T_2T_1\overline{B}V_1^{-1}V_1y_t + T_2T_1\overline{C}(\mathbf{F})Ex_t|I_t$.

Step 4: Conventional System Reduction

It follows that there are $n(f_n) = rank(b_{1\lambda})$ new flows. Put alternatively, if $n(\overline{f})$ is the number of pre-existing flows then the $(n(\overline{f})+n(f_n)) \times (n(\overline{f})+n(f_n))$ leading submatrix of $T_2T_1\overline{B}V_1^{-1}$ is of the form

$$B_{ff} = \begin{bmatrix} I & 0 \\ 0 & R_{11} \end{bmatrix},$$

where R_{11} is the $(n(f_n))$ by $(n(f_n))$ submatrix of the R matrix in the QR factorization of $b_{1\lambda}$. The matrix is B_{ff} is clearly invertible (since R_{11} is invertible) and, hence, we can employ the conventional system reduction procedure detailed in Section (2.3). This yields a new system in the form of (13), but with a smaller matrix a.

Step 5: Terminating the Iterative Scheme

If the resulting new value a is singular, then the sequence of Steps 2–4 is repeated.[10] Otherwise, the resulting dynamic system is $aEd_{t+1}|I_t = bd_t + C_d(\mathbf{F})Ex_t|I_t$, which can be rewritten as $Ed_{t+1}|I_t = Wd_t + \Psi_d(\mathbf{F})Ex_t|I_t$ with $W = a^{-1}b$ and $\Psi_d(\mathbf{F}) = a^{-1}C_d(\mathbf{F})$.

3.3. CONVERGENCE OF THE ALGORITHM

We now establish that this algorithm will produce a non-singular reduced system of the form (11) with $p \leq n(y)$ iterations, i.e., the construction and application of p transformations $T(\mathbf{F})$ for equations and L (reordering transformations) for variables. This results requires two conditions: (i) that there is a non-zero determinantal polynomial, $|Az - B| \neq 0$ for some z and (ii) that for every set of initial conditions (each vector k_0) there exists a solution, i.e., there is a stochastic process $\{y_t\}_{t=0}^{\infty}$ such that the equations of the original model are satisfied at each date.

The first of these conditions is readily verifiable: it can be checked prior to starting on the algorithm. The second of these conditions is an assumption, which is discussed further below. The convergence result is stated formally as:

THEOREM 2. *The singular linear difference equation system* $A\ E_t y_{t+1} = B\ y_t + C(\boldsymbol{F})\ E_t x_t$ *has an equivalent dynamic system representation comprised of* $f_t + K\ d_t + \Psi_f(\boldsymbol{F})\ E_t x_t = 0$ *and* $E_t d_{t+1} = W d_t + \Psi_d(\boldsymbol{F})\ E_t x_t$, *where* f *and* d *are vectors containing the elements of* y, *if: (i) the determinental polynomial,* $|Az - B|$, *is not zero for all values of* z; *and (ii) there exists a solution to the singular linear difference equation system from every set of initial conditions* k_0. *Further, the system reduction algorithm will compute this equivalent, reduced system in a number of iterations* p *that is no more than the number of variables.*

The theorem is proved as follows. First, note that each algorithm iteration involves application of a matrix polynomial $T(\mathbf{F})$ with nonzero determinant τ to the dynamic system: this operation does not alter condition (i) since $|Az - B| \neq 0$ implies $|A^*z - B^*| = |T(z)||Az - B| = \tau|Az - B| \neq 0$. Second, if $\{\widetilde{y}_t\}_{t=1}^{\infty}$ is a solution to the original model, then it is also a solution to any transformed model obtained via such a $T(\mathbf{F})$ operation. (If the original model implies $A\ E_t\widetilde{y}_{t+1} - B\ \widetilde{y}_t - C(\mathbf{F})\ E_t x_t = 0$ then application of $T(\mathbf{F}) = G\mathbf{F} + H$ implies that $A^*\ E_t\widetilde{y}_{t+1} - B^*\ \widetilde{y}_t - T(\mathbf{F})C(\mathbf{F})\ E_t x_t = 0$ with $A^*\mathbf{F} - B^* = T(\mathbf{F})[A\mathbf{F} - B]$ so that $\{\widetilde{y}_t\}_{t=1}^{\infty}$ is a solution to the transformed model.)

The next parts of the proof of the theorem involve details of the algorithm. As stressed above, each iteration eliminates some elements of y from d and moves them into f so long as it is possible to construct another $T(\mathbf{F})$. Thus it is direct to establish an upper bound on the number of algorithm iterations: with one or more elements moved on each iteration, the algorithm must converge in less than $n(y)$ iterations.

The construction of a $T(\mathbf{F})$ at each iteration of the algorithm will be possible unless $n(f_n) = 0$ in Step 3, i.e., unless it is impossible to isolate one or more new flows. Since $n(f_n) = rank(b_{1\lambda})$, if $n(f_n) = 0$ then this implies that the matrix $b_{1\lambda}$ contains only zeros. That is, an equation of the transformed model is:

$$b_{1\lambda}\overline{\lambda}_t + b_{1k}k_t = 0\overline{\lambda}_t + b_{1k}k_t = \Psi_1(\mathbf{F})E_t x_t\ . \tag{16}$$

Now, suppose that, in addition $b_{1k} = 0$: then there is a row of zeros in $\overline{A}z - \overline{B}$ and, hence, $0 = |\overline{A}z - \overline{B}| = |Az - B|$. In this equality, the first equality follows from the implied rows of zeros in $T_1\overline{A}$ and $T_1\overline{B}$, and the second equality follows because the roots of the determinantal equation are unaltered by any of the transformations used in the reduction algorithm. Thus, condition (i) is violated in this case. Thus, condition (i) implies $b_{1k} \neq 0$ if $b_{1\lambda} = 0$.[11]

On the other hand, if $b_{1\lambda} = 0$ and $b_{1k} \neq 0$, then $b_{1k}k_t = \Psi_1(\mathbf{F})E_t x_t$ makes it is possible to solve for a subset of the predetermined variables k_t at each date t that are a function of the other predetermined variables and a distributed lead of the expected values of the x_t's. This outcome violates condition (ii), since it imposes constraints on the initial conditions for the predetermined variables. Hence, under conditions (i) and (ii), there cannot be an equation of the form (16). As a result, each successive iteration of the algorithm eliminates some new flows and ultimate convergence is guaranteed.

The theorem of this section is useful for two related purposes. First, it provides a useful interpretation of the failure of the algorithm in Step 3 in the context of a specific model: if the algorithm fails at this stage and it has been verified that $|Az - B| \neq 0$, then this failure implies that no solution of the model exists from all initial conditions. Second, the theorem establishes that the algorithm can find a reduced dynamic system of the form (11, 12) even when there is not a unique, stable solution of the form studied by Blanchard and Kahn (1980) and King and Watson (1998). Since Theorem 2 only postulates a solution from a set of arbitrary initial conditions, then the solution might include unstable dynamics of economic interest (for example, unit roots) or it might involve a multiplicity of RE equilibria along the lines of Farmer (1993). Hence, Theorem 2 broadens the range of models to which the system reduction algorithm can be applied.

4. Solving the Model

After application of the system reduction algorithm, the transformed dynamic system takes the form:

$$f_t = -[K_{f\lambda} K_{fk}] \begin{bmatrix} \lambda_t \\ k_t \end{bmatrix} - \Psi_f(\mathbf{F}) E x_t | I_t \tag{17}$$

$$E_t \begin{bmatrix} \lambda_{t+1} \\ k_{t+1} \end{bmatrix} = \begin{bmatrix} W_{\lambda\lambda} & W_{\lambda k} \\ W_{k\lambda} & W_{kk} \end{bmatrix} \begin{bmatrix} \lambda_t \\ k_t \end{bmatrix} + \begin{bmatrix} \Psi_\lambda(\mathbf{F}) \\ \Psi_k(\mathbf{F}) \end{bmatrix} E_t x_t \,. \tag{18}$$

This is a nonsingular dynamic system of the form studied by Blanchard and Kahn (1980) and which is at the heart of the model solution approach in KPR-TA.

The solution of this model is discussed in three steps. First, the implications of stability for the initial conditions on λ_t are determined. Second, the construction of perfect foresight solutions is considered. Finally, dynamic decision rules are constructed.

4.1. INITIAL CONDITIONS ON λ_t

The treatment of the determination of the initial conditions on λ_t follows Blanchard and Kahn (1980). To begin, we let V_u be a ($n(u)$ by $n(d)$) matrix that isolates the unstable roots of W.[12] That is, V_u has the property that

$$V_u W = \mu V_u ,$$

where μ is a lower triangular ($n(u)$ by $n(u)$) matrix with the unstable eigenvalues on its diagonal. Applying V_u to (18), then $V_u\, E_t d_{t+1} = V_u\, W d_t + V_u\, \Psi_d(\mathbf{F}) E_t x_t = \mu\, V_u\, d_t + V_u \Psi_d(\mathbf{F}) E_t x_t$.[13] The dynamics of $u_t = V_u d_t$ are then:

$$E_t u_{t+1} = \mu u_t + V_u \Psi_d(\mathbf{F}) E_t x_t . \tag{19}$$

This expression can be used to generate formulae for (a) perfect foresight solutions; and (b) Markov decision rules. In each case, u_t is chosen so that there is a stable system despite the existence of unstable roots. Following Sargent (1979) and Blanchard and Kahn (1980), this is accomplished by unwinding unstable roots forward. Practically, in the computer code discussed below, unstable roots are defined as follows. Suppose that there is a discount factor, β. Then, a root μ_i is unstable if $\beta\mu_i > 1$. This version of the requirement allows us to treat unit roots as stable.[14]

The solutions for u can be used to uniquely determine the date t behavior of the non-predetermined variables λ_t. As stressed by Blanchard–Kahn (1980) this requires that there be the same number of elements of u as there are non-predetermined variables. In terms of the condition $u_t = V_u d_t = V_{u\lambda}\lambda_t + V_{uk} k_t$, this requirement means that we have an equal number of equations and unknowns. However, it is also the case that a unique solution mandates that the (square) matrix $V_{u\lambda}$ is of full rank. This condition on $V_{u\lambda}$ is an implication of the more general rank condition that Boyd and Dotsey (1993) give for higher order linear rational expectations models. Essentially, the rank condition rules out matrices W with unstable roots associated with the predetermined variables rather than nonpredetermined variables.

4.2. PERFECT FORESIGHT (SEQUENCE) SOLUTIONS

Perfect foresight solutions, i.e., the response of the economy to a specified sequence of $x : \{x_s\}_{s=t}^{\infty}$, are readily constructed. Expression (19) implies that:

$$u_t = -[I - \mu^{-1}\mathbf{F}]^{-1}[\mu^{-1} V_u \Psi_d(\mathbf{F})] E_t x_t . \tag{20}$$

Call the result of evaluating the right-hand side of this expression $u_t = \Psi_u(\mathbf{F}) E_t x_t$. Then, $u_t = V_{u\lambda}\lambda_t + V_{u\lambda} k_t$, implies that the initial condition on λ_t is

$$\lambda_t = -V_{u\lambda}^{-1} V_{uk} k_t + V_{u\lambda}^{-1} \Psi_u(\mathbf{F}) E_t x_t . \tag{21}$$

With knowledge of λ_t, expressions (17) and (18) imply that:

$$f_t = -[K_{fk} - K_{f\lambda} V_{u\lambda}^{-1} V_{uk}] k_t - [K_{f\lambda} V_{u\lambda}^{-1} \Psi_u(\mathbf{F}) + \Psi_f(\mathbf{F})] E_t x_t \tag{22}$$

$$k_{t+1} = M_{kk}k_t + [W_{k\lambda}V_{u\lambda}^{-1}\Psi_u(\mathbf{F}) + \Psi_k(\mathbf{F})]E_t x_t \,, \tag{23}$$

where $M_{kk} = (W_{kk} - W_{k\lambda}V_{u\lambda}^{-1}V_{uk})$. Under perfect foresight, $\mathbf{F}^n x_t = x_{t+n}$. These solutions are of interest in their own right and also as inputs into the construction of Markov decision rules, which we consider next.

4.3. MARKOV DECISION RULES

Now suppose that x_t is generated by the process

$$x_t = \Theta\xi_t \tag{drvproc1}$$

and

$$\xi_t = \rho\xi_{t-1} + \theta\varepsilon_t \,, \tag{drvproc2}$$

where ε_t is a martingale difference sequence. Given the first-order character of the driving process, ξ_t is sufficient for $E_t x_{t+k}$ for all $k \geq 0$.

4.3.1. *Evaluating the Driving Process Polynomials*

Consider first the evaluation of $\Psi_d(\mathbf{F})E_t x_t$, and $\Psi_f(\mathbf{F})E_t x_t$. Each of these is a polynomial of the form $\Psi(\mathbf{F}) = \Psi_0 + \Psi_1\mathbf{F} + \Psi_2\mathbf{F}^2 + \cdots + \Psi_n\mathbf{F}^n$. With $E_t x_{t+h} = \Theta\rho^h\xi_t$, it follows that

$$\Psi(\mathbf{F})E_t x_t = [\Psi_0\Theta + \Psi_1\Theta\rho + \Psi_2\Theta\rho^2 + \cdots + \Psi_n\Theta\rho^n]\xi_t \equiv \varphi\xi_t \,. \tag{24}$$

Thus, for example it follows that $V_u\Psi_d(\mathbf{F})E_t x_t = V_u[\Psi_{d0}\Theta + \Psi_{d1}\Theta\rho + \Psi_{d2}\Theta\rho^2 + \cdots + \Psi_{dn}\Theta\rho^n]\xi_t = \varphi_u\xi_t$.

4.3.2. *Computing the Initial Value of λ_t*

To this point, V_u has been treated as a matrix which 'isolated the unstable roots of W'. From a theoretical perspective, the best known such matrices are the eigenvector matrices or, in the case of repeated roots, the matrices in the Jordan form. The computational approach in KPR-TA and related programs – which were designed to handle multiple state variables – simply assumed distinct roots and used the eigenvalue-eigenvector decompositions in GAUSS and MATLAB. However, from the perspective of numerical analysis, a more accurate and stable approach is provided by use of the Schur form, which means that μ is not diagonal but is only lower triangular, with the eigenvalues appearing on the diagonal. There now is an operational difference in our formulas, depending on whether we assume that μ is diagonal or simply lower triangular.

Eigenvalue-Eigenvector Method: In the familiar first case of a diagonal μ, we proceed as follows. Letting φ_{ui} be the i'th row of φ_u, it follows that the unstable canonical variables each evolve according to equations of the form:

$$E_t u_{i,t+1} = \mu_i u_{it} + \varphi_{ui}\xi_t \,. \tag{25}$$

Iterating this expression forward yields

$$u_{it} = -\sum_{j=0}^{\infty} \mu_i^{-j-1} \varphi_{ui} E_t \xi_{t+j} = -\left[\sum_{j=0}^{\infty} \mu_i^{-j-1} \varphi_{ui} \rho^j\right] \xi_t = \varphi_{ui}[\rho - \mu_i I]^{-1} \xi_t \,.$$

For convenience write this solution as $u_{it} = \nu_i \xi_t$, with $\nu_i = \varphi_{ui}[\rho - \mu_i I]^{-1}$ and correspondingly let $u_t = \nu \xi_t$, with the i'th row of ν being ν_i. (Alternatively, one can use an undetermined coefficients representation, $u_{it} = \nu_i \xi_t$ in $E_t u_{i,t+1} = \mu_i u_{it} + \varphi_{ui} \xi_t$ to find that $\nu_i \rho = \mu_i \nu_i I + \varphi_{ui}$ and thus conclude that $\nu_i = \varphi_{ui}[\rho - \mu_i I]^{-1}$: this alternative derivation will be useful in discussing the general lower triangular μ case below.) Thus, we have obtained a solution for u_t, which can be used to solve for λ_t along the same lines as (21).

Schur Decomposition Method: With μ lower triangular, as is the case when the Schur decomposition is used to construct V_u and μ, let the ith row of μ be denoted by $[\mu_{i1} \ldots \mu_{ii} 0 \ldots 0]$. The expression analogous to (25) is:

$$E_t u_{i,t+1} = \sum_{j=1}^{i} \mu_{ij} u_{jt} + \varphi_{ui} \xi_t \,.$$

It follows that the first of these expressions can be solved exactly as previously $u_{1t} = \nu_1 \xi_t$, with $\nu_1 = \varphi_{u1}[\rho - \mu_{11} I]^{-1}$. Given this solution, it follows that $u_{2t} = \nu_2 \xi_t$, with $\nu_2 = (\varphi_{u2} + \mu_{21} \nu_1)[\rho - \mu_{22} I]^{-1}$. Proceeding recursively, the above expression can be written as

$$\begin{aligned} E_t u_{i,t+1} &= \mu_{ii} u_{it} + \sum_{j=1}^{i-1} \mu_{ij} u_{jt} + \varphi_{ui} \xi_t \\ &= \mu_{ii} u_{it} + \left(\sum_{j=1}^{i-1} \mu_{ij} \nu_j + \varphi_{ui}\right) \xi_t \end{aligned}$$

and then solved using the same undetermined coefficients strategy as in the previous case, to conclude that $\nu_i = (\sum_{j=1}^{i-1} \mu_{ij} \nu_j + \varphi_{ui})[\rho - \mu_{ii} I]^{-1}$. Hence, there is an easy-to-implement way of calculating $u_t = \nu \xi_t$, under the Schur decomposition as well.

In either case, then since $u_t = \nu \xi_t$, the requirement that $u_t = V_{u\lambda} \lambda_t + V_{uk} k_t$, implies that the initial condition on λ_t is:

$$\lambda_t = -V_{u\lambda}^{-1} V_{uk} k_t + V_{u\lambda}^{-1} \nu \xi_t \,. \tag{26}$$

4.3.3. *Solutions for the Other Variables*

With knowledge of this solution for λ_t, expressions (17) and (18) imply that:

$$f_t = [K_{fk} - K_{f\lambda} V_{u\lambda}^{-1} V_{uk}] k_t + [K_{f\lambda} V_{u\lambda}^{-1} \nu + \varphi_f] \xi_t \tag{27}$$

$$k_{t+1} = M_{kk} k_t + [W_{k\lambda} V_{u\lambda}^{-1} \nu + \varphi_k] \xi_t \,, \tag{28}$$

where $M_{kk} = (W_{kk} - W_{k\lambda} V_{u\lambda}^{-1} V_{uk})$ as above and φ_f and φ_k are the evaluations of the Ψ_f and Ψ_k polynomials (as in (24)).

4.3.4. *The RE Solution in State Space Form*

These solutions can be grouped together with other information to produce a state space system, i.e., a pair of specifications $Z_t = \Pi S_t$ and $S_{t+1} = M\ S_t + N e_{t+1}$. The specific form is:

$$\begin{bmatrix} f_t \\ \lambda_t \\ k_t \\ x_t \end{bmatrix} = \begin{bmatrix} [K_{fk} - K_{f\lambda} V_{u\lambda}^{-1} V_{uk}] & [K V_{u\lambda}^{-1} \nu + \varphi_f] \\ -V_{u\lambda}^{-1} V_{uk} & V_{u\lambda}^{-1} \nu \\ I & 0 \\ 0 & \Theta \end{bmatrix} \begin{bmatrix} k_t \\ \xi_t \end{bmatrix} \tag{29}$$

$$\begin{bmatrix} k_{t+1} \\ \xi_{t+1} \end{bmatrix} = \begin{bmatrix} M_{kk} & [W_{k\lambda} V_{u\lambda}^{-1} \nu + \varphi_k] \\ 0 & \rho \end{bmatrix} \begin{bmatrix} k_t \\ \xi_t \end{bmatrix} + \begin{bmatrix} 0 \\ \theta \end{bmatrix} \varepsilon_{t+1} \,. \tag{30}$$

Further, given $\begin{bmatrix} f_t \\ d_t \end{bmatrix} = L y_t$, it is easy to restate this representation in terms of the original ordering of the variables.

4.4. HOW IT WORKS IN PRACTICE

The algorithms described above are available in MATLAB and GAUSS.[15] In each case, the researcher specifies the dynamic system of the form (2), specifying the matrices A, B, $[C_0 \ldots C_l]$; the location of the predetermined variables in the vector y_t; and the number of leads l in the C polynomial. The researcher also specifies the driving process (4.3, 4.3). The programs return a solution in the state space form (29, 30), which makes it easy to compute impulse responses, spectra, population moments and stochastic simulations. This solution is also a convenient one for estimation and model evaluation.

We make it easy for the researcher to check the necessary condition for model solvability, $|Az - B| \neq 0$. If the model solution package fails when this condition is satisfied, it must then be for one of two other reasons:

(1) the reduction algorithm terminates without reaching a solution. In our experience, this failure has nearly always meant that the model is misspecified, most

likely as a result of a programming error, so that new flows cannot be isolated. Very occasionally, we have included a behavioral equation of the form,

$$b_{1\lambda}\overline{\lambda}_t + b_{1k}k_t = e\overline{\lambda}_t + b_{1k}k_t = -\Psi_1(\mathbf{F})E_t x_t\,,$$

where e is a very small number or a vector containing zeros and very small numbers. Then, we must decide whether $e = 0$ so that the model is not solvable or to adjust tolerance values to treat this as an equation to be solved for some new flows.

(2) the model solution algorithms reports the wrong number of unstable roots relative to the number of predetermined variables. After excluding the possibility that the locations of the predetermined variables has been erroneously specified, there are two remaining possibilities. First, stability is defined in our code in a relative rather than absolute sense: an adjustment of the critical value for stability, the parameter β, may be necessary. Second, the model may actually not have an RE equilibrium or there may be a multiplicity of equilibria.[16]

5. Extension to Timing Models

An important class of macroeconomic models views the discrete time period t as divided into subperiods that are distinguished by differing information sets. For example, analyses of the real effects of monetary disturbances in flexible price models frequently make such assumptions, as in Svensson (1985) or Christiano and Eichenbaum (1992). An extension of the approach outlined above can be used for these models as well.[17] Thus, consider the a generic timing model in the form:

$$\begin{aligned} &A_0 Ey_{t+1}|I_{0t} + A_1 Ey_{t+1}|I_{1t} + \cdots + A_J Ey_{t+1}|I_{Jt} \\ &\quad = B_0 Ey_t|I_{0t} + B_1 Ey_t|I_{1t} + \cdots + B_J Ey_t|I_{Jt} \\ &\quad + C_0(\mathbf{F})Ex_t|I_{0t} + C_1(\mathbf{F})Ex_t|I_{1t} + \cdots + C_J(\mathbf{F})Ex_t|I_{Jt}\,, \end{aligned} \tag{31}$$

where I_{0t} is no current information, I_{Jt} is full current information, and the information sets are ordered as increasing in information ($I_0 \subset I_{1t} \ldots \subset I_{jt} \ldots \subset I_{Jt}$). In this expression, as earlier in this article, $\mathbf{F}$ is the lead operator defined so as to shift the dating of the variable but not of the information set, so that $C_j(\mathbf{F})Ex_t|I_{jt}$ represents the distributed lead $C_{0j}Ex_{t+1}|I_{jt} + C_{1j}Ex_{t+1}|I_{jt} + \cdots + C_{Jj}Ex_{t+J}|I_{jt}$.[18]

There is a simple two-stage strategy for solving such timing models. First, one solves a common information model that is derived by conditioning on I_{0t}: this solution is straightforward given the results in this paper and constitutes the harder part of the RE solution. Second, one uses that solution to solve for the influences of the particular timing model under study.

5.1. THE COMMON INFORMATION MODEL

In order to solve the general timing model (31), we begin by study the operation of a related common information model, so as to determine the effects of anticipated driving variables. For this purpose, take expectations of (31) conditional on beginning of period t information:

$$\begin{aligned}(A_0 + A_1 + \cdots + A_J)Ey_{t+1}|I_{0t} \\ = (B_0 + B_1 + \cdots + B_J)Ey_t|I_{0t} \\ + (C_0(\mathbf{F}) + C_1(\mathbf{F}) + \cdots + C_J(\mathbf{F}))Ex_t|I_{0t}\,.\end{aligned} \tag{32}$$

This looks just like the singular linear difference system under rational expectations studied above, $AEy_{t+1}|I_t = By_t + C(\mathbf{F})Ex_t|I_t$, with two modifications which are minor. First, the matrices A, B, and $C(\mathbf{F})$ must be defined as appropriate sums of the matrices in the timing model.[19] Second, the information set I_t now corresponds to I_{Jt}, with I_{0t}. But these modifications are (i) unimportant for the mechanics of system reduction and (ii) easy to deal with in terms of the computation of Markov decision rules. In terms of the latter, simply replace ξ_t with $E\xi_t|I_{0t} = \rho\,\xi_{t-1}$. That is, the anticipated driving variable part of the system is:

$$\begin{bmatrix} Ey_t|I_{0t} \\ Ex_t|I_{0t} \end{bmatrix} = \begin{bmatrix} \Pi_{yk} & \Pi_{y\xi} \\ 0 & \Theta \end{bmatrix} \begin{bmatrix} k_t \\ E\xi_t|I_{0t} \end{bmatrix} \tag{33}$$

$$\begin{bmatrix} Ek_{t+1}|I_{0t} \\ E\xi_{t+1}|I_{0t} \end{bmatrix} = \begin{bmatrix} M_{kk} & M_{k\xi} \\ 0 & \rho \end{bmatrix} \begin{bmatrix} k_t \\ E\xi_t|I_{0t} \end{bmatrix}, \tag{34}$$

where the Π and M matrices are exactly those discussed above.

5.2. EFFECTS OF UNANTICIPATED DRIVING VARIABLES

The hard work in a timing model comes in determining the effects of ε_t since it is these shocks about which there is differential information implied by the timing structure. Subtracting (32) from (31) yields an expression that describes the evolution of unanticipated endogenous variables:

$$\begin{aligned}A_1(Ey_{t+1}|I_{1t} - Ey_{t+1}|I_{0t}) + \cdots + A_J(Ey_{t+1}|I_{Jt} - Ey_{t+1}|I_{0t}) \\ = B_1(Ey_t|I_{1t} - Ey_t|I_{0t}) + \cdots + B_J(Ey_t|I_{Jt} - Ey_t|I_{0t}) \\ + C_1(\mathbf{F})(Ex_t|I_{1t} - Ex_t|I_{0t}) + \cdots + C_J(\mathbf{F})(Ex_t|I_{Jt} - Ex_t|I_{0t})\,,\end{aligned} \tag{35}$$

where terms like $(Ey_t|I_{jt} - Ey_t|I_{0t})$ indicate how the receipt of information between subperiod 0 and subperiod j alters the value of the endogenous variables. This expression provides the basic restrictions that are exploited to determine the model economy's response to shocks at date t.

It is useful if these restrictions are written in the following 'one subperiod ahead innovation' form:

$$\begin{aligned}&\underline{A}_1(Ey_{t+1}|I_{1t} - Ey_{t+1}|I_{0t}) + \cdots + \underline{A}_J(Ey_{t+1}|I_{Jt} - Ey_{t+1}|I_{J-1,t})\\ &\quad = \underline{B}_1(Ey_t|I_{1t} - Ey_t|I_{0t}) + \cdots + \underline{B}_J(Ey_t|I_{Jt} - Ey_t|I_{J-1,t})\\ &\quad + \underline{C}_1(\mathbf{F})(Ex_t|I_{1t} - Ex_t|I_{0t}) + \cdots + \underline{C}_J(\mathbf{F})(Ex_t|I_{Jt} - Ex_t|I_{J-1,t})\,,\end{aligned} \tag{36}$$

where $\underline{A}_j = (A_j + \cdots + A_J)$; $\underline{B}_j = (B_j + \cdots + B_J)$; and $\underline{C}_j(\mathbf{F}) = (C_j(\mathbf{F}) + \cdots + C_J(\mathbf{F}))$.

This expression implies that we can concentrate on finding the solution to

$$\begin{aligned}&\underline{A}_j(Ey_{t+1}|I_{jt} - Ey_{t+1}|I_{j-1,t})\\ &\quad = \underline{B}_j(Ey_t|I_{jt} - Ey_t|I_{j-1,t})\\ &\underline{C}_j(\mathbf{F})(Ex_t|I_{jt} - Ex_t|I_{j-1,t})\end{aligned} \tag{37}$$

for each $j = 1, 2, \ldots J$.

5.2.1. *Solving for the Effects of Shocks in Subperiod j*

This subsection discusses how to use (37) to determine the effects of shocks in subperiod j. It is useful to partition the vector y into parts that are nonpredetermined (Λ) and parts that are predetermined (k). That is, write $y = \begin{bmatrix} \Lambda \\ k \end{bmatrix}$.

Since k_t is predetermined, it does not respond to shocks within any of the subperiods j of period t. Hence, the term $\underline{B}_j(Ey_t|I_{jt} - Ey_t|I_{j-1,t})$ in (37) simplifies to $\underline{B}_{j,\Lambda}(E\Lambda_t|I_{jt} - E\Lambda_t|I_{j-1,t})$, where $\underline{B}_{j,\Lambda}$ involves the relevant columns of $\underline{B}_j$. Further, from (34), $Ey_{t+1}|I_{jt} = E[Ey_{t+1}|I_{0,t+1}]|I_{jt} = \Pi_{yk}Ek_{t+1}|I_{jt} + \Pi_{y\xi}E\xi_{t+1}|I_{jt}$ and $E\xi_{t+1}|I_{jt} = \rho E\xi_t|I_{jt} = \rho(\rho\xi_{t-1} + \theta E\varepsilon_t|I_{jt})$ given the driving process in (4.3) above. Accordingly, it follows that:

$$\begin{aligned}&\underline{A}_j(Ey_{t+1}|I_{jt} - Ey_{t+1}|I_{j-1,t})\\ &\quad = \underline{A}_j\Pi_{yk}(Ek_{t+1}|I_{jt} - Ek_{t+1}|I_{j-1,t})\\ &\quad + \underline{A}_j\Pi_{y\xi}\rho\theta(E\varepsilon_t|I_{jt} - E\varepsilon_t|I_{j-1,t})\,.\end{aligned} \tag{38}$$

Since $\underline{C}_j(\mathbf{F}) = \underline{C}_{j,0} + \underline{C}_{j,1}\mathbf{F} + \cdots + \underline{C}_{j,p}\mathbf{F}^p$, it is direct that $\underline{C}_j(\mathbf{F})(Ex_t|I_{jt} - Ex_t|I_{j-1,t}) = (\underline{C}_{j,0}\Theta + \underline{C}_{j,1}\Theta\rho + \cdots + \underline{C}_{j,p}\Theta\rho^p)\,(E\varepsilon_t|I_{jt} - E\varepsilon_t|I_{j-1,t})$: call this solution $\underline{\Psi}_j(E\varepsilon_t|I_{jt} - E\varepsilon_t|I_{j-1,t})$.

Combining these results, (37) can be written as:

$$\begin{aligned}&\left[\, -\underline{B}_{j,\Lambda} \;\; \underline{A}_j\Pi_{yk} \,\right] \begin{bmatrix} E\Lambda_t|I_{jt} - E\Lambda_t|I_{j-1,t} \\ Ek_{t+1}|I_{jt} - Ek_{t+1}|I_{j-1,t} \end{bmatrix}\\ &\quad = \left[\underline{\Psi}_j - \underline{A}_j\Pi_{y\rho}\theta\right]\left[E\varepsilon_t|I_{jt} - E\varepsilon_t|I_{j-1,t}\right]\,.\end{aligned} \tag{39}$$

Solving (39), determines the variables at t as functions of shocks, taking as given aspects of the future solution (as provided by Π_{yk} and $\Pi_{y\xi}$).

What is known about subperiod j that is useful in considering the solution to (39)? The specification of the economic problem shows which variables have been determined in subperiods prior to j, i.e., which elements of

$$q_{jt} = \begin{bmatrix} E\Lambda_t|I_{jt} - E\Lambda_t|I_{j-1,t} \\ Ek_{t+1}|I_{jt} - Ek_{t+1}|I_{j-1,t} \end{bmatrix}$$

are zero by the structure of the model. Further, the model also supplies the information that is revealed in the subperiod, i.e., $E\varepsilon_t|I_{jt} - E\varepsilon_t|I_{j-1,t}$.

From a mathematical point of view, (39) contains more equations ($n(y)$) than unknowns since some of the elements of q_{jt} are determined prior to subperiod j. This is accommodated as follows. First, write (39) as $Mq_{jt} = N(E\varepsilon_t|I_{jt} - E\varepsilon_t|I_{j-1,t})$ and use the fact that there is a permutation matrix P_q which makes the first p elements of q_{jt} the predetermined ones.[20] Second, translate the system to: $(M\ P_q')(P_q q_{jt}) = N(E\varepsilon_t|I_{jt} - E\varepsilon_t|I_{j-1,t})$. Third, partition $\widetilde{M} \equiv M\ P_q'$ as $[\widetilde{M}_1 \widetilde{M}_2]$, with the first p columns of $\widetilde{M}$ being those that are attached to the predetermined variables, q_{jt}^1, and the last $n(y) - p$ columns being those attached to the variables determined in subperiod j and later, q_{jt}^2. Thus, the system takes the form:

$$\widetilde{M}_2 q_{jt}^2 = N(E\varepsilon_t|I_{jt} - E\varepsilon_t|I_{j-1,t})\,.$$

This system is $n(y)$ equations in $n(y) - p$ unknowns, which are coefficients η_j that describe how the elements of q_{jt}^2 respond to the shocks $E\varepsilon_t|I_{jt} - E\varepsilon_t|I_{j-1,t}$. This system can be solved using the QR factorization, checking in the process that there is indeed a consistent solution (by determining that the last p rows of $Q'N$ are all zeros, where Q is the lead matrix in the QR factorization of $\widetilde{M}_2$).

Proceeding similarly through subperiods $j = 1, 2, \ldots J$ yields a matrix η:

$$\begin{bmatrix} \Lambda_t - E\Lambda_t|I_{0t} \\ k_{t+1} - Ek_{t+1}|I_{0t} \end{bmatrix} = \eta(\varepsilon_t - E\varepsilon_t|I_{0t})\,,$$

where η contains many zeros due to the subperiod structure. Partition this matrix as:

$$\eta = \begin{bmatrix} \eta_\Lambda \\ \eta_k \end{bmatrix}$$

and use these matrices in augmenting the solutions for the anticipated components. The overall solution for the observable variables is thus:

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} \Pi_{yk} & \Pi_{y\xi} \\ 0 & \Theta \end{bmatrix} \begin{bmatrix} k_t \\ \rho\xi_{t-1} \end{bmatrix} + \begin{bmatrix} \eta_y \\ 0 \end{bmatrix} \varepsilon_t\,,$$

where $\eta_y = \begin{bmatrix} \eta_\Lambda \\ 0 \end{bmatrix}$ with the last lines of zeros corresponding to the elements of k. The predetermined variables and driving variables evolve according to:

$$\begin{bmatrix} k_{t+1} \\ \xi_{t+1} \end{bmatrix} = \begin{bmatrix} M_{kk} & M_{k\xi} \\ 0 & \rho \end{bmatrix} \begin{bmatrix} k_t \\ \rho\xi_{t-1} \end{bmatrix} + \begin{bmatrix} \eta_k \\ 0 \end{bmatrix} \varepsilon_t + \begin{bmatrix} 0 \\ I \end{bmatrix} \varepsilon_{t+1}$$

These equations may easily be placed in a state space form. The observables evolve according to:

$$z_t = \begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} \Pi_{yk} & \Pi_{y\xi}\rho & \eta_y \\ 0 & \Theta\rho & \Theta \end{bmatrix} \begin{bmatrix} k_t \\ \xi_{t-1} \\ \varepsilon_t \end{bmatrix}$$

and the states evolve according to:

$$\begin{bmatrix} k_{t+1} \\ \xi_t \\ \varepsilon_{t+1} \end{bmatrix} = \begin{bmatrix} M_{kk} & M_{k\xi}\rho & \eta_k \\ 0 & \rho & I \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_t \\ \xi_{t-1} \\ \varepsilon_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \varepsilon_{t+1} .$$

6. Conclusions

In this article, we described an algorithm for the reduction of singular linear difference systems under rational expectations of the form (1) and alternative algorithms for the solution of the resulting reduced dimension system. We also described an extension to 'timing models', i.e., to variants of (1) with distinct informational subperiods. This set of methods is flexible and easy to use.

Appendix: Proof of Theorem 1

The proof relies on results in King and Watson (1998). That paper developed the conditions under which singular linear difference system can be uniquely solved under rational expectations. There were two necessary and sufficient conditions. First, $|Az - B| \neq 0$ for some z. Second, a particular matrix $V_{U\Lambda}$ had full rank: this matrix requirement was necessary to link non-predetermined variables Λ to unstable canonical variables U, where the definition of unstable canonical variables contained both finite and infinite eigenvalues. For the convenience of a reader making the transition from King and Watson (1998), the notation in this section follows that paper and not the main text.

Begin with the solvable dynamic system written in a canonical form employed in King and Watson (1998):[21]

$$(A^*\mathbf{F} - B^*)V\ E_t y_t = C^* E_t x_t ,$$

where V is a variable transformation matrix and

$$[A^*\mathbf{F} - B^*] = \begin{bmatrix} (N\mathbf{F} - I) & 0 \\ 0 & (\mathbf{F}I - J) \end{bmatrix} .$$

with N being a nilpotent matrix.[22] King and Watson (1998, particularly Appendix A) show that such an equivalent system can always be constructed for

any model that possesses a unique, stable rational expectations solution. The transformed variables in this canonical form are

$$\begin{bmatrix} U_t \\ s_t \end{bmatrix} \equiv \begin{bmatrix} i_t \\ u_t \\ s_t \end{bmatrix} = V\ E_t y_t$$

with the partition defined by the magnitude of the eigenvalues. First, there are as many elements of s as there are stable eigenvalues of J. Second, there are as many elements of i_t as there are columns of N. As explained in King and Watson (1998), these are canonical variables associated with 'infinite' eigenvalues.

The intuition behind the proof of Theorem 1 is that the infinite canonical variables, i_t, of King and Watson (1998) are governed by dynamic identities and, consequently, can be related to the f_t variables of the present article. Once this relationship is established and exploited to eliminate the f_t, the remaining variables d_t are governed by a reduced dimension dynamic system that depends on u_t, s_t in the same way as the original model.

Linking i_t and f_t

Premultiplication of the canonical dynamic system by

$$\eta(\mathbf{F}) = \begin{bmatrix} (N\mathbf{F} - I)^{-1} & 0 \\ 0 & I \end{bmatrix}$$

results in

$$\begin{bmatrix} I & 0 \\ 0 & (\mathbf{F}I - J) \end{bmatrix} V\ E_t y_t = \eta(\mathbf{F}) C^* E_t x_t\ .$$

Now, partition the transformation of variables matrix V as follows

$$V = \begin{bmatrix} V_{if} & V_{id} \\ V_{\delta f} & V_{\delta d} \end{bmatrix} = \begin{bmatrix} V_{if} & V_{i\lambda} & V_{ik} \\ V_{uf} & V_{u\lambda} & V_{uk} \\ V_{sf} & V_{s\lambda} & V_{sk} \end{bmatrix},$$

using the notation $d' = [\lambda' k']$ to denote the dynamic variables and $\delta' = [u' s']$. Let

$$V_{U\Lambda} = \begin{bmatrix} V_{if} & V_{i\lambda} \\ V_{uf} & V_{u\lambda} \end{bmatrix}.$$

where λ are those elements Λ left after the removal of f_t that are thus part of the dynamic vector d. Without loss of generality, we will assume that the elements of y are ordered so that V_{if} is square and nonsingular. This is always possible given that $V_{U\Lambda}$ is nonsingular.

Using the first partitions of V, the dynamic system can be written

$$\begin{bmatrix} V_{if} & V_{id} \\ (\mathbf{F}I - J)V_{\delta f} & (\mathbf{F}I - J)V_{\delta d} \end{bmatrix} E_t y_t = \eta(\mathbf{F}) C^* E_t x_t\ .$$

Notice that there are no lead operators in the first $n(i) = n(f)$ rows of this transformed system.

Construction of Reduced System

Now we proceed to construct a reduced dimension dynamic system. To accomplish this, we premultiply by a matrix polynomial $T(\mathbf{F})$ that is partitioned as

$$\begin{bmatrix} T_{11}(\mathbf{F}) & T_{12}(\mathbf{F}) \\ T_{21}(\mathbf{F}) & T_{22}(\mathbf{F}) \end{bmatrix}$$

with components that satisfy:

$$T_{11}(\mathbf{F})V_{if} + T_{12}(\mathbf{F})(\mathbf{F}I - J)V_{\delta f} = I$$

$$T_{11}(\mathbf{F})V_{id} + T_{12}(\mathbf{F})(\mathbf{F}I - J)V_{\delta d} = K$$

$$T_{21}(\mathbf{F})V_{if} + T_{22}(\mathbf{F})(\mathbf{F}I - J)V_{\delta f} = 0$$

$$T_{21}(\mathbf{F})V_{id} + T_{22}(\mathbf{F})(\mathbf{F}I - J)V_{\delta d} = [\mathbf{F}I - W],$$

where K and W are unknown matrices.

The first two of these conditions may be used to set

$$T_{11}(\mathbf{F}) = V_{if}^{-1}$$

$$T_{12}(\mathbf{F}) = 0$$

which jointly imply that $K = V_{if}^{-1}V_{id}$.

Employing the third and fourth condition together

$$T_{22}(\mathbf{F})\{(\mathbf{F}I - J)[V_{\delta d} - V_{\delta f}V_{if}^{-1}V_{id}]\} = \mathbf{F}I - W.$$

Standard results on the determinants and inverses of partitioned matrices imply that $\hat{V} = [V_{\delta d} - V_{\delta f}V_{if}^{-1}V_{id}]$ is an invertible matrix. Accordingly, this condition may be satisfied with

$$T_{22}(\mathbf{F}) = \hat{V}^{-1}$$

and

$$W = \hat{V}^{-1}J\hat{V}.$$

A further implication is then that

$$T_{21}(\mathbf{F}) = -\hat{V}^{-1}(\mathbf{F}I - J)V_{\delta f}V_{if}^{-1}.$$

Hence, it follows that

$$T(\mathbf{F}) = \begin{bmatrix} V_{if}^{-1} & 0 \\ -\widehat{V}^{-1}(\mathbf{F}I - J)V_{\delta f}V_{if}^{-1} & \widehat{V}^{-1} \end{bmatrix}$$

Notice that $|T(\mathbf{F})| = (|V_{if}||\hat{V}|)^{-1}$, which is a nonzero constant. Accordingly, construction of the reduced system involves operating on $(A^*\mathbf{F} - B^*)V E_t y_t = C^* E_t x_t$ with the 'equation transformation' $\widehat{T}(\mathbf{F}) \equiv T(\mathbf{F})\eta(\mathbf{F})$ to produce $(A^{**}\mathbf{F} - B^{**})E_t y_t = T(\mathbf{F})\eta(\mathbf{F})E_t x_t = \Psi(\mathbf{F})E_t x_t$. In this expression, the transformed system matrices take the form:

$$A^{**} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

$$B^{**} = \begin{bmatrix} I & K \\ 0 & W \end{bmatrix}$$

which is the reduced system that we seek.

Properties of the Reduced System

We now demonstrate and discuss key properties of the reduced dimension nonsingular system, which also establish that it satisfies the Blanchard and Kahn (1980) conditions for solvability.

First, we indicate that the finite eigenvalues of the reduced system

$$E_t d_{t+1} = W\, d_t + \Psi_d(\mathbf{F}) E_t x_t$$

are identical to those of the original system. Previously, we have seen that

$$W = \hat{V}^{-1} J \hat{V}.$$

Accordingly, $\hat{V}$ is a left eigenvector matrix of W. The corresponding eigenvalues are the diagonal elements of J. The invariance of the eigenvalues may be viewed as an implication of the fact that $|\widehat{T}(z)| = |T(z)||\eta(z)|$ is a nonzero constant and, hence, this transformation does not affect the determinantal polynomial.

Second, we want to establish that solvability of the original dynamic system implies solvability of the reduced dynamic system. The earlier solvability condition was that $V_{U\Lambda}$ was square and invertible. We now need to establish that this implies that $\hat{V}_{u\lambda}$ is square and nonsingular. To show this, we write

$$\hat{V} \equiv \begin{bmatrix} \widehat{V}_{u\lambda} & \widehat{V}_{uk} \\ \widehat{V}_{u\lambda} & \widehat{V}_{jk} \end{bmatrix} = [V_{\delta d} - V_{\delta f} V_{if}^{-1} V_{id}]\,.$$

Using the second of the partitionings displayed above, we find that

$$\hat{V}_{u\lambda} = V_{u\lambda} - V_{uf} V_{if}^{-1} V_{i\lambda}\,.$$

Since

$$|V_{U\Lambda}| = \begin{vmatrix} V_{if} & V_{i\lambda} \\ V_{u\delta} & V_{u\lambda} \end{vmatrix} = |V_{if}| \cdot |V_{u\lambda} - V_{uf} V_{if}^{-1} V_{i\lambda}|,$$

it follows that $|\hat{V}_{u\lambda}| = |V_{U\Lambda}|/|V_{if}| \neq 0$. Thus, $\hat{V}_{u\lambda}$ is invertible and the reduced dimension (nonsingular) system is solvable according to the Blanchard and Kahn (1980) conditions.

Notes

[1] A reader making comparisons with KPR-TA will notice that we have left two constraints on the optimization problem rather than combining them into one and therefore added a multiplier. It is frequently convenient to do this for simplicity in calculating and approximating the first order conditions. One advantage of the system reduction algorithm developed in Section 3 is that it makes it essentially costless for the researcher to do this.

Also, in this and the subsequent example, we ignore productivity trends for simplicity. KPR-TA shows how to transform the growing economy to a stationary one with some modifications of parameters.

[2] Throughout the paper, we denote the number of elements in an arbitrary vector w_t by $n(w)$. We also use the notation n_s to indicate the number of variables s in several contexts.

[3] Crucini (1991) encountered the sort of singularity discussed in this section while constructing an international real business cycle model with two locations and costless mobility of capital. In the more general model of Baxter and Crucini (1993), which incorporates investment adjustment costs, the singularity arises only in the special case when adjustment costs are zero.

[4] Before leaving this example, it is worth pointing out that it is possible to solve it 'by hand', as in Crucini (1991). To do this, one needs to first be flexible about what is an elment of f_t. The first three linear equations above can clearly be used to solve for $\widehat{c}_t$, $\widehat{p}_t$, and $\widehat{\lambda}_{1t}$ as functions of $\widehat{\lambda}_{2t}$. Solving these out and manipulating remaining equations, it is possible to determine that there is a reduced dynamic system in $\widehat{\lambda}_{2t}$, $\widehat{k}_{1t}$, and $\widehat{k}_{2t}$. These manipulations involve using relationships between variables at $t+1$ (elements of the A matrix) and relationships between variables at t (elements of the B matrix) that are carefully designed to produce the desired result. A further analysis of this model, which stresses the interplay between the model's economic structure and the nature of system reduction is available at http://people.bu.edu/rking/kwre/irbc2.pdf

[5] Another approach to solving singular linear difference systems is provided in work by Sims (1989), who uses the QZ-algorithm to transform the a singular system into an equivalent system that can be solved recursively. All models that can be solved with our technology can also be solved with Sims'. The QZ-algorithm originates in Moler and Stewart (1971) and is discussed by Golub and Van Loan (1989, p. 394).

[6] This question was posed to us by Kei Mu Yi and Andrew John.

[7] We thank Adrian Pagan for pointing out the relationship between our work and Luenberger's work on 'descriptor systems', which are specifications of the form $Ay_{t+1} = By_t + Cx_t$. Luenberger's terminology makes clear that he has in mind systems for which equilibria are not the solution to a control problem, but rather contain some 'descriptive' dynamic equations. While this accords with our view that the analysis of 'suboptimal dynamic equilibria' is a major motivation for 'singular linear difference systems under rational expectations', it is worthwhile stressing that there are benefits to posing even standard control problems in this manner, as suggested by the two location growth model example.

[8] The svd commands in MATLAB and GAUSS produce positive singular values as the initial entries on the diagonal of S so that the results must be transformed in our programs to match the ordering in the text.

[9] The QR factorization of a $p \times m$ matrix M with rank r is given by

$$QR = MP\,, \tag{15}$$

where P is an m by m permutation matrix (i.e., P can be constructed by reordering the rows of I_m); Q is a p by p unitary matrix; and R is a p by m matrix such that

$$R = \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix}.$$

where R_{11} is an r by r upper diagonal, nonsingular matrix and R_{12} is an r by $(m - r)$ matrix.

The QR factorization is useful because it allows us to solve the equation system $My = N$ for r of the elements of y in terms of the remaining $(m - r)$ elements of y and the parameters N. That is, we can write the equation $My = N$ as $RP'y = Q'N$ and partition $P'y = [y_1' y_2']'$. Then, the solution is $y_1 = R_{11}^{-1}(G_1 - R_{12}y_2)$, where G_1 is the first r rows of $Q'M$. The equation system is consistent only if $G_2 = 0$, where G_2 is the last $(m - r)$ rows of $Q'N$.

[10] The matrix a is not necessarily nonsingular because because we may have solved for a smaller number of new flows than actual flows. Or, in some cases, the transformations that eliminate some flows may themselves induce new singularities. To see this latter point, consider the example

$$p_t = \lambda_t$$
$$-E_t p_{t+1} + E_t \lambda_{t+1} = p_t - x_t$$

The associated 2×2 A matrix has rank 1, so that the first step is to eliminate $p_t = \lambda_t$. But when one uses this implication, the resulting 1×1 a matrix is 0, so that it has rank 0. The resulting reduced system involves no dynamic variables, but is simply the pair of equations $p_t = x_t$ and $\lambda_t = x_t$.

[11] Luenberger (1978) uses an argument like this one to establish the inevitable convergence of his 'shuffle' algorithm when $|Az - B| \neq 0$.

[12] Notice that the eigenvalue-eigenvector or Schur decomposition may involve complex numbers. At the same time, the resulting rational expectations solutions are constrained to involve real numbers only. As a double check, our computer programs test for the presence of non-negligible imaginary components.

[13] One interpretation of this transformation is that $u_t = V_u d_t$ is the vector of unstable canonical variables and that V_u is the matrix of left eigenvectors that corresponds to the unstable eigenvalues; in this case, μ is a Jordan matrix with the entries below the main diagonal corresponding to repeated unstable roots. However, a better computational method is to use the Schur decomposition. In this case μ is a lower triangular matrix with the unstable eigenvalues on the diagonal.

[14] It is also consistent with the implications of transversality conditions in dynamic optimizing models, where the requirement is typically that $\beta^{1/2}\mu_i > 1$.

[15] The core programs in GAUSS and MATLAB as well as many example applications are available at http://people.bu.edu/rking/kwre.html.

[16] Given the nature of our reduction algorithm, it is fairly straightforward to calculate restrictions that rational expectations imposes on the solution of the reduced system, $E_t d_{t+1} = W d_t + \Psi_d(\mathbf{F}) E_t x_t$, in the case of multiple equilibria. But we have not yet written code for this purpose.

[17] This approach was employed in King and Watson (1996) to study the empirical implications of financial market frictions models in the style of Christiano and Eichenbaum (1992).

[18] We make use of the fact that $I_{Jt} = I_{0,t+1}$ in some of the derivations below.

[19] We assume that this common information model is solvable in the sense discussed above. This requires that $|Az - B| \neq 0$, which may be verified prior to the solution of the model in question, and also that additional 'rank' conditions are satisfied, which are verified during the solution process.

[20] These are not the same matrices M and N which enter in the state equations.

[21] For the theoretical characterization of solutions and the purposes of this appendix, the restriction to only current x_t is immaterial. This is because its is the properties of A and B that govern the properties of the solutions and the existence of a reduced dimension dynamic subsystem.

[22] A nilpotent matrix has zero elements on and below its main diagonal, with zeros and ones appearing abitrarily above the diagonal. Accordingly, there is a number l such that N^l is a matrix with all zero elements.

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