Uncertainty in Model-Based Seasonal Adjustment Procedures and Construction of Minimax Filters

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Unobserved component autoregressive integrated moving average models are often the cornerstone of model-based seasonal adjustment procedures. Unfortunately these models are inherently underidentified and ad hoc assumptions must be made prior to the analysis. This article investigates the effect of seasonal adjustment filters on a class of observationally equivalent models. Bounds on the mean squared error (MSE) associated with arbitrary linear filters are derived. The article also derives robust seasonal adjustment filters. The filters are robust in the sense that they minimize the maximum MSE from the set of observationally equivalent models. The article shows that the minimax and minimum extraction filters are equivalent for a certain class of models. Empirical results for a number of economic time series are presented.

KEY WORDS: Signal extraction; Unobserved components model; Minimax estimates.

1. INTRODUCTION

The classical model of economic time series represents an observed series as the sum of two unobserved components. The first component is called the *seasonal* component and represents the fairly regular intrayear movements in the series. The seasonal component captures the influence of periodic movements in the weather, the timing of holidays, and so on. The other component, called the *nonseasonal* component, captures the residual variation in the series. (This component is sometimes further decomposed into a "trend" and an "irregular" component. This additional distinction is not necessary in what follows.) Most economic analyses are concerned with nonseasonal variations in economic variables and, therefore, concentrate their attention on the nonseasonal component. The presence of seasonal variation in the series is viewed as a nuisance. Seasonal adjustment techniques are designed to eliminate or at least reduce this nuisance.

If we let x_t be an observed economic time series, then an additive model decomposes x_t as

$$x_t = n_t + s_t, \tag{1.1}$$

where n_t is the nonseasonal component and s_t is the seasonal component. (For some series a multiplicative decomposition is more appropriate. In this case x_t should be viewed as the logarithm of the original series.) Seasonal adjustment can be viewed as a method for estimating n_t, that is, of forming

$$\hat{n}_t = x_t - \hat{s}_t, \tag{1.2}$$

where $\hat{n}_t$ is the estimate of the nonseasonal component and $\hat{s}_t$ is the estimate of the seasonal component.

When we view $\hat{n}_t$ as an estimate of n_t we might naturally want a measure of the precision of the estimate, so, for example, we might want to calculate the mean square of the error $(\hat{n}_t - n_t)$. In unobserved component models the concept of precision or mean squared error (MSE) of an estimate is somewhat different from that in usual models. The difference arises because there is no sample counterpart to the population MSE measure. Since n_t is not observed, we could never calculate a sample MSE. In this sense the model is fundamentally underidentified. There are an infinite number of ways to decompose an observed series into two components. Measures like MSE must be interpreted with caution as they refer to estimates of components that are inherently unobservable.

On the other hand, it still might be possible to define precisely the components in terms of seasonal and nonseasonal generating functions in such a way that the MSE of $\hat{n}_t$ could be calculated and would be a useful measure of precision. Model-based seasonal adjustment procedures, for example, define the unobserved components in terms of generating functions and then use the characteristics of these generating functions to develop optimal seasonal adjustment procedures. A popular specification postulates independent seasonal and nonseasonal univariate autoregressive integrated moving average (ARIMA) models for the components. The models defining the components are constrained so that when summed they describe the dynamic behavior of the observed series x.

If the components are defined in terms of ARIMA generating models and the estimate $\hat{n}_t$ is a linear combination of current, lagged, and future values of x_t, then it is often possible to calculate the MSE (and all of the autocovariances) of $(\hat{n}_t - n_t)$. Examples of such calculations can be found in Hausman and Watson (1985) and Burridge and Wallis (1985). These authors solved the model identification problem by postulating "reasonable" models for the seasonal and nonseasonal components. By "reasonable" I mean that the model chosen for the seasonal component generates a series whose spectrum has most of its mass concentrated near the seasonal frequencies, and the nonseasonal model generates a series with no extra power or peaks near the seasonal frequencies. Unfortunately it is often possible to find other reasonable models that sum to the same observed model and yield different values for the MSE. These observationally equivalent models correspond to different definitions of the seasonal and nonseasonal components.

Another approach to the calculation of the precision of the seasonally adjusted series is to calculate the MSE for

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Journal of the American Statistical Association
June 1987, Vol. 82, No. 398, Applications

all of the "reasonable" definitions of the nonseasonal and seasonal series. That is, one could calculate the MSE over a range of observationally equivalent models. This procedure would make the calculation of bounds on the precision of the seasonally adjusted estimates possible. These bounds are useful since they allow the users of the seasonally adjusted series to conduct conservative inference without knowledge of the specific model used to form the MSE. In many cases it is too ambitious a task to consider all possible reasonable definitions of the seasonal/nonseasonal decomposition. In this article we consider a class of models for the seasonal and nonseasonal components, which is indexed by a single unidentified parameter. The model that we consider has been widely used in connection with model-based seasonal adjustment procedures. These applications suggest that the model is flexible enough to describe the behavior of a wide range of economic time series.

By considering a range of observationally equivalent models for the seasonal and nonseasonal components we can also investigate the robustness of various seasonal adjustment methods. Given any particular linear model it is often possible to derive linear minimum MSE seasonal adjustment filters by using signal extraction methods. Although the performance of a filter may be optimal for one seasonal/nonseasonal component model, its performance may be very poor for another, observationally equivalent model. In model-based seasonal adjustment procedures it is a common practice to concentrate attention on the filter that is optimal for the model that minimizes the contribution of the seasonal component. This gives rise to what Pierce (1976) called the *minimum extraction principle.* In this article I suggest an alternative approach, which involves choosing the seasonal adjustment filter that minimizes the maximum MSE over the class of observationally equivalent models. As it turns out, this "minimax" filter is sometimes the minimum extraction filter, sometimes the maximum extraction filter, and sometimes neither.

In Section 2 I discuss a class of unobserved component models and show that any model in this class can be represented in terms of a single unidentified parameter. Four economic time series are analyzed, including the wholesale price index (WPI) and the civilian labor force (CLF), to investigate empirically the relevant range of observationally equivalent models. In Section 3 I show how bounds on the MSE for linear seasonal adjustment filters can be calculated. This section presents bounds on the MSE for the seasonally adjusted values for four economic time series adjusted by a linear filter approximating Census X-11, the official U.S. Commerce Department method. As a preview of one of the empirical results it is found that an upper bound on the root mean squared error (RMSE) of the official historical value of the WPI inflation rate is .5%, suggesting that inflation rates of 1% per month are not significantly different from zero. In Section 4 I take up the issue of robustness of filters and derive minimax filters. These filters are compared with X-11 and with the minimum extraction filters for the four series. The last section contains some concluding remarks.

2. THE MODEL

We assume that the dynamic behavior of the observed series can be represented in terms of an ARIMA process, so

$$\phi_x(B)x_t = \theta_x(B)e_t, \tag{2.1}$$

where $\phi_x(B)$ and $\theta_x(B)$ are polynomials of order p and q in the backshift operator B, the roots of $\phi_x(B)$ are on or outside the unit circle, the roots of $\theta_x(B)$ are outside the unit circle, and e_t is white noise. We can write the pseudo-autocovariance generating function (pseudo-ACGF) of x, say $A_x(z)$, as

$$A_x(z) = \sigma_e^2 \frac{\theta_x(z)\theta_x(z^{-1})}{\phi_x(z)\phi_x(z^{-1})}. \tag{2.2}$$

Setting $z = e^{-i\omega}$, this is the pseudospectrum for x. An alternative representation for x_t is in terms of the unobserved components n_t and s_t,

$$x_t = n_t + s_t. \tag{2.3}$$

We assume that the innovations in n_t and s_t are uncorrelated at all leads and lags, and their ACGF's will be denoted by $A_n(z)$ and $A_s(z)$. This allows us to write

$$A_x(z) = A_n(z) + A_s(z). \tag{2.4}$$

Since the series x_t is observed, $A_x(z)$ can be consistently estimated. With $A_x(z)$ and Equation (2.4), we see that the model can be completely specified by describing either $A_n(z)$ or $A_s(z)$. Nonseasonal behavior across various economic time series may differ substantially, but by definition, seasonal behavior across series is reasonably similar. For this reason we will specify the model in terms of the seasonal component and leave the nonseasonal component to be determined by Equation (2.4). (We will make the additional assumption that the spectrum of the nonseasonal component does not have spikes at the seasonal frequencies.)

The model that I propose for the seasonal component is the same model used by Hillmer and Tiao (1982) for their model-based seasonal adjustment procedure. To motivate the model it is useful to begin with a deterministic model of seasonality in which $s_t = s_{t-m}$, where m represents the number of time periods per year. If we let $U(B) = 1 + B^1 + B^2 + \cdots + B^{m-1}$, then the sum of seasonal factors over any year is given by

$$U(B)s_t = c, \tag{2.5}$$

where c/m is the mean of the seasonal series. We will attribute the mean of the observed series entirely to the nonseasonal component, and so set $c = 0$. This yields a series s_t with a pseudospectrum that consists solely of spikes at the seasonal frequencies $2\pi j/m$ [$j = 1, 2, \ldots, \frac{1}{2}m$]. This model captures the strict periodic nature of the seasonal component but, in a sense, forces it to be too strict. A model similar to the deterministic model but that allows the seasonal component to evolve slowly over time uses Equation (2.5) as a forecast function, so

$${}_ts_{t+1} + {}_ts_{t+2} + \cdots + {}_ts_{t+m} = 0,$$

which implies that

$$ {}_t s_{t+k} = {}_t s_{t+k-m} \qquad \text{for } k > m, \tag{2.6} $$

where ${}_t s_{t+j}$ is the forecast for s_{t+j}, constructed using data on s up through time t. This specification allows the seasonal component to evolve slowly according to

$$ U(B)s_t = c_t, \tag{2.7} $$

where c_t is an error term. From the forecast function (2.6), the error term is constrained to follow an MA($m - 1$) (MA for moving average) process

$$ c_t = \theta_s(B)f_t \tag{2.8} $$

with f_t white noise. The spectrum of s_t defined in (2.7) and (2.8) is the stochastic analog of the spectrum of s_t defined in (2.5). Most of the power is concentrated around sharp spikes at the seasonal frequencies.

This specification for the seasonal component places testable restrictions on the model (2.1) for the observed series. The autoregressive (AR) polynomial $U(B)$ in (2.7) leads to singularities in the spectrum of s at the seasonal frequencies. This characteristic must be shared by the spectrum of x, so the AR polynomial $\phi_x(B)$ must contain the factor $U(B)$. The ARIMA model for x must satisfy some additional conditions. A set of necessary and sufficient conditions is given in Hillmer and Tiao (1982).

With the specification of the models for x_t and s_t given in (2.1), (2.7), and (2.8), the model for n_t follows from Equation (2.4). The process for n_t will be represented as

$$ \phi_n(B)n_t = \theta_n(B)g_t \tag{2.9} $$

with g_t white noise. From (2.4) we have

$$ \sigma_e^2 \frac{\theta_x(z)\theta_x(z^{-1})}{\phi_x(z)\phi_x(z^{-1})} = \sigma_g^2 \frac{\theta_n(z)\theta_n(z^{-1})}{\phi_n(z)\phi_n(z^{-1})} + \sigma_f^2 \frac{\theta_s(z)\theta_s(z^{-1})}{U(z)U(z^{-1})}. \tag{2.10} $$

We will impose one restriction on the process for n_t. Since n_t is nonseasonal it should not have peaks in its spectrum at the seasonal frequencies. This implies that $U(B)$ and $\phi_n(B)$ have no common roots.

Given this specification for the components we are now in a position to investigate the identifiability of the model. We will assume that the polynomial $[\phi_n(z)\theta_s(z) + U(z)\theta_n(z)]$ shares no common factors with $\phi_n(z)$ or $U(z)$, so Equation (2.10) implies that

$$ \phi_x(B) = \phi_n(B)U(B) \tag{2.11} $$

so that the AR parameters are identified. Identification problems arise from the MA parts of the component models. The nature of the identification problem is easy to see. Suppose that we have one set of components that satisfy Equations (2.7)–(2.10) and we then add an independent white noise component to the seasonal component. Since the sum of an ARMA($m - 1$, $m - 1$) (ARMA for autoregressive moving average) plus independent white noise follows an ARMA($m - 1$, $m - 1$), we have generated a new seasonal component satisfying (2.7) and (2.8). [Of course the values of the coefficients in $\theta_s(B)$ and the value of σ_f^2 will change.] To complete the specification of this new model we merely choose a nonseasonal process so that (2.10) is satisfied. The structure imposed on the components by Equations (2.7)–(2.10) allows us to represent the entire class of observationally equivalent models in terms of one unidentified parameter that measures the variance of the independent white noise process that is added to the seasonal component.

To show this formally we follow Burman (1976; 1980) and construct a unique partial fraction expansion of the ACGF of x_t to yield

$$ \sigma_e^2 \frac{\theta_x(z)\theta_x(z^{-1})}{\phi_x(z)\phi_x(z^{-1})} = \frac{\tilde{\theta}_s(z)\tilde{\theta}_s(z^{-1})}{U(z)U(z^{-1})} + \frac{\alpha(z)\alpha(z^{-1})}{\phi_n(z)\phi_n(z^{-1})} + \lambda(z)\lambda(z^{-1}), \tag{2.12} $$

where $\tilde{\theta}_s(z)$ is a polynomial of order $m - 2$, $\alpha(z)$ is a polynomial of order $p - m$, and $\lambda(z)$ is a polynomial of order max(0, $q - p$). We can combine the last two terms to form

$$ \frac{\tilde{\theta}_n(z)\tilde{\theta}_n(z^{-1})}{\phi_n(z)\phi_n(z^{-1})} $$

and define s_t^γ as a process with ACGF

$$ \frac{\tilde{\theta}_s(z)\tilde{\theta}_s(z^{-1})}{U(z)U(z^{-1})} + \gamma $$

and n_t^γ as a process with ACGF

$$ \frac{\tilde{\theta}_n(z)\tilde{\theta}_n(z^{-1})}{\phi_n(z)\phi_n(z^{-1})} - \gamma, $$

and so we can represent x_t as

$$ x_t = s_t^\gamma + n_t^\gamma \tag{2.13} $$

for any value of γ that gives rise to legitimate ACGF's from n_t^γ and s_t^γ. Bounds are placed on γ by the constraint that the spectra of s_t^γ and n_t^γ must be nonnegative. We will denote the lower bound by γ^l and the upper bound by γ^u.

As is clear from Equation (2.12), the degree of underidentification can be indexed by the range $\gamma^u - \gamma^l$. We can think of $\gamma^u - \gamma^l$ as the ACGF (equal to the variance) of a white noise component r_t. Identification problems arise because we do not know what fraction of r_t to attribute to the seasonal and what fraction to attribute to the nonseasonal component. Some model-based seasonal adjustment procedures attribute all of this component to the nonseasonal. [Examples can be found in Pierce (1976), Box, Hillmer, and Tiao (1976), Burman (1980), Hillmer and Tiao (1982), and Bell and Hillmer (1984).] The most thorough discussion of this choice for the identification of the seasonal component can be found in Bell and Hillmer (1984). They made three important points: (a) a precise definition of the seasonal component is necessary if seasonal adjustment is to be viewed as a well-posed statistical estimation problem, (b) some arbitrary choice must be made to define the seasonal component precisely, and (c)

Table 1. Model Uncertainty

Series	Model for observed series	$\hat{\sigma}_e$	$(\gamma^u - \gamma^l)^{1/2}$	ψ
CRF	$(1 - B)(1 - B^{12})x_t = (1 - .61B)(1 - .53B^{12})e_t$	.0901	.0551	.37
CLF	$(1 - B)(1 - B^{12})x_t = (1 - .27B - .22B^5)(1 - .72B^{12})e_t$	.0034	.0016	.21
WPI	$(1 - B)(1 - B^{12})x_t = (1 + .15B^2 + .15B^3 + .17B^5 + .07B^6)(1 - .93B^{12})e_t$	.0077	.0033	.19
DDC72	$(1 - .20B - .18B^2 - .3B^3)(1 - B)(1 - B^{12})x_t = (1 + .43B)(1 - .91B^{12})e_t$	.0033	.0010	.09

given that an arbitrary choice must be made, a set of reasonable arguments support the decomposition in which r_t is attributed (in full) to the nonseasonal component. There is an alternative to making an arbitrary choice of one of the observational equivalent models. This alternative approach explicitly incorporates model uncertainty into the seasonal adjustment process. The remainder of this article will investigate this approach.

The foregoing results show that the degree of model uncertainty depends on the ACGF of the observed series. To investigate the empirical relevance of model uncertainty, unobserved component models for four series have been estimated using monthly data from 1956:1 to 1979:12. The four series considered were the logarithm of the civilian labor force (CLF), the logarithm of the wholesale price index (WPI), the logarithm of total cash receipts from farming (CRF), and the logarithm of demand deposits and currency in 1972 dollars (DDC72). [All series are available in the Capsule Data Bank at Data Resources, Inc. The Current Population Survey error component of CLF, documented in Hausman and Watson (1985), was ignored.] As an index of uncertainty, the following was calculated:

$$\psi = (\gamma^u - \gamma^l)/\sigma_e^2.$$

The numerator is the variance of the white noise component r_t, and the denominator is the variance of the (univariate) innovation variance in x_t. Since r_t is one component of the innovation in x_t, the index shows the percentage of the innovation in the observed series that is attributed to the current value of r_t. (Since it is impossible to unscramble perfectly the values of the unobserved components given data on x alone, the univariate innovation in x is composed of both current and lagged values of the shocks to the components.) The results for the four series are shown in Table 1. Although the forms of the ARIMA models vary from series to series, they all contain seasonal and nonseasonal differencing operators. The calculated values for the index ψ suggest that the degree of model uncertainty can be substantial and can vary considerably from series to series. The index is highest for CRF where it takes on the value .31 and is smallest for DDC72 where it takes on the value .09.

3. SEASONAL ADJUSTMENT MEAN SQUARED ERROR

Most seasonal adjustment procedures can be represented or well approximated by a time-invariant linear filter applied to the observed time series. When the model generating the components is known, the properties of the filter and the autocovariances of the components can be used in a straightforward manner to determine the seasonal adjustment MSE. If we denote the seasonal adjustment filter by

$$V(B) = \sum_{i=-\infty}^{\infty} v_i B^i, \tag{3.1}$$

then the seasonally adjusted value is

$$\hat{n}_t = V(B)x_t \tag{3.2}$$

and the seasonal adjustment error is

$$a_t = \hat{n}_t - n_t. \tag{3.3}$$

Letting $A_a(z)$ denote the pseudo-ACGF of a_t, Equations (3.2) and (3.3) imply that

$$A_a(z) = W(z)W(z^{-1})A_n(z) + V(z)V(z^{-1})A_s(z), \tag{3.4}$$

where

$$W(z) = 1 - V(z). \tag{3.5}$$

For the models we are considering the seasonal and nonseasonal components are nonstationary with a variance increasing over time without bound. This implies that the mean square of a_t will remain bounded (as t grows large) for a certain class of filters only. These filters were investigated by Pierce (1979). From the second term on the right side of (3.4) it is clear that $V(B)$ must contain the factor $U(B)$ if the MSE is to remain bounded. When n_t is nonstationary, further conditions must be imposed. If we write the AR polynomial for the nonseasonal component as

$$\phi_n(z) = \phi_n^1(z)\phi_n^2(z), \tag{3.6}$$

where $\phi_n^1(z)$ has all of its roots on the unit circle and $\phi_n^2(z)$ has all of its roots outside the unit circle, then the first term on the right side of (3.4) implies that $W(B)$ must contain the factor $\phi_n^1(B)$. Bell (1984) gave conditions that ensure that a_t is covariance stationary. In addition to the characteristics of $V(B)$ just listed, his assumption A on the initial conditions of the process leads to a stationary seasonal adjustment error. We will assume that $V(B)$ and $W(B)$ contain the factors $U(B)$ and $\phi_n^1(B)$, respectively, and make Bell's assumption A concerning the initial conditions for x. These imply that the process describing a_t is covariance stationary.

In the example considered in Section 2 the ACGF of n_t was known only up to an unknown constant γ. If we let

Table 2. RMSE's (measured in percentage) of Seasonal Adjustment Filters for the CRF Series

Variable being estimated	Total cash receipts from farming				
	Historical adjustment filter				
	X-11	M^{γ^l}	M^{γ^*}	$M^{\gamma^{**}}$	M^{γ^u}
n^{γ^l}	2.96	2.88	3.47	3.25	5.18
n^{γ^u}	5.31	5.00	3.47	3.78	2.55
Δn^{γ^l}	4.03	3.93	4.95	4.58	7.73
Δn^{γ^u}	7.26	6.84	3.97	4.58	1.57
	Current adjustment filter				
	X-11 ARIMA	H^{γ^l}	H^{γ^*}	$H^{\gamma^{**}}$	H^{γ^u}
n^{γ^l}	4.05	4.02	4.30	4.19	5.25
n^{γ^u}	5.44	5.24	4.42	4.57	4.01
Δn^{γ^l}	5.34	5.25	5.44	5.33	6.68
Δn^{γ^u}	7.09	6.74	4.78	5.19	2.85

Table 3. RMSE's (measured in percentage) of Seasonal Adjustment Filters for the CLF Series

Variable being estimated	Civilian labor force				
	Historical adjustment filter				
	X-11	M^{γ^l}	M^{γ^*}	$M^{\gamma^{**}}$	M^{γ^u}
n^{γ^l}	.13	.11	.15	.13	.15
n^{γ^u}	.18	.18	.15	.15	.15
Δn^{γ^l}	.13	.12	.19	.16	.20
Δn^{γ^u}	.22	.22	.15	.16	.15
	Current adjustment filter				
	X-11 ARIMA	H^{γ^l}	H^{γ^*}	$H^{\gamma^{**}}$	H^{γ^u}
n^{γ^l}	.17	.16	.17	.17	.17
n^{γ^u}	.21	.20	.19	.19	.19
Δn^{γ^l}	.17	.16	.17	.17	.18
Δn^{γ^u}	.24	.24	.19	.20	.19

n_t^γ denote the nonseasonal component with γ in its ACGF and $a_t^\gamma = \hat{n}_t - n_t^\gamma$, then the ACGF of a_t^γ, say $A_a^\gamma(z)$, is

$$A_a^\gamma(z) = W(z)W(z^{-1})\left[\frac{\tilde{\theta}_n(z)\tilde{\theta}_n(z^{-1})}{\phi_n(z)\phi_n(z^{-1})} - \gamma\right] + V(z)V(z^{-1})\left[\frac{\tilde{\theta}_s(z)\tilde{\theta}_s(z^{-1})}{U(z)U(z^{-1})} + \gamma\right] \quad (3.7)$$

$$= A_a^{\gamma^l}(z) + (\gamma - \gamma^l)[V(z) + V(z^{-1}) - 1], \quad (3.8)$$

so the MSE of a_t^γ is

$$\text{MSE}[a_t^\gamma] = \text{MSE}[a_t^{\gamma^l}] + (\gamma - \gamma^l)(2v_0 - 1). \quad (3.9)$$

Bounds on the MSE follow directly from (3.9). When $v_0 > \frac{1}{2}$, the $\text{MSE}[a_t^\gamma]$ is bounded below by $\text{MSE}[a_t^{\gamma^l}]$ and bounded above by $\text{MSE}[a_t^{\gamma^u}]$. The opposite occurs when $v_0 < \frac{1}{2}$. When $v_0 = \frac{1}{2}$, the MSE is constant for all values of γ.

Interest is often focused on the change in the series rather than its level. The MSE for the change in the seasonally adjusted value is also easily calculated from the ACGF for a_t^γ. Like the MSE for the level, it is a linear function of γ, where the slope of the function depends on the coefficients in the seasonal adjustment filter. In particular,

$$\text{MSE}[\Delta a_t^\gamma] = \text{MSE}[\Delta a_t^{\gamma^l}] + 2(\gamma - \gamma^l)[2v_0 - v_1 - v_{-1} - 1].$$

Again, bounds on the MSE depend only on $\text{MSE}[\Delta a_t^{\gamma^l}]$ and $\text{MSE}[\Delta a_t^{\gamma^u}]$.

The official seasonal adjustment method for most U.S. economic time series is the Census X-11 program. This program contains a menu of seasonal adjustment filters as well as adjustments for trading day variation and outliers. [For a complete description see Shiskin, Young, and Musgrave (1967).] If the corrections for outliers and trading day variation are disregarded, then the linear filter given in Wallis (1974) is a good approximation to the standard options version of the filter used to adjust monthly historical data additively. (The filter given in Wallis uses the 13-term Henderson MA filter for the trend–cycle component. This is the option chosen by X-11 for most series.) The linear approximation is an 84-term symmetric filter. The value of v_0 is .82, and the value of v_1 is .02. In the first column of each of Tables 2–5 bounds on the MSE are shown for the seasonally adjusted levels and changes in the four series described in Section 2. (The computational details concerning these calculations can be found in Appendix A.) The range encompassed by these bounds can be quite large. For example, the MSE for CRF corresponding to n^{γ^u} is over three times the MSE for n^{γ^l}. Confidence intervals for the level and the change in the nonseasonal component are nearly twice as wide using n^{γ^u} than they are using n^{γ^l}. Inferences concerning the nonseasonal component using the minimal variance representation (i.e., γ^l) may be quite inappropriate for other observationally equivalent models. When the seasonal adjustment filter used is X-11, inferences using the MSE calculated for the maximal seasonal variation representation will be conservative (the size of tests will be overstated and confidence intervals will be too wide) for any of the other observationally equivalent models of the seasonal/nonseasonal decomposition.

Table 4. RMSE's (measured in percentage) of Seasonal Adjustment Filters for the WPI Series

Variable being estimated	Wholesale price index				
	Historical adjustment filter				
	X-11	M^{γ^l}	M^{γ^*}	$M^{\gamma^{**}}$	M^{γ^u}
n^{γ^l}	.27	.14	—	.22	.25
n^{γ^u}	.38	.35	—	.28	.28
Δn^{γ^l}	.28	.13	—	.32	.38
Δn^{γ^u}	.46	.48	—	.32	.31
	Current adjustment filter				
	X-11 ARIMA	H^{γ^l}	H^{γ^*}	$H^{\gamma^{**}}$	H^{γ^u}
n^{γ^l}	.30	.19	—	.22	.24
n^{γ^u}	.43	.38	—	.35	.35
Δn^{γ^l}	.29	.19	—	.24	.26
Δn^{γ^u}	.51	.49	—	.41	.39

Table 5. RMSE's (measured in percentage) of Seasonal Adjustment Filters for the DDC72 Series

Variable being estimated	Demand deposits and currency in 1972 dollars				
	Historical adjustment filter				
	X-11	$M^{\gamma l}$	M^{γ^*}	$M^{\gamma^{**}}$	$M^{\gamma u}$
$n^{\gamma l}$	.13	.08	—	—	.09
$n^{\gamma u}$	.16	.12	—	—	.11
$\Delta n^{\gamma l}$	.12	.06	—	—	.11
$\Delta n^{\gamma u}$	.16	.15	—	—	.12
	Current adjustment filter				
	X-11 ARIMA	$H^{\gamma l}$	H^{γ^*}	$H^{\gamma^{**}}$	$H^{\gamma u}$
$n^{\gamma l}$	.15	.11	—	—	.12
$n^{\gamma u}$	.18	.15	—	—	.15
$\Delta n^{\gamma l}$	.13	.09	—	—	.09
$\Delta n^{\gamma u}$	.18	.16	—	—	.15

Since the two-sided symmetric X-11 filter requires future as well as past data, it cannot be used to adjust current values. Various modifications have been suggested to carry out current seasonal adjustment. The X-11 program includes special "end-weights" that can be used to adjust data when 7 years of future data are not available. An alternative procedure, suggested by Dagum (1975), is to apply the symmetric X-11 filter to a series augmented by forecasts of the future values. The forecasts can be constructed from the ARIMA models for the series, and the procedure is referred to as X-11 ARIMA. (The X-11 ARIMA procedure typically augments the series with 12 months of forecasts, and the special nonsymmetric X-11 lag 12 filter is used. The results here use the symmetric filter applied to the series augmented by 84 forecasts.) Geweke (1978) and Pierce (1980) showed that, when the model for x_t is known, this procedure has the desirable property of minimizing the mean square of seasonal adjustment revisions that take place as more data become available. Combining the linear forecast filter implicit in the ARIMA forecasting procedure with the symmetric X-11 filter produces a linear one-sided filter. Bounds on the seasonal adjustment MSE for the four series were calculated for these current adjustment filters. The results are shown in the bottom portions of Tables 2–5. It is important to note that in all cases except one the MSE's for current adjusted values are larger than the values for historical data. (This increase in MSE always occurs with the use of an *optimal* filter, but it need not occur with the use of a nonoptimal filter like X-11.) In general the X-11 filter uses future values of the observed data in a way that reduces the MSE.

The calculated RMSE values reported in Tables 2–5 should be of interest to users of these series. Since the data are logarithms, the RMSE's are measured in percentage. The upper bound on the RMSE for the rate of change in the currently adjusted WPI suggests that changes of 1% per month may be insignificantly different from zero. The results suggest that the rate of change in the nonseasonal component of DDC is quite accurately estimated by X-11, with an RMSE of no more than .18% per month. The rate of increase of the nonseasonal component in CRF is poorly estimated by X-11. Its RMSE is 7.26% for historical data and 7.09% for current data. This increase in MSE that occurs when future data are added implies that X-11 is not efficiently using the observed data to adjust the data seasonally. The data are used efficiently by an optimal filter. The construction of optimal filters is the subject of the next section.

4. MINIMAX SEASONAL ADJUSTMENT FILTERS

The construction of optimal seasonal adjustment filters has received considerable attention in the statistics and econometrics literature. Most of the modern literature postulates independent ARIMA models for the components and then forms optimal seasonal adjustment filters by using signal extraction methods. When the models for the components are not completely specified, or are specified up to some set of unidentified parameters, as they are in this article, the common practice is to choose a particular model first and then to form the optimal filter for this model. The usefulness of this filter for extracting the seasonal in any of the other observationally equivalent models is not considered. [A notable exception can be found in Bell and Hillmer (1984, sec. 4.3.2).] The results of Section 3 show that filter performance may vary substantially over the observationally equivalent models. A filter that is optimal for one model may perform quite poorly for another. In this section we construct seasonal adjustment filters that have desirable properties for all of the observationally equivalent models. In particular we construct filters that minimize the maximum MSE over the entire class of observationally equivalent models.

As a starting point we will consider the optimal filter for estimating the nonseasonal component n_t given a complete realization of the x process. Bell (1984) showed that the model for the components given in Section 2 together with his assumption A concerning the distribution of the initial conditions for the components imply that standard Wiener filtering formulas can be used to construct the optimal seasonal adjustment filter. The key feature of the model that makes this possible is the fact that the AR polynomials for the seasonal and nonseasonal components share no common roots on the unit circle. The results of Burridge and Wallis (1983) made this clear by showing that the effect of initial conditions will be transient, in the sense that the Kalman filter and smoothing recursions will converge, so long as the uncertainty surrounding the initial conditions is finite. (These Kalman filter and smoothing techniques are recursive approaches for forming optimal filters.) For exactness we will make Bell's assumption A concerning the initial conditions, so the optimal seasonal adjustment filter is given by the symmetric two-sided filter

$$M^{\gamma}(B) = \left[\frac{\tilde{\theta}_n(B)\tilde{\theta}_n(F)}{\phi_n(B)\phi_n(F)} - \gamma\right]\left[\sigma_e^2\,\frac{\theta_x(B)\theta_x(F)}{\phi_x(B)\phi_x(F)}\right]^{-1}, \tag{4.1}$$

where $F = B^{-1}$. Our assumption concerning the initial conditions, the assumption that $\theta_x(B)$ is invertible, and the assumption that the variance σ_e^2 is positive and finite imply that $M^{\gamma}(B)$ is the unique linear minimum MSE filter.

The linear minimum MSE estimate of n_t^{γ} is given by

$$\hat{n}_t^{\gamma} = M^{\gamma}(B)x_t. \quad (4.2)$$

We will define $a_t^{\gamma i,\gamma j}$ as the seasonal adjustment error in estimating $n_t^{\gamma i}$ using the optimal estimate of $n_t^{\gamma j}$, so

$$a_t^{\gamma i,\gamma j} = M^{\gamma j}(B)x_t - n_t^{\gamma i} = \hat{n}_t^{\gamma j} - n_t^{\gamma i}. \quad (4.3)$$

We will now derive the minimax filter. The minimax filter will be chosen so that its weights, w_i, satisfy

$$\min_{w_i} \left\{ \max_{\gamma \in [\gamma^l, \gamma^u]} ms\left[\sum_{i=k}^{j} w_i x_{t-i} - n_t^{\gamma} \right] \right\}.$$

Setting the bounds in the summation equal to $-\infty$ and ∞, we can derive the appropriate minimax filter for an entire realization of x.

To derive this filter it is useful to consider three cases:

(1) $m_0^{\gamma^l} < \frac{1}{2}$,

(2) $m_0^{\gamma^u} > \frac{1}{2}$,

(3) $m_0^{\gamma^l} \geq \frac{1}{2} \geq m_0^{\gamma^u}$.

To see that these three cases are mutually exclusive and exhaustive, notice from Equation (4.1) that

$$\frac{\partial m_0^{\gamma}}{\partial \gamma} = -\sigma^2 < 0, \quad (4.4)$$

where σ^2 is the variance of a random variable generated from an ARMA process with AR polynomial $\theta_x(B)$, MA polynomial $\phi_x(B)$, and innovation variance σ_e^{-2}. We can now derive the minimax filter for three cases. [The results on uniqueness of the minimax filter were first presented in Findley (1985).]

Case 1.

$$m_0^{\gamma^l} < \tfrac{1}{2}.$$

From Equation (3.9), setting $v_0 = m_0^{\gamma^l}$, MSE$[a_t^{\gamma,\gamma^l}]$ is a strictly decreasing linear function of γ in the interval $[\gamma^l, \gamma^u]$, so max{MSE$[a_t^{\gamma,\gamma^l}]$} occurs at $\gamma = \gamma^l$. But when $\gamma = \gamma^l$ we know that $M^{\gamma^l}(B)$ is the unique linear minimum MSE filter. This implies that $M^{\gamma^l}(B)$ is the unique linear minimax filter for Case 1.

Case 2.

$$m_0^{\gamma^u} > \tfrac{1}{2}.$$

An analogous argument shows that $M^{\gamma^u}(B)$ is the unique linear minimax filter for Case 2.

Case 3.

$$m_0^{\gamma^l} \geq \tfrac{1}{2} \geq m_0^{\gamma^u}.$$

Since m_0^{γ} is continuous and monotonically decreasing in γ over $[\gamma^l, \gamma^u]$, there exists a unique $\gamma^* \in [a_t^{\gamma,\gamma^u}]$ such that $m_0^{\gamma^*} = \frac{1}{2}$. From Equation (3.9) MSE$[a_t^{\gamma,\gamma^*}]$ is constant as γ varies over $[\gamma^l, \gamma^u]$, so

$$\max\{\text{MSE}[a_t^{\gamma,\gamma^*}]\} = \text{MSE}[a_t^{\gamma^*,\gamma^*}].$$

But the unique linear MSE filter for estimating $n_t^{\gamma^*}$ is $M^{\gamma^*}(B)$. This implies that $M^{\gamma^*}(B)$ is the unique linear minimax filter for Case 3.

The foregoing argument derives the minimax filter for the estimate of the level of the nonseasonal component given a complete realization of the observed series. This choice of γ will not necessarily coincide with that for the minimax filter for the change in the series or that for the minimax filter for the level of the nonseasonal component constructed from a limited amount of observed data. The results of Section 3 imply that variations in MSE for different values of γ depend only on the coefficients m_0^{γ} and m_1^{γ}.

(Recall that the filter is symmetric, so $m_{-1}^{\gamma} = m_1^{\gamma}$.) By using the same kind of argument that was used in the previous paragraph it is possible to show that the minimax filter for the changes in the nonseasonal component applied to the level of x_t is given by

$(1 - B)M^{\gamma^l}(B)$ if $m_0^{\gamma^l} - m_1^{\gamma^l} < \frac{1}{2}$,

$(1 - B)M^{\gamma^u}(B)$ if $m_0^{\gamma^u} - m_1^{\gamma^u} > \frac{1}{2}$,

$(1 - B)M^{\gamma^{**}}(B)$ where $m_0^{\gamma^{**}} - m_1^{\gamma^{**}} = \frac{1}{2}$ otherwise.

When less than a complete realization of the observed series is available the filter $M^{\gamma}(B)$ cannot be used. So, for example, the most recent value of x cannot be seasonally adjusted using $M^{\gamma}(B)$, since future values of x are required by the filter. By applying the law of iterated projections (e.g., Riesz and Nagy 1955, p. 268) it is easy to show that the optimal estimate of the current value of the nonseasonal n_t^{γ} can be constructed by applying the optimal "full information" filter, $M^{\gamma}(B)$, to a series made up of the actual current and lagged values of x and optimal (linear minimum MSE) forecasts of future values of x. (Any necessary presample values of x can be replaced by backcasts.) Since the forecasts of the unknown future values of x are linear combinations of the observed current and lagged values of x, this new "limited information" seasonal adjustment filter can be written as a function of the optimal full information filter and the filters for forming the optimal forecasts. We will call this optimal limited information filter $H^{\gamma}(B)$. Since $H^{\gamma}(B)$ is an optimal filter for estimating n_t^{γ} using the restricted information set, the arguments used previously to derive minimax filters carry over if $\partial(h_0^{\gamma}/\partial\gamma) < 0$. This derivative condition is proved for a wide class of filters, which includes $H^{\gamma}(B)$, in Findley (1985). Therefore, given the limited information set, the minimax filters are given by

$H^{\gamma^l}(B)$ if $h_0^{\gamma^l} < \frac{1}{2}$,

$H^{\gamma^u}(B)$ if $h_0^{\gamma^u} > \frac{1}{2}$,

$H^{\bar{\gamma}^*}(B)$ where $h_0^{\bar{\gamma}^*} = \frac{1}{2}$ otherwise.

The value of γ indexing the minimax filter may change as information is acquired. So, for example, it is possible that $H^{\gamma^l}(B)$ is the minimax filter for current adjustment and that $M^{\gamma^u}(B)$ is the minimax filter for adjusting his-

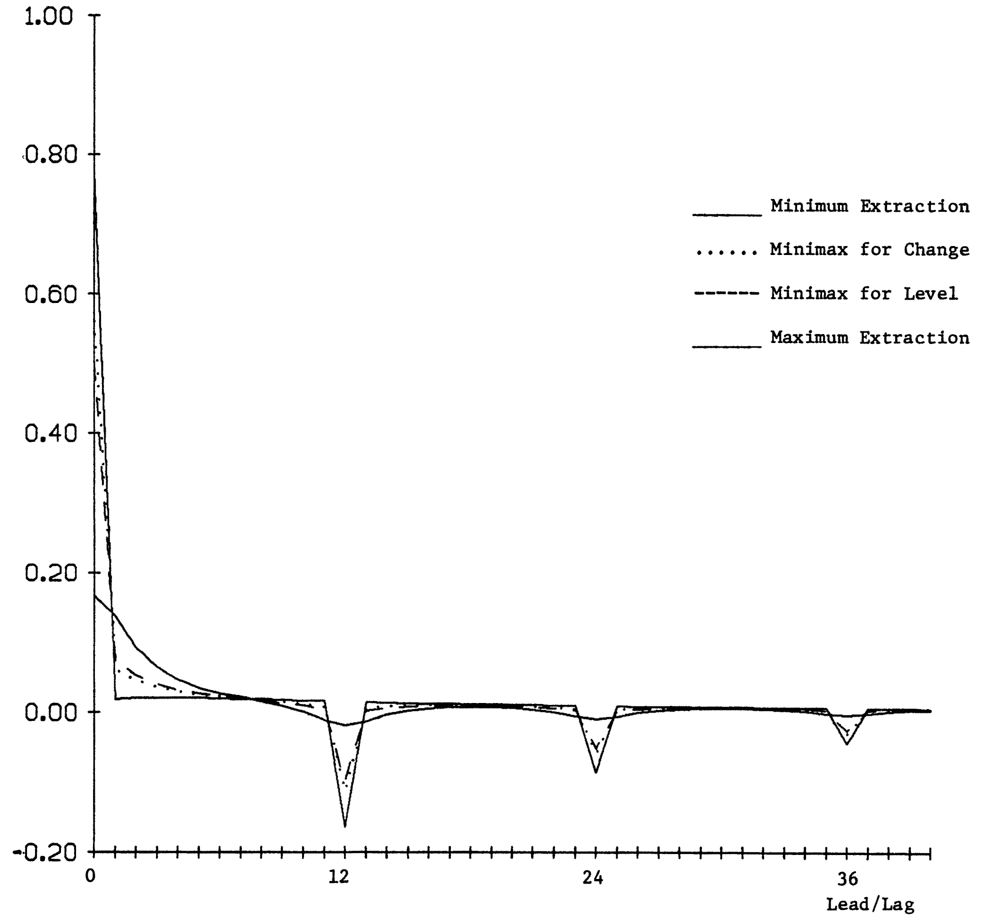

Figure 1. Seasonal Adjustment Filters for the Cash Receipts From Farming Series.

torical values. This can occur because h_0^γ is a function of all of the coefficients in $M^\gamma(B)$ and the parameters generating forecasts of future values of the x's. In general, $h_0^\gamma \neq m_0^\gamma$. Changing the value of γ for the optimal filter as more data become available will increase the magnitude of revisions. These revisions errors will arise because of the change in the value of γ as well as the unavoidable forecast errors in x. [This point was made in Geweke (1978) in another context.] A reasonable solution is to use the minimax filter for the historically adjusted series as the underlying filter for the current adjustment.

Derivations of minimax filters in more general settings are possible. Findley (1985) considered a signal extraction problem that is related, but is more general, than the problem considered here. In his model the range of observationally equivalent models was indexed by the allocation of a white noise component as it is in this model, but he allowed the signal and noise to be correlated and the optimal filter to be time varying. He showed that the basic results in Cases 1–3 here carry over to these more general settings.

A variety of optimal filters was calculated for each of the economic time series discussed in Section 3. In Figure 1 we plot the symmetric filter coefficients versus lag/lead of the CRF series. The figure contrasts four filters. The first is the minimum extraction filter—the optimal filter for estimating $n_t^{\gamma^l}$. This symmetric filter has a large positive weight on the current observation and large negative weights at the seasonal leads and lags. All other coefficients are small. The filter with the smallest weight on the current observation corresponds to the maximal extraction filter—the optimal filter for estimating $n_t^{\gamma^u}$. Rather than place a large weight on the current observation this filter uses a smooth centered MA of the first few leads and lags. Since less weight is placed on the current observation, less needs to be subtracted at the seasonal leads and lags. The middle two filters correspond to the minimax filters for the level and the change in the nonseasonal component. These filters are nearly identical.

The performances of these filters across the range of observationally equivalent models are compared in Tables 2–5. The minimum extraction filter produces an optimal estimate of $n_t^{\gamma^l}$ with an RMSE of 2.88%. It produces an estimate of $n_t^{\gamma^u}$ with an RMSE of 5.00% and an estimate of $\Delta n_t^{\gamma^u}$ with an RMSE of 6.84%. The maximal extraction filter, on the other hand, produces an estimate of $n_t^{\gamma^u}$ with an RMSE of only 2.55%, but as an estimator of $n_t^{\gamma^l}$ it has an RMSE of 5.18%. Both of these filters perform well for one extreme of the observationally equivalent models and poorly for the other extreme. The minimax filters perform

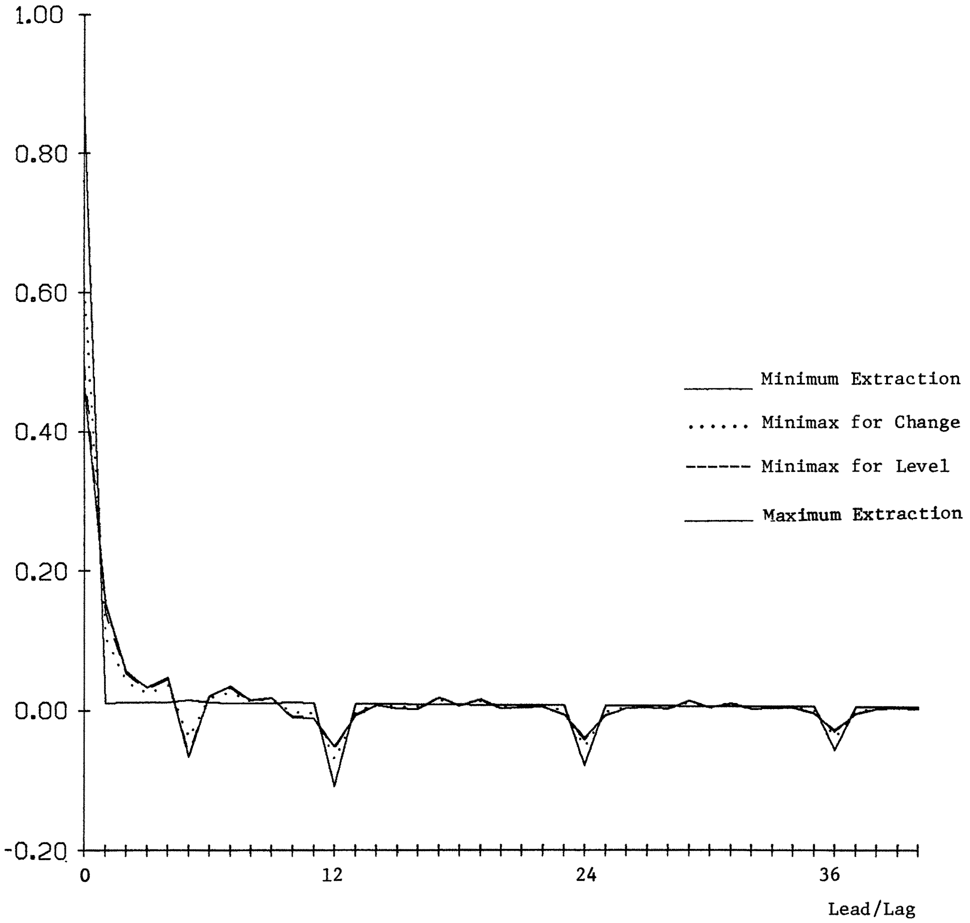

Figure 2. Seasonal Adjustment Filters for the Civilian Labor Force Series.

moderately well for all of the observationally equivalent models. The filter constructed to yield the minimax estimate of the levels (denoted by M^{γ^*} in Tables 2–5) yields an estimate with an RMSE of 3.47% for the level of the nonseasonal component regardless of the model generating the nonseasonal component. It also performs reasonably well for estimating the change in the nonseasonal component with an RMSE between 3.97% and 4.95%. The final filter, $M^{\gamma^{**}}$, is the minimax filter for the changes in the nonseasonal component. It produces an estimate of the change in the nonseasonal component with an RMSE of 4.58% regardless of the model for the nonseasonal component and an estimate of the level with an RMSE between 3.25% and 3.78%.

Tables 2–5 also compare the performance of the various optimal filters with X-11. For the CRF series the upper bounds on the RMSE using X-11 are 5.31% for the level and 7.26% for the changes. Using the minimax levels filter the corresponding values are 3.47 and 4.95. If we define relative efficiency as the ratio of the upper bounds on MSE, then the efficiency of X-11 relative to the minimax filter is .43 [$= (3.47/5.31)^2$] for the level of the nonseasonal component and .40 [$= (4.58/7.26)^2$] for the change.

The bottom panels of Tables 2–5 show the RMSE's of the current adjusted values, that is, the estimates of the nonseasonal components at time t using data up through time t only. Here we see that there can be substantial information in the future values of the observed series concerning the current value of the unobserved components. The values reported for γ^* and γ^{**} correspond to the minimax values of γ for the two-sided filter. The upper bounds on the RMSE for the minimax one-sided filters would, of course, be smaller. Even using γ^* and γ^{**} calculated from the two-sided filter we see substantial gain in terms of the upper bound of the RMSE. Using X-11 the upper bound is 5.44% for the level and 7.09% for the change. The corresponding values for the minimum extraction filter are 5.24% and 6.74%, and for the maximum extraction filter they are 5.25% and 6.68%. Using the minimax filters the upper bound for the level is reduced to 4.42%, and the value for the change is reduced to 5.33%.

In Figures 2–4 the analogous filters for the other three series are shown. For the WPI and DDC72 the minimax filter for the level is the maximal extraction filter, since in both cases the maximal extraction filter has a weight on the current observation greater than .5. For the DDC72 series this is also the minimax filter for the change in the nonseasonal component. Tables 2–5 contrast the performance of the various filters for each of the series. For all of the series the minimax filters provide a useful alternative

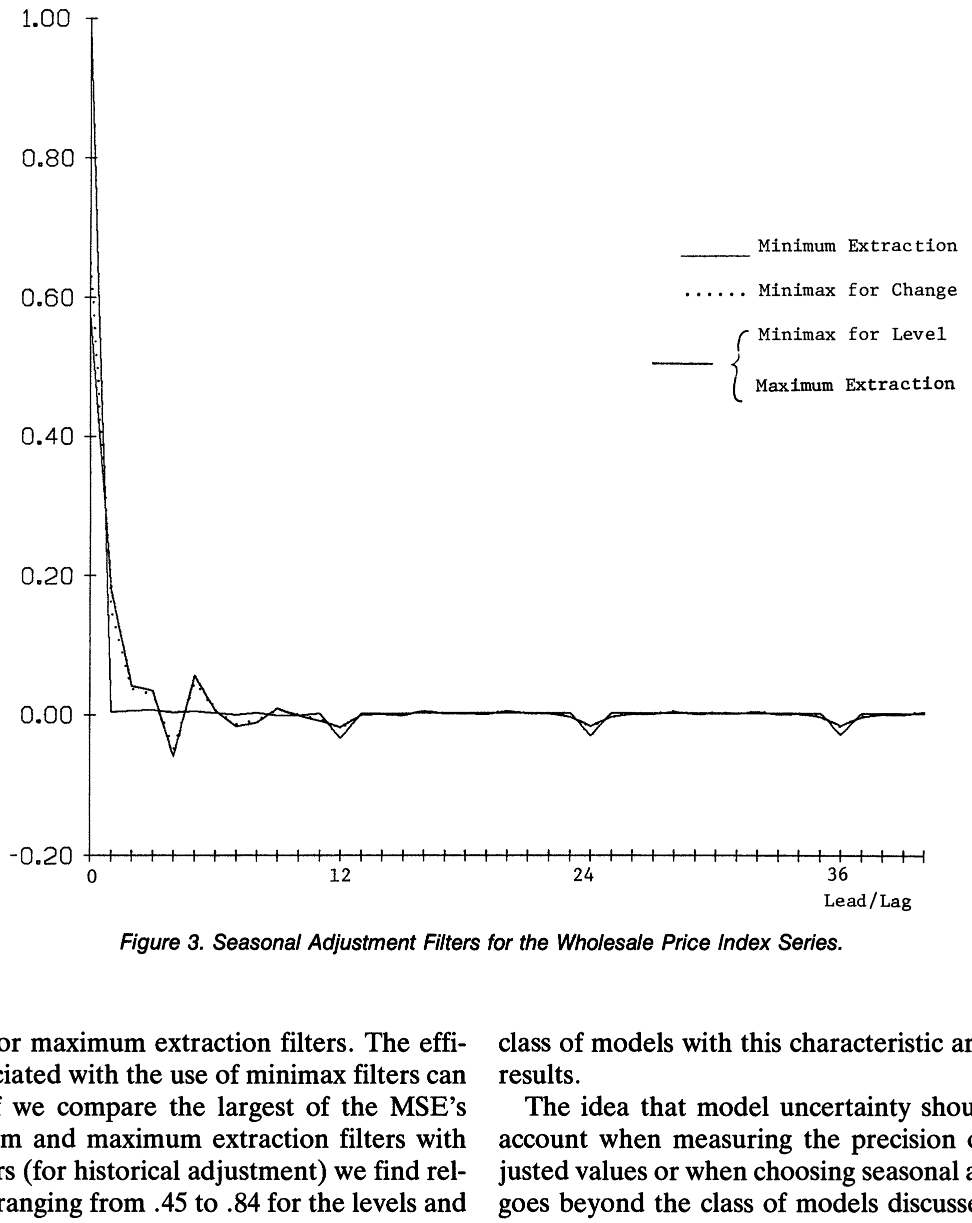

Figure 3. Seasonal Adjustment Filters for the Wholesale Price Index Series.

to the minimum or maximum extraction filters. The efficiency gains associated with the use of minimax filters can be substantial. If we compare the largest of the MSE's from the minimum and maximum extraction filters with the minimax filters (for historical adjustment) we find relative efficiencies ranging from .45 to .84 for the levels and from .35 to .64 for the changes.

5. CONCLUDING REMARKS

In this article I discussed uncertainty in seasonally adjusted values that arises from the lack of identification in unobserved component models. The results of Section 2 show that this source of uncertainty can be substantial. For a certain class of models for the seasonal component it was demonstrated that this source of uncertainty can be measured, so bounds on the MSE of the seasonally adjusted values can be calculated. I also incorporated this source of uncertainty in the evaluation of filters and showed how minimax seasonal adjustment filters can be formed.

Although I concentrated on a fairly narrow class of models for the seasonal component, many of the results in this article are easy to generalize. Technically, the results required that the model uncertainty could be expressed in terms of the allocation of a white noise component to either the seasonal or the nonseasonal component. Any class of models with this characteristic are covered by the results.

The idea that model uncertainty should be taken into account when measuring the precision of seasonally adjusted values or when choosing seasonal adjustment filters goes beyond the class of models discussed in this article. Summers (1981), for example, attempted to measure the effect of model uncertainty over a much wider class of seasonal models, although many of the models he considered were not observationally equivalent.

A relevant class of observationally equivalent models that we have not considered begins with the same additive decomposition of x,

$$x_t = s_t + n_t,$$

but postulates a seasonal AR model for s_t of the form

$$s_t = \rho s_{t-m} + \varepsilon_t^s.$$

This model of seasonality (typically with some additional MA terms) has been used by many researchers including Nerlove, Grether, and Carvalho (1979), Pierce (1976), and Hausman and Watson (1985). Since this model produces a seasonal component with a peak in its spectrum at zero frequency, it has been argued that it is an "inappropriate" model for the seasonal component. The argument suggests

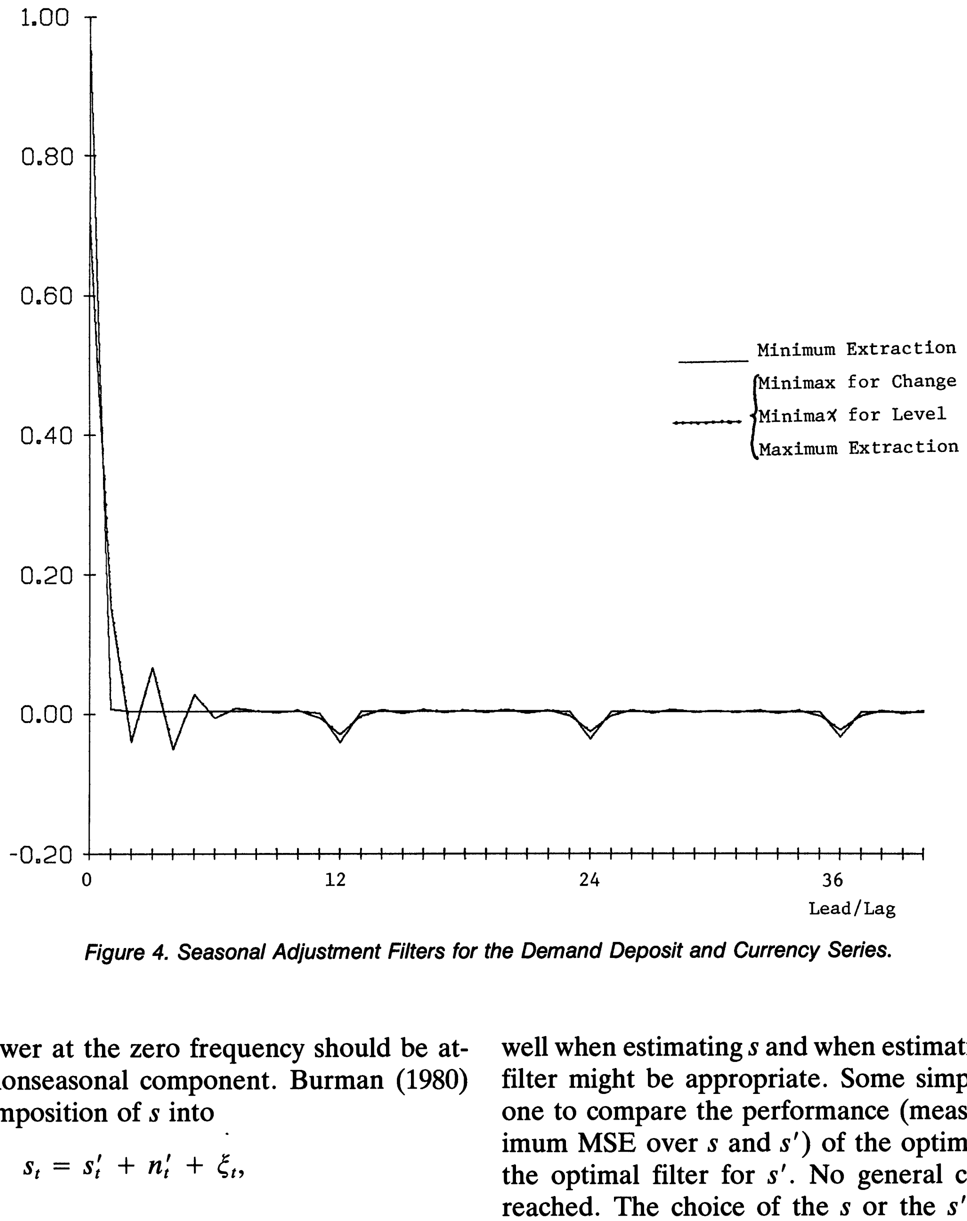

Figure 4. Seasonal Adjustment Filters for the Demand Deposit and Currency Series.

that the extra power at the zero frequency should be attributed to the nonseasonal component. Burman (1980) discussed a decomposition of s into

$$s_t = s_t' + n_t' + \xi_t,$$

where

$$(1 - \rho_1 B)n_t' = e_t^n,$$
$$(1 + \rho_1 B + \rho_2 B^2 + \cdots + \rho_{m-1} B^{m-1})s_t' = e_t^s$$

with

$$\rho_k = \rho^{k/m}, \qquad k = 1, 2, \ldots, m - 1,$$

and ξ_t, e_t^n, and e_t^s are uncorrelated white noise components. He then suggested that s' be viewed as the seasonal component, rather than s, and that $n + n' + \xi$ be viewed as the nonseasonal component, rather than n.

We have two observationally equivalent models of the seasonal component. One defines the seasonal component as s, and the other defines the seasonal component as s'. Since we have two models of the seasonal component, we will have two estimates of the X-11 RMSE and two optimal model-based seasonal adjustment filters. If both definitions of the seasonal s and s' are reasonable, then it makes sense to choose a seasonal adjustment filter that performs well when estimating s and when estimating s'. A minimax filter might be appropriate. Some simple algebra allows one to compare the performance (measured by the maximum MSE over s and s') of the optimal filter for s and the optimal filter for s'. No general conclusion can be reached. The choice of the s or the s' optimal filter as minimax will depend on the process generating x.

APPENDIX A: COMPUTATIONAL ISSUES

This appendix provides an outline of the methods used to calculate the numbers shown in Tables 2–5.

A.1 MSE for the X-11 Two-Sided Filter

The calculations were performed using the linear approximation to X-11 presented in Wallis (1974). This filter will be denoted by $X11(B)$, and we will let $W11(B) = 1 - X11(B)$. An examination of the filter reveals that $X11(B)$ contains the factor $U(B)$ and $W11(B)$ contains the factor $(1 - B)^2$. Let

$$\tilde{X}11(B) = [U(B)]^{-1}X11(B)$$

and

$$\tilde{W}11(B) = [(1 - B)^2]^{-1}W11(B).$$

(Numerically, these polynomials are easily formed using polynomial long division.) Let

$$\bar{\theta}_n(B) = [\phi_n^2(B)]^{-1}\tilde{\theta}_n(B)$$

denote the MA polynomial for the moving average representation for $(1 - B)^2 n_t$. This was formed numerically by truncating $[\phi_n^2(B)]^{-1}$ after 300 terms. [Only DDC72 contained a $\phi_n^2(B) \neq 1$. The roots of its polynomial were well outside the unit circle, so approximation based on this 300-term truncation is quite good.] The ACGF of $a_{t,X11}^\gamma = n_t^\gamma - X11(B)x_t$ is

$$A_{X11}^\gamma(z) = \tilde{W}11(z)\tilde{W}11(z^{-1})[\bar{\theta}_n(z)\bar{\theta}_n(z^{-1}) - (1 - z)^2(1 - z^{-1})^2\gamma] + \tilde{X}11(z)\tilde{X}11(z^{-1})[\tilde{\theta}_s(z)\tilde{\theta}_s(z^{-1}) + U(z)U(z^{-1})\gamma]. \quad (A.1)$$

The results reported in Table 2 were calculated by using polynomial multiplication to form $A_{X11}^{\gamma 1}(z)$. The results for all other values of γ were calculated from

$$A_{X11}^\gamma = A_{X11}^{\gamma 1}(z) + (\gamma - \gamma^1)[X11(z) + X11(z^{-1}) - 1], \quad (A.2)$$

which follows from Equation (3.8) in the text.

A.2 MSE Associated With the M^γ (optimal two-sided) Filters

First, some preliminaries. Let

$$a_t^{\gamma 1, \gamma 2} = n_t^{\gamma 1} - M^{\gamma 2}(B)x_t.$$

The standard formulas yield

$$\text{ACGF}(a_t^{\gamma 2,\gamma 2}) = A_n^{\gamma 2}(z)A_s^{\gamma 2}(z)/A_x(z), \quad (A.3)$$

where $A_n^{\gamma 2}(z)$, $A_s^{\gamma 2}(z)$, and $A_x(z)$ are the pseudo-ACGF's for $n_t^{\gamma 2}$, $s_t^{\gamma 2}$, and x_t, respectively. By using Equations (2.10), (2.11), and (2.13) in the text and simplifying we have

$$\begin{aligned}\text{ACGF}(a_t^{\gamma 2,\gamma 2}) = [&\tilde{\theta}_n(z)\tilde{\theta}_n(z^{-1})\tilde{\theta}_s(z)\tilde{\theta}_s(z^{-1}) \\ &- \gamma\phi_n(z)\phi_n(z^{-1})\tilde{\theta}_s(z)\tilde{\theta}_s(z^{-1}) \\ &+ \gamma\tilde{\theta}_n(z)\tilde{\theta}_n(z^{-1})U(z)U(z^{-1}) \\ &- \gamma^2 U(z)U(z^{-1})\phi_n(z)\phi_n(z^{-1})]\pi(z)\pi(z^{-1}),\end{aligned} \quad (A.4)$$

where

$$\pi(z) = [\sigma_e\theta_x(z)]^{-1}.$$

Equation (3.7) in the text implies that

$$\text{ACGF}(a^{\gamma 1,\gamma 2}) = \text{ACGF}(a^{\gamma 2,\gamma 2}) + (\gamma^1 - \gamma^2)[M^{\gamma 2}(z) + M^{\gamma 2}(z^{-1}) - 1]. \quad (A.5)$$

Finally, from Equation (4.1)

$$M^{\gamma 2}(z) = U(z)U(z^{-1})[\tilde{\theta}_n(z)\tilde{\theta}_n(z^{-1}) - \gamma^2\phi_n(z)\phi_n(z^{-1})]\pi(z)\pi(z^{-1}) \quad (A.6)$$

Equations (A.4)–(A.6) provide all of the ingredients for forming the MSE associated with the M^γ filters. All of the calculations are straightforward except the inversion of $\theta_x(z)$ that is needed to form $\pi(z)$. The polynomial $\sigma_e\pi(z)$ is the AR polynomial of the autoregressive representation of $\phi_n^1(B)U(B)x_t$. For the calculations it was approximated by truncating the powers of $[\theta_x(z)]^{-1}$ higher than 720. [Two of the series contained factors of $(1 - .9B^{12})$, so very long AR polynomials were necessary for a good approximation.]

A.3 MSE for the X-11 One-Sided Filter

The procedure that was used amounted to forming the one-sided X-11 filter (which is model specific) and then applying the same kind of calculations that are outlined in Section A.1 of this appendix. All of the models that we considered had $\phi_n^1(B) = (1 - B)^2$. The first step in forming the one-sided X-11 filter is to construct the autoregressive representation for x_t. This is given by

$$\tilde{\phi}_n^2(B)\phi_n^1(B)U(B)x_t = e_t, \quad (A.7)$$

where

$$\tilde{\phi}_n^2(B) = \phi_n^2(B)[\theta_x(B)]^{-1}.$$

For the calculations the polynomial $[\theta_x(B)]^{-1}$ was approximated by a 720-term polynomial as described in Section A.2.
Let

$$\psi(B) = \tilde{\phi}_n^2(B)\phi_n^1(B)U(B) \quad (A.8)$$

and

$$\alpha_i(k) = \frac{\partial x_{t+k/t}}{\partial x_{t-i}}, \qquad k = 1, 2, \ldots; i = 0, 1, \ldots.$$

From (A.7) the $\alpha_i(k)$ can be calculated by the recursion

$$\alpha_i(k) = \sum_{j=1}^{k-1} \psi_j\alpha_i(k - j) + \psi_{k+i}. \quad (A.9)$$

Recall that the one-sided X-11 ARIMA procedure is to apply the two-sided X-11 filter to the historical series padded into the future with optimal forecasts. If we denote the one-sided X-11 filter by $\overline{X}11(B)$ $(= \sum \overline{X}11_i B^i)$, then

$$\overline{X}11_i = X11_i + \sum_{k=1}^{84} \alpha_i(k)X11_{-k}, \qquad i = 0, 1, \ldots. \quad (A.10)$$

In Appendix B it is shown that $\overline{X}11(B)$ contains the factor $U(B)$ and that $\overline{W}11(B)$ $[= 1 - \overline{X}11(B)]$ contains the factor $(1 - B)^2$. This allows us to calculate the MSE associated with $\overline{X}11(B)$ by using the same techniques discussed in Section A.1 of this appendix.

A.4 MSE Associated With the H^γ (optimal one-sided) Filters

Rather than directly calculate the MSE associated with this filter, we will use the properties of optimal predictors and the MSE associated with the M^γ that we formed in Section A.2 to find the MSE associated with the H^γ filter. Again, some preliminaries. For $k = 1, 2, \ldots$ it is straightforward to form

$$x_{t+k} - x_{t+k/t} = \sum_{i=0}^{k-1} \delta_i e_{t+k-i}. \quad (A.11)$$

Let $\hat{n}_t^\gamma = M^\gamma(B)x_t$ denote the optimal two-sided estimate of n_t^γ, and let $\tilde{n}_t^\gamma = H^\gamma(B)x_t$ denote the optimal one-sided estimate. We have

$$\begin{aligned}\hat{n}_t^\gamma &= \sum_{i=-\infty}^{\infty} m_i^\gamma x_{t-i} \\ &= \sum_{i=0}^{\infty} m_i^\gamma x_{t-i} + \sum_{k=1}^{\infty} m_k^\gamma x_{t+k/t} + \sum_{k=1}^{\infty} m_k^\gamma[x_{t+k} - x_{t+k/t}] \\ &= \tilde{n}_t + \sum_{k=1}^{\infty} m_k^\gamma \sum_{i=0}^{k-1} \delta_i e_{t+k-i}\end{aligned}$$

or

$$\hat{n}_t^\gamma = \tilde{n}_t^\gamma + \sum_{i=1}^{\infty} \lambda_i^\gamma e_{t+i}, \quad (A.12)$$

where

$$\lambda_i^\gamma = \sum_{j=0}^{\infty} m_{j+i}^\gamma \delta_j, \qquad i = 1, 2, \ldots, \quad (A.13)$$

and we have used the fact that $m_k^\gamma = m_{-k}^\gamma$. From (A.12) we have

$$(n_t^\gamma - \tilde{n}_t^\gamma) = (n_t^\gamma - \hat{n}_t^\gamma) + \sum_{i=1}^{\infty} \lambda_i^\gamma e_{t+i},$$

where the two terms on the right side are uncorrelated because $\hat{n}_t^\gamma$ is a linear minimum MSE predictor of n_t^γ given an information set that includes the e_{t+i}. This implies that

$$\text{MS}(n_t^\gamma - \tilde{n}_t^\gamma) = \text{MS}(n_t^\gamma - \hat{n}_t^\gamma) + \left[\sum_{i=1}^{\infty} (\lambda_i^\gamma)^2\right] \sigma_e^2, \quad \text{(A.14)}$$

where MS denotes mean square. Since $\text{MS}(n_t^\gamma - \hat{n}_t^\gamma)$ was calculated in Section A.2, we can calculate $\text{MS}(n_t^\gamma - \tilde{n}_t^\gamma)$ by calculating the sum of the squared λ_i^γ's. Each λ_i^γ was calculated using Equation (A.13), and the sum in Equation (A.14) was truncated after 720 terms.

In Tables 2–5 values are presented for $\text{MS}(n_t^{\gamma^1} - \tilde{n}_t^{\gamma^2})$ for various combinations of γ^1 and γ^2. These were calculated using

$$\text{MS}(n_t^{\gamma^1} - \tilde{n}_t^{\gamma^2}) = \text{MS}(n_t^{\gamma^2} - \tilde{n}_t^{\gamma^2}) + (\gamma^1 - \gamma^2)(2h_0^{\gamma^2} - 1),$$

which follows from Equation (3.8) of the text.

The MSE's for various estimates of the change in n_t^γ are also presented in Tables 2–5. These were calculated using a straightforward modification of the procedure outlined previously.

APPENDIX B: PROPERTIES OF THE ONE-SIDED X-11 FILTER

In this appendix it will be shown that the one-sided X-11 filter, denoted by $\overline{X}11(B)$, contains the factor $U(B)$, and that $\overline{W}11(B) = 1 - \overline{X}11(B)$ contains the factor $(1 - B)^2$. These two results were used in Section A.3 of Appendix A. I begin by presenting the $\overline{X}11(z)$ polynomial.

In Section A.3 of Appendix A the set of constants $\alpha_i(k)$ was introduced. Let

$$\alpha^k(z) = \sum_{i=0}^{\infty} \alpha_i(k) z^i \quad \text{(B.1)}$$

so that

$$x_{t+k/t} = \alpha^k(B) x_t.$$

Using Equation (A.10) we have

$$\overline{X}11(z) = \sum_{i=0}^{84} X11_i z^i + \sum_{i=0}^{\infty}\sum_{k=1}^{84} X11_{-k}\alpha_i(k) z^i$$

$$= \sum_{i=0}^{84} X11_i z^i + \sum_{k=1}^{84} X11_{-k}\alpha^k(z). \quad \text{(B.2)}$$

The results in this appendix are easily derived from Equation (B.2), given the following important property of $\alpha^k(z)$.

Lemma B.1. If $\phi_n^1(z) = (1 - z)^2$, then $1 - z^k\alpha^k(z)$ contains the factor $(1 - z)^2 U(z)$ for $k = 1, 2, 3, \ldots$.

Proof (supplied by an anonymous referee). The proof is by induction. For $k = 1$, we have

$$1 - z^1\alpha^1(z) = \psi(z) = \tilde{\phi}_n^2(z)(1 - z)^2 U(z)$$

when $\phi_n^1(z) = (1 - z)^2$ [see (A.8)]. Next, write

$$z^{k+1}\alpha^{k+1}(z) = \sum_{i=0}^{\infty} z^{k+1+i}\alpha_i(k + 1)$$

$$= \sum_{i=0}^{\infty} z^{k+1+i}\left\{\sum_{j=1}^{k} \psi_j\alpha_i(k + 1 - j) + \psi_{k+1+i}\right\} \quad \text{(B.3)}$$

from the definition of $\alpha_i(k + 1)$ given in (A.9). By rearranging terms in (B.3) and noting that $\psi_0 = 1$, we have

$$1 - z^{k+1}\alpha^{k+1}(z) = -\psi(z) - \sum_{j=1}^{k} \psi_j z^j[1 - z^{k+1-j}\alpha^{k+1-j}(z)],$$

from which the result follows directly.

By using the result in Lemma B.1 the main result of this appendix is easily shown. First it is shown that $\overline{X}11(z)$ contains $U(z)$. From Lemma B.1 we have $1 - (z^*)^k\alpha^k(z^*) = 0$, where z^* is a root of $U(z)$. This implies that $\alpha^k(z^*) = (z^*)^{-k}$, so [from (B.2)]

$$\overline{X}11(z^*) = \sum_{k=0}^{84} X11_k z^{*i} + \sum_{k=1}^{84} X11_{-k} z^{*-k} = X11(z^*) = 0.$$

This shows that $U(z)$ is a factor of $\overline{X}11(z)$.

To show that $(1 - z)^2$ is a factor of $1 - \overline{X}11(B)$, write

$$1 - \overline{X}11(z)$$

$$= 1 - \sum_{k=0}^{84} X11_k z^k - \sum_{k=1}^{84} X11_{-k}\alpha^k(z)$$

$$= 1 - \sum_{k=0}^{84} X11_k z^k - \sum_{k=1}^{84} X11_{-k} z^{-k} + \sum_{k=1}^{84} X11_{-k}(z^{-k} - \alpha^k(z))$$

$$= [1 - X11(z)] + \sum_{k=1}^{84} X11_{-k} z^{-k}(1 - z^k\alpha^k(z)).$$

The first term in brackets has a factor of $(1 - z)^2$, as discussed in Appendix A. The second term has a factor of $(1 - z)^2$ by Lemma B.1.

[*Received June 1984. Revised June 1986.*]

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