TESTING FOR COMMON TRENDS

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Testing for Common Trends:

Technical Appendix

by

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1. Summary

This technical report contains the statements and proofs of the lemmas and theorems in Stock and Watson (1988).

Cointegrated multiple time series share one or more common trends. In Stock and Watson (1988), we develop two tests for the number of common stochastic trends (i.e. for the order of cointegration) in a multiple time series with and without drift. Both tests involve the roots of the OLS coefficient matrix obtained by regressing the series onto its first lag. Critical values for the tests are tabulated, and their power is examined in a Monte Carlo study.

Economic time series are often modeled as having a unit root in their autoregressive representation, or (equivalently) as containing a stochastic trend. But both casual observation and economic theory suggest that many series might contain the *same* stochastic trends, so that they are cointegrated. If each of n series is integrated of order one but can be jointly characterized by $k < n$ stochastic trends, then the vector representation of these series will have k unit roots and n-k distinct stationary linear combinations. Our tests can be viewed as tests of the number of common trends, the number of autoregressive unit roots, or the number of linearly independent cointegrating vectors.

Both of the proposed tests are asymptotically similar but differ in their treatment of the nuisance parameters of the process. The first test (q_f) is developed under the assumption that certain components of the process have a finite order VAR representation, and the nuisance parameters are handled by

estimating this VAR. The second test (q_c) entails computing the eigenvalues of a "corrected" sample first order autocorrelation matrix, where the correction is essentially a sum of the autocovariance matrices.

Previous researchers have found that U.S. postwar interest rates, taken individually, appear to be integrated of order one. In addition, the theory of the term structure (equating the expected future spot rate to the implicit forward rate, plus a stationary risk premium) implies that yields on similar assets of different maturities will be cointegrated. Applying these tests to postwar U.S. data on the Federal Funds rate and the three- and twelve- month Treasury Bill rates, we find support for this prediction: the three interest rates are found to be cointegrated, possessing a single common trend.

2. Theorems and Lemmas

This section contains the statements of the theorems and lemmas concerning the proposed tests of the number of common trends. For definitions and discussion, see Stock and Watson (1988).

Theorem 3.1. Suppose that $\hat{D} \xrightarrow{p} RD$, that W_t is generated by (3.1) with $W_0=\gamma=0$, that $\hat{\Pi}(L) \xrightarrow{p} R_2\Pi(L)R_2^{-1}$, and that $\max_i E(\eta_{it}^4) \le \mu_4 < \infty$. Then

(i) $T(\hat{\Phi}_f - I_k) \Rightarrow R_2\Psi_k'\Gamma_k^{-1}R_2^{-1}$

(ii) $T(\hat{\lambda}_f - \iota) \Rightarrow \lambda_*$

(iii) $T(|\hat{\lambda}_f| - \iota) \Rightarrow Re(\lambda_*)$.

Lemma 4.1. If $\max_i E(\nu_{it}^4) \le \mu_4 < \infty$ and $\beta_1 = \beta_2 = 0$ in (2.7), then

$$T(\hat{\Phi} - I_k) - [\tilde{C}(1)\Psi_{nT}'\tilde{C}(1)' + M'][\tilde{C}(1)\Gamma_{nT}\tilde{C}(1)']^{-1} \overset{p}{\to} 0$$

where $\Psi_{nT} = T^{-1}\sum\xi_{t-1}\nu_t'$, $\Gamma_{nT} = T^{-2}\sum\xi_t\xi_t'$, and

$M = [\sum_{j=0}^{\infty}(\tilde{C}_j^* - \tilde{C}_j)\tilde{C}_j' + \tilde{C}(1)\tilde{C}(1)'] - E\sum_{j=1}^{\infty}u_{t-j}u_t'$.

Lemma 4.2. Let Ω be a k×k matrix such that $\Omega\Omega' = \tilde{C}(1)\tilde{C}(1)'$. Then, under the conditions of Lemma 4.1,

(i) $T(\tilde{\Phi}_c - I_k) \Rightarrow \Omega\Psi_k'\Gamma_k^{-1}\Omega^{-1}$

(ii) $T(\tilde{\lambda}_c - \iota) \Rightarrow \lambda_*$.

Theorem 4.1. Suppose that $\hat{D} \overset{p}{\to} RD$ and $\hat{M} \overset{p}{\to} R_2MR_2'$. Then under the assumptions of Lemma 4.1,

(i) $T(\hat{\Phi}_c - I_k) \Rightarrow R_2\Omega\Psi_k'\Gamma_k^{-1}\Omega^{-1}R_2^{-1}$

(ii) $T(\hat{\lambda}_c - \iota) \Rightarrow \lambda_*$.

Theorem 5.1. Suppose that $\hat{D} \xrightarrow{p} RD$, that W_t is generated by (3.1), that $\hat{\Pi}(L) \xrightarrow{p} R_2\Pi(L)R_2^{-1}$, and that $\max_i E(\eta_{it}^4) \le \mu_4 < \infty$. Then

a. if $\gamma = 0$ and W_0 is an arbitrary constant,

(a.i) $$T(\hat{\Phi}_f^{\mu} - I_k) \Rightarrow R_2 \Psi_k^{\mu\prime} (\Gamma_k^{\mu})^{-1} R_2^{-1}$$

(a.ii) $$T(\hat{\lambda}_f^{\mu} - \iota) \Rightarrow \lambda_*^{\mu}$$

(a.iii) $$T(|\hat{\lambda}_f^{\mu}| - \iota) \Rightarrow \mathrm{Re}(\lambda_*^{\mu})$$

b. if γ and W_0 are arbitrary constants,

(b.i) $$T(\hat{\Phi}_f^{\tau} - I_k) \Rightarrow R_2 \Psi_k^{\tau\prime} (\Gamma_k^{\tau})^{-1} R_2^{-1}$$

(b.ii) $$T(\hat{\lambda}_f^{\tau} - \iota) \Rightarrow \lambda_*^{\tau}$$

(b.iii) $$T(|\hat{\lambda}_f^{\tau}| - \iota) \Rightarrow \mathrm{Re}(\lambda_*^{\tau}) .$$

Lemma 5.1. If $\max_i E(\nu_{it}^4) \le \mu_4 < \infty$ and

(i) if $\beta_2 = 0$ in (2.7) and β_1 is an arbitrary constant, then

$$T(\hat{\Phi}^{\mu} - I) - [\tilde{C}(1)\Psi_{nT}^{\mu\prime}\tilde{C}(1)' + M'][\tilde{C}(1)\Gamma_{nT}^{\mu}\tilde{C}(1)']^{-1} \xrightarrow{p} 0$$

(ii) if β_1 and β_2 in (2.7) are arbitrary constants, then

$$T(\hat{\Phi}^{\tau} - I) - [\tilde{C}(1)\Psi_{nT}^{\tau\prime}\tilde{C}(1)' + M'][\tilde{C}(1)\Gamma_{nT}^{\tau}\tilde{C}(1)']^{-1} \xrightarrow{p} 0$$

where $\Psi_{nT}^{\mu} = T^{-1}\sum \xi_{t-1}^{\mu} \Delta \xi_t^{\mu\prime}$, $\Gamma_{nT}^{\mu} = T^{-2}\sum \xi_t^{\mu}\xi_t^{\mu\prime}$, $\Psi_{nT}^{\tau} = T^{-1}\sum \xi_{t-1}^{\tau}\Delta\xi_t^{\tau\prime}$, and $\Gamma_{nT}^{\tau} = T^{-2}\sum \xi_t^{\tau}\xi_t^{\tau\prime}$, where $\xi_t^{\mu} = \xi_t - T^{\frac{1}{2}}\theta_{0T}$ and $\xi_t^{\tau} = \xi_t - T^{\frac{1}{2}}\theta_{1T} - T^{-\frac{1}{2}}\theta_{2T}t$, where $\theta_{iT} = T^{-3/2}\sum_{t=1}^{T} a_{it}\xi_t$, $i=0,1,2$, with $a_{0t}=1$, $a_{1t}=4-6(t/T)$, and $a_{2t}=-6+12(t/T)$, and where $M = E\sum_{j=1}^{\infty} u_{t-j}u_t'$.

Theorem 5.2. If $\hat{D} \xrightarrow{p} RD$ and $\hat{M} \xrightarrow{p} R_2 M R_2'$, then under the assumptions of Lemma 5.1,

a. if $\beta_2 = 0$ in (2.7) and β_1 is an arbitrary constant, then

(a.i) $$T(\hat{\Phi}_c^\mu - I) \Rightarrow R_2 \Omega \Psi_k^{\mu\prime} (\Gamma_k^\mu)^{-1} \Omega^{-1} R_2^{-1}$$

(a.ii) $$T(\hat{\lambda}_c^\mu - \iota) \Rightarrow \lambda_*^\mu$$

(a.iii) $$T(|\hat{\lambda}_c^\mu| - \iota) \Rightarrow \mathrm{Re}(\lambda_*^\mu)$$

b. if β_1 and β_2 in (2.7) are arbitrary constants, then

(b.i) $$T(\hat{\Phi}_c^\tau - I) \Rightarrow R_2 \Omega \Psi_k^{\tau\prime} (\Gamma_k^\tau)^{-1} \Omega^{-1} R_2^{-1}$$

(b.ii) $$T(\hat{\lambda}_c^\tau - \iota) \Rightarrow \lambda_*^\tau$$

(b.iii) $$T(|\hat{\lambda}_c^\tau| - \iota) \Rightarrow \mathrm{Re}(\lambda_*^\tau) \ .$$

3. Proofs

Proof of Theorem 3.1.

(i) We first show that $T[\hat{\Phi}_f - I_k] - TR_2[\tilde{\Phi}_f - I_k]R_2^{-1} \xrightarrow{p} 0$. From the definitions of $\tilde{\Phi}_f$ and $\hat{\Phi}_f$ and the (almost sure) invertibility of Γ_k, this follows if

$$T^{-1}\sum \hat{\zeta}_{t-1} \Delta \hat{\zeta}_t' - R_2 T^{-1} \sum \zeta_{t-1} \Delta \zeta_t' R_2' \xrightarrow{p} 0 \qquad \text{(A.1a)}$$

$$T^{-2}\sum \hat{\zeta}_t \hat{\zeta}_t' - R_2 T^{-2} \sum \zeta_t \zeta_t' R_2' \xrightarrow{p} 0 \ . \qquad \text{(A.1b)}$$

To show (A.1a), since $\zeta_t = \Pi(L) S_k D X_t$ and $\hat{\zeta}_t = \hat{\Pi}(L) S_k \hat{D} X_t$,

$$\begin{aligned} &T^{-1}\sum \hat{\zeta}_{t-1} \Delta \hat{\zeta}_t' - R_2 T^{-1} \sum \zeta_{t-1} \Delta \zeta_t' R_2' \\ &\quad = \textstyle\sum_{j=1}^p \sum_{i=1}^p \hat{\Pi}_i S_k \hat{D} (T^{-1} \sum_{t=p+2}^T X_{t-i-1} \Delta X_{t-j}') \hat{D}' S_k' \hat{\Pi}_j' \\ &\quad\quad - R_2 \textstyle\sum_{j=1}^p \sum_{i=1}^p \Pi_i S_k D (T^{-1} \sum_{t=p+2}^T X_{t-i-1} \Delta X_{t-j}') D' S_k' \Pi_j' R_2' \ . \end{aligned}$$

(A.2)

For fixed (i,j), under the stated conditions it is straightforward to show that $T^{-1}\sum_{t=p}^{T} X_{t-i-1}\Delta X'_{t-j}=O_p(1)$ (e.g. Stock 1987). By assumption, $\hat{D}\overset{p}{\to}RD$ and $\hat{\Pi}_i\overset{p}{\to}R_2\Pi_i R_2^{-1}$, so that $\hat{\Pi}_i S_k\hat{D}\overset{p}{\to}R_2\Pi_i R_2^{-1}S_k RD$. Since $S_k R=R_2 S_k$, the difference on the right hand side of (A.2) thus converges in probability to zero, proving (A.1a). The proof of (A.1b) is analogous, using the fact that $T^{-2}\sum X_{t-i}X'_{t-j}=O_p(1)$ for fixed (i,j). Since, from (3.2), $T[\tilde{\Phi}_f-I_k]\Rightarrow\Psi'_k\Gamma_k^{-1}$, it follows that $T[\hat{\Phi}_f-I_k]\Rightarrow R_2\Psi'_k\Gamma_k^{-1}R_2^{-1}$.

(ii) Let $\hat{\lambda}_f^{\dagger}$ denote the vector of ordered eigenvalues of $T[\hat{\Phi}_f-I_k]$. It follows from (i) and the continuity of the eigenvalues as functions of the elements of $T[\hat{\Phi}_f-I_k]$ that $\hat{\lambda}_f^{\dagger}\Rightarrow\lambda_*$. But $T^k\det[\hat{\Phi}_f-\lambda I_k]=\det[T(\hat{\Phi}_f-I_k)-T(\lambda-1)I_k]$, so $\hat{\lambda}_f^{\dagger}=T(\hat{\lambda}_f-\iota)$, from which it follows that $T(\hat{\lambda}_f-\iota)\Rightarrow\lambda_*$.

(iii) Write $T(\hat{\lambda}_{fj}-1)=a_j+ib_j$, where $i=\sqrt{-1}$. By (ii), a_j and b_j are $O_p(1)$ random variables. Now

$$T[|\hat{\lambda}_{fj}|-1] = T[|1+(a_j+ib_j)/T| - 1] = T[\{(1+a_j/T)^2+b_j^2/T^2\}^{1/2} - 1]$$
$$= T[(1+a_j/T) + O_p(T^{-2}) - 1] = a_j + o_p(1)$$

so that $T(|\hat{\lambda}_f|-\iota)-\mathrm{Re}[T(\hat{\lambda}_f-\iota)]\overset{p}{\to}0$. Also, from part (ii) of this Lemma, $\mathrm{Re}[T(\hat{\lambda}_f-\iota)]\Rightarrow\mathrm{Re}(\lambda_*)$. Thus $T(|\hat{\lambda}_f|-1)\Rightarrow\mathrm{Re}(\lambda_*)$, the desired result. □

Proof of Lemma 4.1.

First write $T(\tilde{\Phi}-I_k)$ as

$$T(\tilde{\Phi}-I_k)=TU_T'V_T^{-1} \qquad (A.3)$$

where $U_T=T^{-2}\sum_2^T W_{t-1}\Delta W_t'$ and $V_T=T^{-2}\sum_2^T W_{t-1}W_{t-1}'$. Using (2.7) with $\beta_1=\beta_2=0$, V_T can be written,

$$V_T = T^{-2}\sum_{t=1}^{T-1}[\tilde{C}(1)\xi_t+\tilde{C}^*(L)\nu_t][\tilde{C}(1)\xi_t+\tilde{C}^*(L)\nu_t]'.$$

Under the stated conditions, Stock's (1987) Theorem 1 applies directly, and

$$V_T = \tilde{C}(1)\Gamma_{nT}\tilde{C}(1)' + o_p(1). \qquad (A.4)$$

where Γ_{nT} is defined in the statement of the Theorem. Using Chan and Wei (1988, Theorem 2.4), $\Gamma_{nT}\Rightarrow\Gamma_n=\int_0^1 B_n(t)B_n(t)'dt$, where $B_n(t)$ is a n-dimensional Wiener process and => denotes converence on $C[0,1]^n$. Since $\tilde{C}(1)$ has full row rank under the null, $[\tilde{C}(1)\Gamma_{nT}\tilde{C}(1)']$ is almost surely invertible in the limit.

Turning to U_T, write $TU_T=T^{-1}\sum W_t\Delta W_t'-T^{-1}\sum\Delta W_t\Delta W_t'$. Now

$$T^{-1}\sum\Delta W_t\Delta W_t' \xrightarrow{p} \sum_{j=0}^{\infty}\tilde{C}_j\tilde{C}_j' \qquad (A.5)$$

from (2.5). Using (2.5) and (2.7),

$$T^{-1}\sum W_t\Delta W_t' = \tilde{C}(1)T^{-1}\sum\xi_t[\tilde{C}(L)\nu_t]' + T^{-1}\sum[\tilde{C}^*(L)\nu_t][\tilde{C}(L)\nu_t]'. \qquad (A.6)$$

The second term in (A.6) converges to a constant matrix:

$$T^{-1}\sum[\tilde{C}^{*}(L)\nu_t][\tilde{C}(L)\nu_t]' \xrightarrow{p} \sum_{j=0}^{\infty}\tilde{C}_j^{*}\tilde{C}_j' \quad . \tag{A.7a}$$

The first term in (A.6) can be treated using the decomposition in Stock's (1987) Theorem 1, yielding

$$\tilde{C}(1)T^{-1}\sum\xi_t[\tilde{C}(L)\nu_t]' - \tilde{C}(1)[\Psi_{nT}+I_n]\tilde{C}(1)' \xrightarrow{p} 0 \ . \tag{A.7b}$$

Combining (A.5), (A.6) and (A.7),

$$TU_T - [\tilde{C}(1)(\Psi_{nT}+I_n)\tilde{C}(1)' - \sum_{j=0}^{\infty}\tilde{C}_j\tilde{C}_j' + \sum_{j=0}^{\infty}\tilde{C}_j^{*}\tilde{C}_j'] \xrightarrow{p} 0 \ . \tag{A.8}$$

Combining (A.3), (A.4) and (A.8) yields the desired result with $M=[\tilde{C}(1)\tilde{C}(1)'+\sum_{j=0}^{\infty}(\tilde{C}_j^{*}-\tilde{C}_j)\tilde{C}_j']$. The second expression for M in the Theorem obtains by direct calculation. □

Proof of Lemma 4.2.

(i) Using Lemma 4.1 (i) and the definitions of U_T and V_T given in its proof,

$$\begin{aligned} T(\hat{\Phi}_c-I_k) &= (TU_T - M)'V_T^{-1} \\ &= [\tilde{C}(1)\Psi_{nT}\tilde{C}(1)'-M]'[\tilde{C}(1)\Gamma_{nT}\tilde{C}(1)']^{-1} + o_p(1). \end{aligned} \tag{A.9}$$

From Theorem 2.4 of Chan and Wei (1988), $(\Psi_{nT},\Gamma_{nT}) \Rightarrow (\Psi_n,\Gamma_n)$. Letting $\Omega\Omega'=\tilde{C}(1)\tilde{C}(1)'$ (where Ω is k×k), $\{\tilde{C}(1)B_n(t)\}$ has the same distribution as

$\{\Omega B_k(t)\}$. Thus $(\tilde{C}(1)\Psi_n\tilde{C}(1)',\tilde{C}(1)\Gamma_n\tilde{C}(1)')$ has the same distribution as $(\Omega\Psi_k\Omega',\Omega\Gamma_k\Omega')$. Since $\tilde{C}(1)$ has full row rank by construction, $(\tilde{C}(1)\Gamma_n\tilde{C}(1)')^{-1}$ exists almost surely. It follows that $T(\hat{\Phi}_c-I_k)\Rightarrow(\Omega\Psi_k\Omega')'(\Omega\Gamma_k\Omega')^{-1}$ $=\Omega\Psi_k'\Gamma_k^{-1}\Omega^{-1}$.

(ii) The proof of Theorem 3.1 (ii) applies directly. □

Proof of Theorem 4.1.

(i) Let $\hat{U}_T=T^{-2}\sum\hat{W}_{t-1}\Delta\hat{W}_t'$ and $\hat{V}_T=T^{-2}\sum\hat{W}_{t-1}\hat{W}_{t-1}'$. Then:

$$T[\hat{\Phi}_c-I_k] = (T\hat{U}_T - \hat{M})'\hat{V}_T^{-1} . \qquad (A.10)$$

Comparing (A.9) and (A.10), one finds that the result (i) follows if

$$T\hat{U}_T-R_2TU_TR_2' \overset{p}{\to} 0 \qquad (A.11a)$$

$$\hat{V}_T-R_2V_TR_2' \overset{p}{\to} 0 \quad . \qquad (A.11b)$$

To show (A.11a), use $W_t=S_kDX_t$, $\hat{W}_t=S_k\hat{D}X_t$, and $S_kR=R_2S_k$ to write

$$T\hat{U}_T-R_2TU_TR_2' = S_k\hat{D}T^{-1}\sum X_{t-1}\Delta X_t'\hat{D}'S_k' - S_kRDT^{-1}\sum X_{t-1}\Delta X_t'D'R'S_k' .$$

Since $\hat{D}\overset{p}{\to}RD$ by assumption and since $T^{-1}\sum X_{t-1}\Delta X_t=O_p(1)$, (A.11a) follows. The proof of (A.11b) is similar. Thus, from Lemma 4.2 (i),

$$T[\hat{\Phi}_c-I_k] = [R_2\tilde{C}(1)\Psi_{nT}\tilde{C}(1)'R_2']'[R_2\tilde{C}(1)\Gamma_{nT}\tilde{C}(1)'R_2']^{-1} + o_p(1) .$$

The argument used to prove Lemma 4.2 (i) implies that

$$T[\tilde{\Phi}_c - I_k] \Rightarrow [R_2\Omega\Psi_k\Omega' R_2']'[R_2\Omega\Gamma_k\Omega' R_2']^{-1}$$
$$= R_2\Omega\Psi_k'\Gamma_k^{-1\prime}\Omega^{-1}R_2^{-1} .$$

(ii) This result is an immediate consequence of (i) since $\Psi_k'\Gamma_k^{-1}$ and $R_2\Omega\Psi_k'\Gamma_k^{-1\prime}\Omega^{-1}R_2^{-1}$ are similar matrices. □

It is convenient to prove Lemma 5.1 before proving Theorem 5.1.

Proof of Lemma 5.1.

We prove (ii) first. Let $a_{1t}=4-6(t/T), a_{2t}=-6+12(t/T)$, and $\omega_t \equiv \beta_3\xi_t + \beta_4(L)\nu_t$ (so that from (2.7) $Y_t = \beta_1 + \beta_2 t + \omega_t$). Using the definition of ω_t and Chebyschev's inequality one obtains

$$\begin{bmatrix} T^{-\frac{1}{2}}(\tilde{\beta}_1-\beta_1) \\ T^{\frac{1}{2}}(\tilde{\beta}_2-\beta_2) \end{bmatrix} = \begin{bmatrix} T^{-3/2}\sum a_{1t}\omega_t \\ T^{-3/2}\sum a_{2t}\omega_t \end{bmatrix} + o_p(1) = \begin{bmatrix} \beta_3\theta_{1T} \\ \beta_3\theta_{2T} \end{bmatrix} + o_p(1) \qquad \text{(A.12)}$$

where $\theta_{iT} \equiv T^{-3/2}\sum a_{it}\xi_t$, $i=1,2$.

Turning to the moment matrices comprising $\tilde{\Phi}_c$, write $Y_t^{\tau} = \omega_t - (\tilde{\beta}_1-\beta_1) - (\tilde{\beta}_2-\beta_2)t$. Since $W_t^{\tau} = S_k Y_t^{\tau}$,

$$T^{-2}\sum W_t^{\tau} W_t^{\tau\prime} = S_k T^{-2}\sum[\omega_t - (\tilde{\beta}_1-\beta_1) - (\tilde{\beta}_2-\beta_2)t][\omega_t - (\tilde{\beta}_1-\beta_1) - (\tilde{\beta}_2-\beta_2)t]' S_k' . \qquad \text{(A.13)}$$

Using (A.12), Chebyschev's inequality, and the bound on the fourth moment of ν_t, direct calculation of each of the nine terms in (A.13) shows that

$$T^{-2}\sum W_t^{\tau}W_t^{\tau\prime} = \tilde{C}(1)[\Gamma_{nT}-(\theta_{0T}\theta_{1T}'+\theta_{1T}\theta_{0T}')-(\theta_{3T}\theta_{2T}'+\theta_{2T}\theta_{3T}') + \tfrac{1}{2}(\theta_{1T}\theta_{2T}'+\theta_{2T}\theta_{1T}')+\theta_{1T}\theta_{1T}'+1/3\theta_{2T}\theta_{2T}']\tilde{C}(1)' + o_p(1) \qquad (A.14)$$

where $\theta_{0T}=T^{-3/2}\sum\xi_t$ and $\theta_{3T}=T^{-3/2}\sum(t/T)\xi_t$. This can be rewritten to give the desired result,

$$T^{-2}\sum W_t^{\tau}W_t^{\tau\prime} - \tilde{C}(1)\Gamma_{nT}^{\tau}\tilde{C}(1)' \overset{p}{\to} 0 \qquad (A.15)$$

where $\Gamma_{nT}^{\tau}=T^{-2}\sum\xi_t^{\tau}\xi_t^{\tau\prime}$, where ξ_t^{τ} is defined in the statement of the Lemma.

To obtain a limiting representation in terms of functionals of Wiener processes, note that $\theta_{it}\Rightarrow\int_0^1 a_i(s)B(s)ds=\theta_i$, $i=0,..,3$, by the continuous mapping theorem, where $a_0(s)=1$, $a_1(s)=4-6s$, $a_2(s)=-6+12s$, and $a_3(s)=s$. Thus the terms in (A.14) converges to their counterparts expressed as θ_i and Γ_n, which can be rewritten as $\Gamma_{nT}^{\tau}\Rightarrow\Gamma_n^{\tau}=\int_0^1 B_n^{\tau}(t)B_n^{\tau}(t)'dt$, where $B_n^{\tau}(t)=B_n(t)-\theta_1-\theta_2 t$.

Turning to the term $T^{-1}\sum W_{t-1}^{\tau}\Delta W_t^{\tau\prime}$, use $\Delta W_t^{\tau}=S_k[\Delta w_t-(\tilde{\beta}_2-\beta_2)]$ to write,

$$T^{-1}\sum W_{t-1}^{\tau}\Delta W_t^{\tau\prime} = S_k T^{-1}\sum[w_{t-1}-(\tilde{\beta}_1-\beta_1)-(\tilde{\beta}_2-\beta_2)(t-1)][\Delta w_t-(\tilde{\beta}_2-\beta_2)]'S_k' . \qquad (A.16)$$

Expanding (A.16) using (A.12), defining $\theta_{4T}=T^{-1/2}\xi_T$ and $\theta_{5T}=T^{-1/2}\sum(t/T)\nu_t$, and using Chebyschev's inequality, one obtains:

$$T^{-1}\sum W_{t-1}^{\tau}\Delta W_t^{\tau\prime} = \tilde{C}(1)[\Psi_{nT} - \Theta_{1T}\Theta_{4T}' - \Theta_{2T}\Theta_{5T}' - \Theta_{0T}\Theta_{2T}'$$
$$+ \Theta_{1T}\Theta_{2T}' + \tfrac{1}{2}\Theta_{2T}\Theta_{2T}']\tilde{C}(1)' + M + o_p(1) \qquad \text{(A.17)}$$
$$=\tilde{C}(1)\Psi_{nT}^{\tau}\tilde{C}(1)' + M + o_p(1)$$

which in combination with (A.15) gives the desired result. By direct calculation and Theorem 2.4 of Chan and Wei (1988), $\Psi_{nT}^{\tau} \Rightarrow \Psi_n^{\tau} = \int_0^1 B_n^{\tau}(t)dB_n^{\tau}(t)'$.

(i) The proof of (i) is similar to the proof of (ii) but simpler. By assumption, $\beta_2=0$ so that $Y_t=\beta_1+\omega_t$. Letting $\tilde{\beta}_1=T^{-1}\sum Y_t$, $Y_t^{\mu}=Y_t-\tilde{\beta}_1=\omega_t-(\tilde{\beta}_1-\beta_1)$, one obtains

$$T^{-\frac{1}{2}}(\tilde{\beta}_1-\beta_1) = T^{-3/2}\sum\omega_t = \beta_3\Theta_{0T} + o_p(1) \Rightarrow \beta_3\Theta_0 \; .$$

Thus the limits of the two matrices comprising Φ_c^{μ} can be computed as in the proof of (ii):

$$T^{-2}\sum W_t^{\mu}W_t^{\mu\prime} = S_kT^{-2}\sum[\omega_t-(\tilde{\beta}_1-\beta_1)][\omega_t-(\tilde{\beta}_1-\beta_1)]'S_k'$$
$$= \tilde{C}(1)[\Gamma_{nT}-\Theta_{0T}\Theta_{0T}']\tilde{C}(1)' + o_p(1)$$
$$= \tilde{C}(1)\Gamma_{nT}^{\mu}\tilde{C}(1)' + o_p(1)$$
$$\Rightarrow \tilde{C}(1)\Gamma_n^{\mu}\tilde{C}(1)' \qquad \text{(A.18)}$$

$$T^{-1}\sum W_{t-1}^{\mu}\Delta W_t^{\mu\prime} = S_kT^{-1}\sum[\omega_{t-1}-(\tilde{\beta}_1-\beta_1)]\Delta\omega_t'S_k'$$
$$= \tilde{C}(1)[\Psi_{nT}-\Theta_{0T}\Theta_{4T}']\tilde{C}(1)' + M' + o_p(1)$$
$$= \tilde{C}(1)\Psi_{nT}^{\mu}\tilde{C}(1)' + M + o_p(1)$$
$$\Rightarrow \tilde{C}(1)\Psi_n^{\mu}\tilde{C}(1)' + M \qquad \text{(A.19)}$$

where Γ^{μ}_{nT} and Ψ^{μ}_{nT} are given in the statement of the theorem and where $\Gamma^{\mu}_{n}=\int_0^1 B^{\mu}_{n}(t)B^{\mu}_{n}(t)'dt$ and $\Psi^{\mu}_{n}=\int_0^1 B^{\mu}_{n}(t)dB^{\mu}_{n}(t)'$, where $B^{\mu}_{n}(t)=B_n(t)-\theta_0$. The desired result obtains from (A.18), (A.19), and the definition of $\bar{\Phi}^{\mu}_{c}$. □

Proof of Theorem 5.1.

(b.i) We prove (b.i) first, initially considering the case that D and $\Pi(L)$ are known. To examine the OLS coefficient matrix based on W^{τ}_{t}, let $T(\bar{\Phi}^{\tau}_{f}-I_k)=T\bar{U}^{\tau\prime}_{fT}(\bar{V}^{\tau}_{fT})^{-1}$, where $\bar{U}^{\tau}_{fT} = T^{-2}\sum\zeta^{\tau}_{t-1}\Delta\zeta^{\tau\prime}_{t}$, and $\bar{V}^{\tau}_{fT} = T^{-2}\sum\zeta^{\tau}_{t-1}\zeta^{\tau\prime}_{t-1}$, where $\zeta^{\tau}_{t}=\Pi(L)W^{\tau}_{t}$. Use (3.1) and the definition $\zeta_t=\sum^{t}_{s=1}\eta_s$ to write $\Pi(L)W_t=\Pi(1)W_0-\Pi(L)\gamma t+\zeta_t$. Also let $\tilde{W}_0$ and $\tilde{\gamma}$ denote the coeficients from a regression of W_t onto $(1,t)$. Noting that $\Pi(L)t=\Pi(1)t+\Pi^*(1)$, where $\Pi^*(L)=(1-L)^{-1}[\Pi(L)-\Pi(1)]$, one obtains:

$$\zeta^{\tau}_{t} = \zeta_t - \Pi(1)(\tilde{W}_0-W_0) - \Pi(1)(\tilde{\gamma}-\gamma)t - \Pi^*(1)(\tilde{\gamma}-\gamma) .$$

Analysis like that leading to (A.12) shows that

$$\begin{bmatrix} T^{-\frac{1}{2}}(\tilde{W}_0-W_0) \\ T^{\frac{1}{2}}(\tilde{\gamma}-\gamma) \end{bmatrix} = \begin{bmatrix} \Pi(1)^{-1}\Xi_{1T} \\ \Pi(1)^{-1}\Xi_{2T} \end{bmatrix} + o_p(1) \qquad \text{(A.21)}$$

where $\Xi_{iT}=T^{-3/2}\sum a_{it}\zeta_t$, $i=1,2$. Using (A.21) to provide rates of convergence for $\tilde{W}_0-W_0$ and $\tilde{\gamma}-\gamma$ and applying Chebyschev's inequalty, one obtains

$$\begin{aligned} T\bar{U}^{\tau}_{fT} &= T^{-1}\sum[\zeta_{t-1}-\Xi_{1T}-\Xi_{2T}(t-1)][\eta_t-\Xi_{2T}]' + o_p(1) \\ &= \Psi^{\tau}_{kT} + o_p(1) \Rightarrow \Psi^{\tau}_{k} \end{aligned}$$

$$\tilde{V}^{\tau}_{fT} = T^{-2}\sum[\zeta_{t-1}-\Xi_{1T}-\Xi_{2T}(t-1)][\zeta_{t-1}-\Xi_{1T}-\Xi_{2T}(t-1)]' + o_p(1)$$
$$= \Gamma^{\tau}_{kT} + o_p(1) \Rightarrow \Gamma^{\tau}_{k} .$$

Thus $T(\tilde{\Phi}^{\tau}_{f}-I_k) \Rightarrow \Psi^{\tau\prime}_{k}(\Gamma^{\tau}_{k})^{-1}$.

The extension of this result to the case that D and $\Pi(L)$ are consistently estimated (up to the normalization matrix R) parallels the proof of Theorem 3.1 (i) and is omitted.

(a.i) The proof of (a.i) is similar to the proof of (b.i), with modifications like those used to extend the proof of Lemma 5.1 (ii) to Lemma 5.1 (i).

(a.ii), (a.iii), (b.ii), (b.iii). The proofs of (a.ii), (a.iii), (b.ii), (b.iii) parallel the proofs of Theorem (3.1) (ii) and (iii) and are omitted.

Proof of Theorem 5.2.

The proof of Theorem 5.2 parallels the proofs of Theorem 4.1 and is omitted.

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